3.9. Normal Subgroups and Quotient Groups.

Definition

A subgroup H of G is called Normal subgroup of G if $aH = Ha$ for all $a \in G$.

Example:

(i) In S_3, the subgroup $\{e, P_1, P_2\}$ is normal.

$$S_3 = \{e, P_1, P_2, P_3, P_4, P_5\}$$

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$P_3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad P_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \quad P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

Let $H = \{e, P_1, P_2\}$ is a subgroup of S_3

$$eH = He$$

$$P_1 H = \{P_1 e, P_1 \cdot P_1, P_1 \cdot P_2\}$$

$$= \{P_1, P_2, e\}$$

$$HP_1 = \{e \cdot P_1, P_1 \cdot P_1, P_2 \cdot P_1\}$$

$$= \{P_1, P_2, e\}$$

Hence $P_1 H = HP_1$. H is a Normal subgroup.

(ii)

In S_3, the subgroup $\{e, p_1\}$ is not a normal subgroup

$$S_3 = \{e, p_1, p_2, p_3, p_4, p_5\}$$

Let $H = \{e, p_1\}$ is a subgroup of S_3

$$eH = He$$

$$p_1 H = \{p_1 e, p_1 p_1\} = \{p_1, p_4\}$$

$$H p_1 = \{e p_1, p_3 \cdot p_1\} = \{p_1, p_5\}$$

$$p_1 H \neq H p_1$$

Hence the subgroup H is not a normal subgroup.

Theorem 3.39

Every subgroup of an abelian group is a normal subgroup.

Let G be an abelian group and H be a subgroup of G. $a \in G$. We claim that

$$aH = Ha$$

Let

$$x \in aH$$

$$x = ah \quad \text{for all } h \in H.$$

$$x = ha$$

Since G is abelian, by commutative law, $ha \in G$.

$$x \in Ha.$$

Hence $aH \subseteq Ha$

Similarly,

$x \in Ha$

$x = ha$ for all $h \in H$

$x = ah$

$x \in aH$

Hence $Ha \subseteq aH$

$\therefore aH = Ha$

Hence H is a normal subgroup of G.

$\boxed{\text{Theorem} \quad 3.40}$

Let H be a subgroup of index 2 in a group G. Then H is a normal subgroup of G.

Let H be a subgroup of index 2 in G. That is $[G:H] = 2$.

if $a \in H$ then $H = aH = Ha$.

if $a \notin H$ then aH is a left coset different from H.

Hence $H \cap aH = \phi$.

further, since $[G:H] = 2$

$H \cup aH = G$

$H = G - aH$, $aH = G - H$

Similarly.

$H \cup Ha = G$.

$Ha = G - H$.

Hence $aH = Ha$.

$\therefore H$ is a normal subgroup of G.

[Theorem 3.4]

Let N be a subgroup of G. Then the following are equivalent

(i) N is a normal subgroup of G

(ii) $aNa^{-1} = N$ for all $a \in G$.

(iii) $aNa^{-1} \subseteq N$ for all $a \in G$.

(iv) $ana^{-1} \in N$ for all $n \in N$ and $a \in G$.

Let N be a subgroup of G.

(i) $\Rightarrow$ (ii)

Suppose N is a normal subgroup of G.

$$aN = Na \quad \text{for all } a \in G.$$

post multiply by a^{-1} on both sides

$$aNa^{-1} = Naa^{-1}$$
$$aNa^{-1} = Ne$$
$$aNa^{-1} = N.$$

(ii) $\Rightarrow$ (iii)

$$aNa^{-1} = N \quad \text{for all } a \in G.$$

then $aNa^{-1} \subseteq N$. for all $a \in G$.

(iii) $\Rightarrow$ (iv)

$$aNa^{-1} \subseteq N. \quad \text{for all } a \in G.$$

then $ana^{-1} \in N$ for all $n \in N$.

(iv) $\Rightarrow$ (i)

Suppose that $ana^{-1} \in N$ for all $n \in N$. and $a \in G$.

We claim that $aN = Na$

Let $x \in aN$

$x = an$ for all $n \in N$

$x = an(a^{-1}a)$

$x = (ana^{-1})a$ for all $n \in N$

$x \in Na$ (since $ana^{-1} \in N$)

Hence $aN \subseteq Na$.

