8·2016

DIFFERENCIATION.

Definition:

Let f be defined on $[a,b]$ for any $x \in [a,b]$ define, $f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$ provided limit exists then f' is said to be derivative of f.

Theorem 1:

Let f be defined on $[a,b]$ if f is differentiable at a point $x \in [a,b]$ Then f is continuous at x. (i.e Differenciation $\Rightarrow$ continuous]

proof:

Since f is differentiable at x.

$$\therefore f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \longrightarrow (1)$$

Suppose $t \to x$,

Consider,

$$f(t) - f(x) = \left[\frac{f(t) - f(x)}{t - x} \right] (t - x)$$

$$= f'(x)(t - x)$$

$$= f'(x)(0) \quad [\because t \to x]$$

$$= 0$$

$f(t) - f(x) \to 0.$

$\Rightarrow f(t) = f(x)$ whenever $t \to x$

$\lim f(t) = f(x).$

Theorem 2:

Suppose f, g are defined on $[a,b]$ and differentiable at $x \in [a,b]$. Then P.T $f+g$, ~~$f-g$~~, fg, f/g are differentiable at x. P.T

(i) $(f+g)'x = f'(x) + g'(x)$

(ii) $(fg)'x = f(x)g'(x) + f'(x)g(x)$

(iii) $(f/g)'x = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$.

Proof:

(i) consider,

$$(f+g)t - (f+g)x = f(t) + g(t) - [f(x) + g(x)]$$

$\div$ $t-x$ on both sides,

$$\frac{(f+g)t - (f+g)x}{t-x} = \frac{f(t) - f(x)}{t-x} + \frac{g(t) - g(x)}{t-x}$$

Taking $\lim_{t \to x}$ on both sides,

$$\lim_{t \to x} \frac{(f+g)t - (f+g)x}{t-x} = \lim_{t \to x} \frac{f(t) - f(x)}{t-x} + \lim_{t \to x} \frac{g(t) - g(x)}{t-x}$$

$$\Rightarrow (f+g)'x = f'(x) + g'(x)$$

(ii) To P.T $(fg)'x = f(x)g'(x) + g(x)f(x)$.

Let $h = fg$.

consider, $h(t) - h(x) = (fg)t - (fg)x$

$$= f(t)g(t) - f(x)g(x)$$

$$= f(t)g(t) + f(t)g(x) - f(t)g(x) - f(x)g(x)$$

$$= f(t)[g(t) - g(x)] + g(x)[f(t) - f(x)].$$

$\div\ t-x$ and taking $\lim\limits_{t\to x}$ on both sides,

$$\lim_{t\to x} \frac{h(t) - h(x)}{t-x} = \lim_{t\to x} \frac{f(t)[g(t)-g(x)]}{t-x} + \lim_{t\to x} \frac{g(x)[f(t)-f(x)]}{t-x}$$

$$h'(x) = f(x)g'(x) + g(x)f'(x)$$

(i.e) $(fg)'x = f(x)g'(x) + g(x)f'(x)$

(iii) Let $h = f/g$

$$h(t) - h(x) = (f/g)t - (f/g)x$$

$$= \frac{f(t)}{g(t)} - \frac{f(x)}{g(x)}.$$

$$= \frac{g(x)f(t) - f(x)g(t)}{g(t)g(x)}$$

$$= \frac{1}{g(t)g(x)}\left[f(t)g(x) - f(x)g(x) + f(x)g(x) - g(t)f(x)\right]$$

$$h(t) - h(x) = \frac{1}{g(t)g(x)}\left[g(x)(f(t) - f(x)) - f(x)(g(t) - g(x))\right]$$

$\div\ t - x$ and taking $\lim\limits_{t\to x}$ on both sides,

$$\Rightarrow \lim_{t\to x}\frac{h(t)-h(x)}{t-x} = \lim_{t\to x}\frac{1}{g(t)g(x)}\left[g(x)\lim_{t\to x}\frac{f(t)-f(x)}{t-x} - f(x)\lim_{t\to x}\frac{g(t)-g(x)}{t-x}\right]$$

$$\Rightarrow h'(x) = \frac{1}{g(x)g(x)}\left[g(x)f'(x) - f(x)g'(x)\right]$$

$$\Rightarrow (f/g)'x = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}.$$

Theorem 3:

chain rule for differentiation.

