# Algebra

## 3.11 Homomorphisms

**Definition:** A map $f : G \to G'$ is called a homomorphism if $f(ab) = f(a) \cdot f(b)$ for all $a, b \in G$. ($G$ & $G'$ are groups).

**Ex: 1.** $f : (\mathbb{Z}, +) \to (\mathbb{Z}, +)$ defined by $f(x) = 2x$.

Now, $f(x+y) = 2(x+y) = 2x + 2y = f(x) + f(y)$

Hence $f$ is a homomorphism.

**Ex: 2** $f : (\mathbb{R}^*, \cdot) \to (\mathbb{R}^*, \cdot)$ defined by $f(x) = |x|$.

Now, $f(xy) = |(xy)| = |x| \cdot |y| = f(x) \cdot f(y)$

Thus, $f$ is a homomorphism.

**Definition:** Let $f : G \to G'$ be a homomorphism

(i) If $f$ is onto, then it is called an epimorphism.

(ii) If $f$ is 1-1, then it is called a monomorphism.

(iii) A homomorphism of a group itself is called an endomorphism.

Theorem: (1)

Let $f: G \to G'$ be a homomorphism.
Then

(i) $f(e) = e'$

(ii) $f(a^{-1}) = [f(a)]^{-1}$

(iii) If $H$ is a subgroup, then $f(H)$ is a subgroup of $G'$.

(iv) If $H$ is normal in $G$, then $f(H)$ is normal in $f(G)$.

(v) If $H'$ is a subgroup of $G'$, then $f^{-1}(H')$ is a subgroup of $G$.

(vi) If $H'$ is normal in $f(G)$ then $f^{-1}(H')$ is normal in $G$.

Proof:

(i): Let $a \in G$.

Then $f(a) = f(ae) = f(a) \cdot f(e)$ [since $f$ is homomorphism]

Hence $\boxed{f(e) = e'}$

(ii) $f(a) f(a^{-1}) = f(aa^{-1}) = f(e) = e'$

Thus $\boxed{f(a^{-1}) = [f(a)]^{-1}}$

(iii) Let $H$ be a subgroup of $G$.

To P.T $f(H)$ is subgroup of $G'$.

Since $H \neq \phi$, $f(H)$ is also non-empty.

Now, let $x, y \in f(H)$.

Then, $x = f(a)$, $y = f(b)$ where $a, b \in H$.

$$xy^{-1} = f(a)\cdot[f(b)]^{-1}$$
$$= f(a)\cdot f(b^{-1}) = f(ab^{-1}) \in f(H)$$
[Since $H$ is subgroup, $a,b \in H \Rightarrow ab^{-1} \in H$].

Thus $xy^{-1} \in f(H)$

$\Rightarrow f(H)$ is a subgroup of $G'$.

(iv) Let $H$ be a normal in $G$.

To P.T $f(H)$ is normal in $f(G)$.

Let $x \in f(H)$ & $y \in f(G)$.

To P.T $yxy^{-1} \in f(H)$.

Now, $x = f(a)$ & $y = f(b)$ where $a \in H$ & $b \in H$.

Since $H$ is normal in $G$, $bab^{-1} \in H \Rightarrow f(bab^{-1}) \in f(H)$

But $f(bab^{-1}) = f(b)f(a)f(b^{-1}) \in f(H)$

$$\Rightarrow yxy^{-1} \in f(H)$$

Thus $f(H)$ is normal in $G'$.

(v). If $H'$ is subgroup of $G'$, we have to P.T $f^{-1}(H')$ is a subgroup of $G$.

Since $f(e) = e' \in H'$.

$$\Rightarrow e \in f^{-1}(H').$$

$$\therefore f^{-1}(H') \neq \phi.$$

Let $a, b \in f^{-1}(H') \Rightarrow f(a), f(b) \in H'$

Since $H'$ is a subgroup in $G'$,

$\therefore f(a)[f(b)]^{-1} \in H'$

$\Rightarrow f(a) f(b^{-1}) \in H'$ ($\because$ f is homomorphism)

$\Rightarrow f(ab^{-1}) \in H'$

$\Rightarrow ab^{-1} \in f^{-1}(H')$

Thus $f^{-1}(H')$ is a subgroup of $G$.

(vi): If $H'$ is normal in $f(G)$ then we have to P.T $f^{-1}(H')$ is normal in $G$.