Now, let $x \in Na$

$x = na$

$x = (aa^{-1})na$

$x = a(a^{-1}na)$

$x = a(a^{-1}n(a^{-1})^{-1})$ (since $ana^{-1} \in N$)

$x \in aN$

$\therefore Na \subseteq aN$

Hence $Na = aN$.

Hence N is a normal subgroup of G.

PROBLEMS.

01. prove that the intersection of two normal subgroups of a group G is normal subgroup of G.

Let H and K be two normal subgroups of G. Then prove $H \cap K$ is a subgroup of G.

To prove $H \cap K$ is normal subgroup of G.

Now, let $a \in G$ and $x \in H \cap K$

$x \in H \cap K$ then

$x \in H$ and $x \in K$

Since H and K are two normal Subgroups of G.

$axa^{-1} \in H$ and $axa^{-1} \in K$

Hence $axa^{-1} \in H \cap K$.

Thus $H \cap K$ is a normal subgroup of G.

2. The centre H of a group G is a normal Subgroup of G.

The centre H of G is given by

$H = \{a / a \in G,\ ax = xa \text{ for all } x \in G\}$

(Commutative law is true)

Now let

$x \in H$ and $a \in G$.

Hence $ax = xa$

post multiply by a^{-1} on both sides

$axa^{-1} = xaa^{-1}$

$axa^{-1} = x$

$axa^{-1} \in H$

Hence H is a normal subgroup of G.

3. Let H be a subgroup of G. Let $a \in G$ Then aHa^{-1} is a subg[illegible]

Let $e = aea^{-1} \in aHa^{-1}$.

Hence aHa^{-1} is not empty. $aHa^{-1} \neq \phi$

Now, let $x, y \in aHa^{-1}$.

$x \in aHa^{-1}$

$x = ah_1a^{-1}$ for all $h_1 \in H$

$y \in aHa^{-1}$

$y = ah_2a^{-1}$ for all $h_2 \in H$

$$xy^{-1} = (ah_1a^{-1})(ah_2a^{-1})^{-1}$$
$$= (ah_1a^{-1})((a^{-1})^{-1}h_2^{-1}a^{-1})$$
$$= ah_1(a^{-1}a)h_2^{-1}a^{-1}$$
$$= ah_1h_2^{-1}a^{-1}$$
$$xy^{-1} \in aHa^{-1}$$

Since H is a subgroup of G.

$h_1h_2^{-1} \in H$.

Hence aHa^{-1} is a subgroup of G.

4. Show that if a group G has exactly one subgroup H of given order, then H is a normal subgroup of G.

Let the order of H be m

$o(H) = m$

Let $a \in G$. and aHa^{-1} is also a subgroup of G.

we claim that $o(H) = o(aHa^{-1}) = m$

Now Consider $f : H \longrightarrow aHa^{-1}$.

defined by $f(h) = aha^{-1}$

f is 1-1 for.

$$f(h_1) = f(h_2)$$
$$ah_1a^{-1} = ah_2a^{-1}$$

$$a^{-1}a\,h_1\,a^{-1}a = a^{-1}a\,h_2\,a^{-1}a$$

$$e\,h_1\,e = e\,h_2\,e$$

$$h_1 = h_2$$

Pre and post multiply by a^{-1} and a on both sides

f is onto, for

let $x \in aHa^{-1}$

$x = aha^{-1}$ for all $h \in H$

then $f(h) = aha^{-1}$

$f(h) = x$.

Hence f is bijection.

$$|H| = |aHa^{-1}| = m.$$

But H is the only subgroup of G of Order m. So.

$$aHa^{-1} = H.$$

$$aHa^{-1}a = Ha \quad \text{(post multiply by } a \text{ on both sides)}$$

$$aH = Ha$$

Hence H is a normal subgroup of G.

5. Show that if H and N are subgroups of a group G and N is normal in G, then $H \cap N$ is normal in H. Show by an example that $H \cap N$ need not be normal in G.

let $x \in H \cap N$ and $a \in H$

we claim that $axa^{-1} \in H \cap N$.

Now $x \in N$ and $a \in H$

$\Rightarrow axa^{-1} \in N$, since N is normal subgroup

Also $x \in H$ and $a \in H$

$\Rightarrow \quad axa^{-1} \in H$.

Since H is a group

Hence $axa^{-1} \in H \cap N$.

$H \cap N$ is a normal subgroup of H.