Suppose f is continuous on $[a,b]$, $f'(x)$ exists at some point $x \in [a,b]$. g is defined on an interval I, which contains. The range of f and g is differentiable at the point $f(x)$.

If $h(t) = g(f(t))$ $(a \le t \le b)$. Then h is differentiable at x & $h'(x) = g'(f(x)) \cdot f'(x)$.

proof:

Given that f is differentiable at x and g is differentiable at $f(x)$.

$$\text{(i.e) } \lim_{t \to x} \frac{f(t) - f(x)}{t - x} = f'(x) \longrightarrow (1)$$

$$\lim_{f(t) \to f(x)} \frac{g(f(t)) - g(f(x))}{f(t) - f(x)} = g'(f(x)) \longrightarrow (2)$$

Let $u(t) \to 0$ as $t \to x$ and

$v(f(t)) \to 0$ as $f(t) \to f(x)$

$$(1) \Rightarrow \lim_{t \to x} \frac{f(t) - f(x)}{t - x} = f'(x) + 0$$

$$= f'(x) + u(t) \longrightarrow (3)$$

$$\Rightarrow f(t) - f(x) = (t - x)\left[f'(x) + u(t)\right]$$

$$(2) \Rightarrow \lim_{f(t) \to f(x)} \frac{g(f(t)) - g(f(x))}{f(t) - f(x)} = g'(f(x)) + 0$$

$$= g'(f(x)) + v(f(t)) \longrightarrow (4)$$

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$\Rightarrow g[f(t)-f(x)] = [f(t)-f(x)][g'(f(x))+v(f(t))]$ $\to$ (5)

Now,

$$h(t)-h(x) = g[f(t)]-g[f(x)]$$

$$= [f(t)-f(x)][g'(f(x))+v(f(t))]$$

$$\Rightarrow h(t)-h(x) = (t-x)\{(f'(x)+u(t))\}(g'(f(x))+v(f(t)))$$ [by 3]

$$\frac{h(t)-h(x)}{t-x} = f'(x)+u(t)[g'(f(x))+v(f(t))]$$

$$= f'(x)g'f(x)+f'(x)v(f(t))+u(t)g'f(x)+u(t)v(f(t))$$ $\to$ (6)

Now, if $\lim t\to x$,

since f is continuous,

$$t\to x \Rightarrow f(t)\to f(x)$$

$$\therefore t\to x \Rightarrow u(t)=0$$

$$f(t)-f(x) \Rightarrow v(f(t))=0$$

(6) $\Rightarrow \lim_{t\to x}\frac{h(t)-h(x)}{t-x} = f'(x)g'f(x)+f'(x)\cdot 0 + 0\cdot g(f(x))+0\cdot 0$

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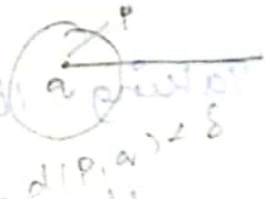
Local maximum:

Let f be a real valued function define on a metric space x. Then f has a local maximum at a point $p \in x$. if there exists $\delta > 0$ such that $f(q) \leq f(p)$ $\forall q \in x$ with $d(p,q) < \delta$.



Local minimum:

Let f be a real valued function define on a metric space x. Then f has a local minimum at a point $p \in x$. if there exists $\delta > 0$ such that $f(q) \geq f(p)$ $\forall q \in x$ with $d(p,q) < \delta$.

