Definition

Let A and B be two sets. Each element x of A there is associated, an element of B, which we denote by f(x). Then f is said to be a function from A to B (or a mapping of A into B). The set A is called the domain of f and the elements f(x) are called the values of f. The set of all values of f is called the range of f.

Definition

Let A and B be two sets and let f be a mapping of A into B. If E $\subset$ A, f(E) is defined to be the set of all elements f(x), for x$\in$ E. We call f(E) the image of E under f. In this notation, f(A) is the range of f. It is clear that f(A)$\subset$ B.
If f(A) $=$ B, we say that f maps A onto B.

Definition

A relation $\sim$ on a set A is said to be equivalence relation if

- Reflexive. a$\sim$a.
- Symmetric. If a$\sim$b then b$\sim$a.
- Transitive. If a$\sim$b, b$\sim$c then a$\sim$c.

Example

Let $S = Z$. $a \sim b$ means $a \equiv b \pmod{m}$. To Prove that $\sim$ is an equivalence relation. Proof. $a \equiv b \pmod{m}$ means a-b is multiple of m. (i). Reflexive.

Clearly a - a = 0 which is multiple of m.

$a \equiv a \pmod{m}$. $a \sim a$. Hence reflexive is true.

(ii). Symmetric.

Let $a \sim b$. To prove that $b \sim a$.

Since $a \sim b$, we have a -b is multiple of m.

Therefore, b-a is also multiple of m.

$b \equiv a \pmod{m}$. Thus $b \sim a$.

(iii) Transitive.
Let a$\sim$ b, b $\sim$ c then we have to prove that a $\sim$ c.
a $\sim$ b implies a-b is multiple of m.
a b = km (1)where k is an integer.
b $\sim$ c implies b c is multiple of m.
i.e b-c = k_1m..(2), where k_1 is an integer.
Now, a -c = km +b -c = km + k_1m = (k+k_1) m
Therefore, a c = k_2 m, where k2 = k+k1 is also an integer.
Hence a - c is multiple of m.
i.e a $\equiv$ c (mod m)
Hence a$\sim$ c. Thus transitive is true.
Thus $\sim$ satisfies Reflexive, Symmetric and Transitive.
Hence $\sim$ is an equivalence relation.

A relation $\sim$ on a set A is said to be Partial order relation if

- Reflexive. $a \sim a$.
- Antisymmetric If $a \sim b$, $b \sim a$ then $a = b$
- Transitive. If $a \sim b$, $b \sim c$ then $a \sim c$.

Definition

Let J be set of positive integers. For any positive integers n, Let $J_n = \{1, 2, ...n\}$
The set A is finite if $A \sim J_n$

Definition

Countable

Then the set A is said to be **Countable**, if there exists 1-1 correspondence between the set A and J i.e $A \sim J$

Definition

at most countable

The set A is said to be **at most countable**, if A is finite or countable.

Definition

Un Countable

The set A is said to be **Un Countable**, if A is neither finite nor countable.

Example Let $A = \{...... -4, -3, -2, -1, 0, 1, 2, 3, 4, ...\}$ be set of all integers.
Let $J = \{1, 2, ...\}$ be set of positive integers.
Define $f : J \to A$ as:
$f(n) = \frac{n}{2}$ if n is even
$f(n) = \frac{(n-1)}{2}$ if n is odd
Then f is 1-1 correspondence between J and A.
Hence A is countable.
Thus set of all integers is countable.

Definition

Sequence: The sequence is a function f defined on the set J of all positive integers.
If $f(n) = x_n$, $n \in J$ is a sequence f denoted by $\{x_n\}$.
The elements of $\{x_n\}$ are called terms of the sequence.