(ii) $H \cap N$ is need not be normal in G.

Let $G = S_3$. Take $N = G$. and

$H = \{e, P_3\}$

$N = S_3 = \{e, P_1, P_2, P_3, P_4, P_5\}$

$H = \{e, P_3\}$

$H \cap N = \{e, P_3\} = H$.

which is not normal in G.

6. If H is a subgroup of G and N is a normal subgroup of G then HN is a subgroup of G.

To prove that HN is a subgroup of G it is enough to prove $HN = NH$.

Let $x \in HN$

$x = hn$ where $h \in H$ and $n \in N$

$x \in hN$

But $hN = Nh$ since N is normal

$x \in Nh$

$x = n_1 h$ where $n_1 \in N$

$x \in NH$ Hence

$HN \subseteq NH$

Similarly, $x \in NH$

$x = nh$ where $n \in N$ and $h \in H$.

$x \in Nh$

But $hN = Nh$ since N is normal

$x \in hN$.

$x = h n_2$ where $n_2 \in N$

$x \in HN$

Hence $NH \subseteq HN$.

Therefore $HN = NH$.

Hence HN is a subgroup of G

7\. M and N are normal subgroups of a group G such that $M \cap N = \{e\}$. Show that every element of M commutes with every element of N.

Let M and N be a normal subgroup of G. such the $M \cap N = \{e\}$.

Let $a \in M$ and $b \in N$.

To prove that

$$ab = ba.$$

Consider the element $aba^{-1}b^{-1}$.

Since $a^{-1} \in M$ and M is normal

$$ba^{-1}b^{-1} \in M.$$

Also $a \in M$, $ba^{-1}b^{-1} \in M$. then

$$aba^{-1}b^{-1} \in M.$$

Again since $b \in N$ and N is normal

$$aba^{-1} \in N$$

Also $b^{-1} \in N$ and $aba^{-1} \in N$. then

$$aba^{-1}b^{-1} \in N.$$

Thus $aba^{-1}b^{-1} \in M \cap N$.

$aba^{-1}b^{-1} \in \{e\}$

$aba^{-1}b^{-1} = e$

Post Multiply by b

$aba^{-1}b^{-1}b = eb$

$aba^{-1} = b$

post Multiply by a

$aba^{-1}a = ba$

Hence, $ab = ba$

Theorem 3-42.

A subgroup N of G is normal iff the product of two right cosets of N is again a right coset of N.

Given N is normal subgroup of G. Then to prove product of 2 right coset is a right coset. let Na, Nb be two r. coset

Then, $NaNb = N(aN)b$

$= N(Na)b$ since $Na = aN$

$= NNab$

$= Nab$ since $NN = N$.

Converse part

Given the product of two right coset is again a right coset. to prove N is normal subgroup.

let NaNb be the right Coset of N.

further $ab = ea\,eb$.

$\in NaNb$.

Hence NaNb is right coset containing ab.

$\therefore NaNb = Nab$

Let $a \in G$ and $n \in N$ Then

$ana^{-1} = eana^{-1}$

$\in NaNa^{-1}$

$NaNa^{-1} = Naa^{-1}$

$= N$

$\therefore ana^{-1} \in N.$

Hence N is a normal subgroup of G.

Theorem 2.43

Let N be a normal subgroup of a group G. Then G/N is a group under the operation defined by $NaNb = Nab$.

By our known theorem

Let N is normal subgroup. then the product of two right coset is again a right coset of N.

$NaNb = Nab$. is a well defined operation in G/N

Now,

let $Na, Nb, Nc \in G/N$.

$$Na(NbNc) = Na(Nbc)$$
$$= Na(bc)$$
$$= N(ab)c$$
$$= NaNbNc.$$

The binary operation is Associative

$\because Ne = N \in G/N$

$NaNe = Nae = Na$

$NeNa = Nea = Na$

$\therefore Ne$ is the identity element

Also $NaNa^{-1} = Naa^{-1} = Ne$.

$Na^{-1}Na = Na^{-1}a = Ne$

$\therefore Na^{-1}$ is the inverse of Na.

$\therefore G/N$ is group.

Definition.

Quotient group.

Let N be a normal subgroup of G. Then the group G/N is called the quotient group (factor group) of G Modulo N.

$3Z$ is a normal subgroup of $(Z,+)$.