Theorem 4:

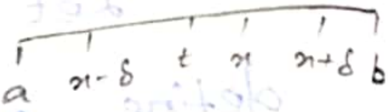
Let f be defined on $[a,b]$ if f has local maximum (or) local minimum at a point $x \in [a,b]$ and if $f'(x)$ exists Then $f'(x) = 0$.

Proof:

Suppose f has a local minimum at $x \in (a,b)$, then there exists $\delta > 0$ such That $f(t) \leq f(x)$, $\forall t \in (x-\delta, x+\delta) \rightarrow (1)$

case (i)

If $x-\delta < t < x$, then $t-x < 0$.



By ① $f(t)-f(x) \le 0$.

$$\frac{f(t)-f(x)}{t-x} \ge 0.$$

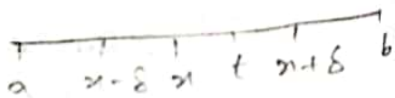
Taking lt $t \to x$, we get

$$\lim_{t \to x} \frac{f(t)-f(x)}{t-x} \ge 0.$$

$$\therefore f'(x) \ge 0 \longrightarrow ②$$

case (ii)

If $x < t < x+\delta$ then $t-x > 0$.



$$\therefore \frac{f(t)-f(x)}{t-x} \le 0.$$

Taking lt $t \to x$, we get

$$\lim_{t \to x} \frac{f(t)-f(x)}{t-x} \le 0.$$

$$\therefore f'(x) \le 0 \longrightarrow ③$$

From ② and ③ we get

$$f'(x) = 0.$$

Theorem 5: Generalised mean value theorem

If f and g are continuous real value functions on $[a,b]$ which are differentiable in (a,b) then there is a point $x \in (a,b)$ at which $[f(b)-f(a)]g'(x) = [g(b)-g(a)]f'(x)$

Proof:

Let $h(t) = [f(b)-f(a)]g(t) - [g(b)-g(a)]f(t)$

$a \le t \le b$.

Since f and g are continuous on $[a,b]$ and differentiable on (a,b).

$\therefore$ h is also continuous on $[a,b]$ & differentiable on (a,b).

To p.t $h'(x) = 0$ for some $x \in (a,b)$

case (i):

If h is constant, then $h'(x) = 0 \ \forall \ x \in (a,b)$.

case (ii):

If $h(t) > h(a)$ for some $t \in (a,b)$

Let x be a point on $[a,b]$ at which h attains its maximum.

(i.e) h has a local minimum at

By above Theorem, $h'(x) = 0$.

Case (iii)

Similarly If $h(t) < h(a)$ for some $t \in (a,b)$

Then h attains local minimum at $x \in (a,b)$

Again by above Theorem, $h'(x) = 0$.

Hence In are cases $h'(x) = 0$ for some $x \in (a,b)$.

$$\therefore [f(b)-f(a)]g'(x) - [g(b)-g(a)]f'(x) = 0$$
$a \le x \le b$.

$$\therefore [f(b)-f(a)]g'(x) = [g(b)-g(a)]f'(x).$$

Mean Value Theorem:

If f is a real continuous function on $[a,b]$ which is differentiable in (a,b) then there is a point $x \in (a,b)$ at which $f(b)-f(a) = (b-a)f'(x)$.

Proof:

By Generalized mean value theorem, we have $[f(b)-f(a)]g'(x)$
$= [g(b)-g(a)]f'(x)$ $\to$ ①

$\forall x \in (a,b)$

Let $g(x) = x$

$g'(x) = 1$

$g(b) = b$

$g(a) = a$

Hence from (1) we get

$f(b) - f(a) = (b-a) f'(x)$.

Theorem 7 :-

Suppose f is differentiable on (a,b) then P.T (i) If $f'(x) \geq 0$, Then f is monotonically increasing.

(ii) If $f'(x) = 0 \cdot \forall x \in (a,b)$. Then f is constant.

(iii) If $f'(x) \leq 0 \; \forall x \in (a,b)$ Then f is monotonically decreasing.