Theorem:1 Every infinite subset of a countable set A is countable.
Proof. Let A be countable.
Let E be infinite subset of A.
We have to prove that E is countable. Now, arrange the elements x of A as a sequence $\{x_n\}$ of distinct elements.
Construct a sequence $\{n_k\}$ as follows:
Let n_1 be the smallest integer such that $x_{n_1} \in E$.
Choose n_2 be the smallest integer $> n_1$ such that $x_{n_2} \in E$
Choose n_3 be the smallest integer $> n_2$ such that $x_{n_3} \in E$
Continuing in this way, Let n_k be the smallest integer $> n_{k-1}$ such that $x_{n_k} \in E$ etc..
Take $f(k) = x_{n_k}$, $k = (1, 2, 3, ...)$.
We get a 1-1 correspondence between E and J (the set of positive integers).
Hence E is countable.
Thus infinite subset of a countable set is countable.

Theorem: 2 Let $\{E_n\}$, n = 1,2,....... be a sequence of countable sets.
Let $S = \bigcup_{n=1}^{\infty} E_n$. Then S is countable. (i.e., countable union of countable sets is countable.)
Proof.. Let every set E_n be arranged in a sequence $\{x_{n_k}\}$, k =1,2,3,.......
Consider the array,

x_{11}	x_{12}	x_{13}	x_{14}	....
x_{21}	x_{22}	x_{23}	x_{24}	....
x_{31}	x_{32}	x_{33}	x_{44}	....
x_{41}	x_{42}	x_{43}	x_{44}	....
....	....	....	....	....
....	....	....	....	....
....	....	....	....	....
x_{n1}	x_{n2}	x_{n3}	x_{n4}	....
....	....	....	....	....

Now, the elements of E_n form the n^{th} row.
Also, the array contains all the elements of S.
These elements and be arranged in a sequence,
$x_{11}; x_{21}, x_{12}; x_{31}, x_{22}, x_{13}; x_{41}, x_{32}, x_{23}, x_{14}; ....(1)$
If any two elements of E_n have elements in common, these will appear more than once in (1).
Since $E_1 \subset S$ and E_1 is infinite, thus S is infinite .
Hence,there is a subset T of the set of all positive integers such that $S \sim T$. Thus S is countable.
Hence countable union of countable sets is countable.

Corollary. Suppose A is at most countable, and for every $\alpha \in A$, B_α is at most countable.
Take $T = \bigcup_{\alpha \in A} B_\alpha$. Then T is at most countable
Proof. Clearly T is a subset of S (as in theorem 2). Since S is countable, and $T \subset S$, By theorem 1 " infinite subset of a countable set is countable",
Hence, we have T is countable.

Theorem. 3 Let A be a countable set. Let B_n be the set all of n-tuples $(a_1, a_2, ...a_n)$.
and the elements $a_1, a_2, ...a_n$ need not be distinct. Then B_n is countable.

Proof.

This theorem is prove by mathematical induction on n.
Since, B_1 *subsetA* and A is countable, hence we have B_1 is countable.
Hence theorem is true for n=1. Assume that result is true for B_{n-1}. (i.e) B_{n-1} is countable. We have to prove that theorem for B_n.
Now, the elements of B_n are of the form,
(b, a) $(b \in B_{n-1}, a \in A)$.
For every fixed b, the set of pairs $(b, a) \sim A$ and hence countable.
Thus B_n is union of countable sets of countable sets, By theorem (2), B_n is countable.

Corollary. The set of all rational numbers is countable.
Proof..
By theorem 3, take n = 2.
We know that every rational number r is of the form $\frac{b}{a}$, where a and b are integers.
The set of pairs (a,b) is countable and hence the fraction $\frac{b}{a}$ is countable.
Thus set of all rational numbers are countable.

Theorem. 4 Let A be the set of all sequences whose elements are the digits 0 and 1. Then the set A is countable.
(The elements of sequences like 1,0,0,1,1,0,0,1,0,1,1,1,0,.........)
Proof. Let E be the countable subset of A.
Let E consist of the sequences $s_1, s_2, ...s_n, ...$
Let us now construct a sequence s as follows:
If the n^{th} digit in s_n is 1, then n^{th} digit of s is 0 and vice-versa.
Thus we get a sequence s differs from every member of E at least one place.
Hence $s \notin E$.
Clearly $s \in A$. Hence E is a proper subset of A.
i.e. we have prove that every countable subset of A is proper subset of A.
Since A not at all proper subset of A,Thus A is uncountable.
Hence the Theorem.