Let $x \in f^{-1}(H')$ and $a \in G$.

Then $f(x) \in H'$ & $f(a) \in f(G)$.

Since $H'$ is normal in $f(G)$,

$f(a) f(x) [f(a)]^{-1} \in H'$

$\Rightarrow f(a) f(x) f(a^{-1}) \in H'$ ($\because$ f is homomorphism)

$\Rightarrow f(axa^{-1}) \in H'$ ($\because$ f is homomorphism)

$\Rightarrow axa^{-1} \in f^{-1}(H')$

Hence $f^{-1}(H')$ is normal in $G$.

**Definition:**

Let $f: G \to G'$ be a homomorphism.
Let $K = \{x / x \in G, f(x) = e'\}$.
Then $K$ is called kernel of $f$ and is denoted by $\ker f$.

**Theorem: 2**

Let $f: G \to G'$ be a homomorphism. Then the kernel $K$ of $f$ is a normal subgroup of $G$.

**Proof:**

Since $f(e) = e'$

$\Rightarrow e \in \ker f$.

$\therefore \ker f \neq \phi$.

Let $x, y \in \ker f$.

$\Rightarrow f(x) = e', \quad f(y) = e' \rightarrow (1)$

$$\text{Now } f(xy^{-1}) = f(x) \cdot f(y^{-1}) \quad (\because f \text{ is homo})$$
$$= f(x)\,[f(y)]^{-1}$$
$$= e' \cdot (e')^{-1} \quad (\text{by } (1))$$
$$= e'$$

$\therefore f(xy^{-1}) = e' \Rightarrow xy^{-1} \in \ker f$.

Thus $\ker f$ is a subgroup of $G$.

Next to P.T $\ker f$ is normal in $G$.

Let $x \in Ker\, f$, $a \in G$.

Then, $f(axa^{-1}) = f(a) f(x) f(a^{-1})$

$$= f(a)\, e' \,[f(a)]^{-1}$$
$$= f(a)\, f(a^{-1})$$
$$= f(aa^{-1})$$
$$= f(e)$$
$$= e'$$

$\therefore f(axa^{-1}) = e'$

$\Rightarrow axa^{-1} \in Ker\, f$.

Hence Ker $f$ is a normal subgroup of $G$.

i) **Theorem: Fundamental theorem of Homomorphism.**

Let $f : G \to G'$ be an epimorphism.

Let $K$ be the kernel of $f$.

Then $\frac{G}{K} \cong G'$.

**Proof:** Define $\phi : \frac{G}{K} \to G'$ by $\phi(Ka) = f(a)$

**Step: 1)** $\phi$ is well defined.

Let $Kb = Ka$.

Then $b \in Ka$.

Hence $b = ka$, where $k \in K$.

Now, $f(b) = f(ka) = f(k) f(a)$

$= e' f(a) = f(a)$.

$\therefore \phi(Kb) = f(b) = f(a) = \phi(Ka)$.

Hence $\phi(Ka) = \phi(Kb)$.

Thus $\phi$ is well defined.

Step:(ii) $\phi$ is 1-1

$$\phi(Ka) = \phi(Kb) \Rightarrow f(a) = f(b)$$
$$\Rightarrow f(a)[f(b)]^{-1} = e'$$
$$\Rightarrow f(ab^{-1}) = e'$$
$$\Rightarrow ab^{-1} \in K$$
$$\Rightarrow a \in Kb$$
$$\Rightarrow Ka = Kb.$$

Hence, $\phi$ is 1-1.

Step:(iii) $\phi$ is onto

Let $a' \in G'$. Since $f$ is onto, there exists $a \in G$ such that $f(a) = a'$.

Thus $\phi(Ka) = f(a) = a'$.

$\Rightarrow \phi$ is onto.

Step: iv) $\phi$ is a homomorphism.

$$\phi(Ka\,Kb) = \phi(Kab) = f(ab)$$

$$= f(a)\,f(b)$$

(since $f$ is homomorphism)

$$= \phi(Ka)\,\phi(Kb)$$

$$\therefore \phi(Ka\,Kb) = \phi(Ka)\,\phi(Kb).$$

Thus $\phi$ is a homomorphism.

Hence $\phi$ is an isomorphism from $\frac{G}{K}$ onto $G'$.

$$\therefore \boxed{\frac{G}{K} \cong G'}$$

Hence Proved.

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