Proof :-

(i) By mean value theorem, we have

$f(x_2) - f(x_1) = (x_2 - x_1) f'(x) \longrightarrow (1)$

for each pair x_1, x_2 in (a,b).

since $f'(x) \geq 0$.

$\therefore$ If $x_2 > x_1$, R.H.S of (1) ≥ 0

Hence L.H.S of (1) ≥ 0.

(i.e) $f(x_2) \geq f(x_1)$

$\therefore$ f is monotonic increasing.

(ii) If $f'(x) = 0$

① $\Rightarrow f(x_2) - f(x_1) = 0$.

$\therefore f(x_2) = f(x_1)$ for each values x_1, x_2 in (a,b).

$\Rightarrow f$ is constant.

(iii) Since $f'(x) \le 0$ and if $x_2 \le x_1$

① $\Rightarrow$ $\therefore$ R.H.S of ① ≥ 0.

Hence L.H.S of ① ≥ 0.

$\therefore f(x_2) - f(x_1) \ge 0$.

$\therefore f(x_2) \ge f(x_1)$ whenever $x_2 \le x_1$

$\Rightarrow f$ is monotonically decreasing.

Theorem 8:

Suppose f is real differentiable function on $[a,b]$ and suppose $f'(a) \le \lambda < f'(b)$. Then there is a point $x \in (a,b)$ such that $f'(x) = \lambda$ (Similarly a result holds of the case $f'(a) > f'(b)$).

proof :

Let $g(t) = f(t) - \lambda t$. $\longrightarrow$ (1)

Then $g'(t) = f'(t) - \lambda$ $\longrightarrow$ (2)

$g'(a) = f'(a) - \lambda$ $\longrightarrow$ (3)

$g'(b) = f'(b) - \lambda$ $\longrightarrow$ (4)

(3) $\Rightarrow$

Since $f'(a) < \lambda < f'(b)$

$f'(a) - \lambda < 0 \Rightarrow g'(a) < 0$ (by (3))

and $f'(b) - \lambda > 0 \Rightarrow g'(b) > 0$ (by (4))

$\therefore g'(a) < 0$ such that $g(t_1) < g(a)$ and

$g'(b) > 0$ such that $g(t_2) < g(b)$ $\forall t_1, t_2 \in (a,b)$

Hence by known theorem g attains its minimum on $[a,b]$ at some point $x \in (a,b)$.

$\therefore$ By Theorem (A) $g'(x) = 0$

(i.e) $f'(x) - \lambda = 0$

$\therefore f'(x) = \lambda$.

Rolle's Theorem:

Suppose f has a derivative at each point of (a,b) and f is continuous at end points a and b if $f(a) = f(b)$ then there exists atleast one point $x \in (a,b)$ such that $f'(x) = 0$.

Proof:

Suppose $f' \neq 0$ in each point of (a,b)

Given that differentiable on (a,b) $\Rightarrow$ f is continuous on (a,b) also f is continuous at a and b.

$f'(b)$

t

$\therefore f$ is continuous on $[a,b]$

$\therefore f$ attains minimum say m and maximum say M such that $f(a) = m$ and $f(b) = M$.

Given $f(a) = f(b)$

$\therefore \quad m = M$.

$\therefore f$ is constant function on (a,b).

Hence $f' = 0$ in (a,b)

This is a contradiction.

Hence there exists $x \in (a,b)$ such that $f'(x) = 0$.

L' HOSPITALS RULE:

Suppose f and g are real and differentiable in (a,b) and $g'(x) \neq 0 \ \forall x \in (a,b)$ where $-\infty \leq a < b \leq \infty$ and suppose $\frac{f'(x)}{g'(x)} \to A$ as $x \to a$. If $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$ or if $g(x) \to +\infty$ as $x \to a$. Then $\frac{f(x)}{g(x)} \to A$ as $x \to a$.

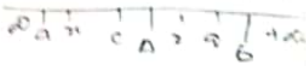
Proof:

Consider the case in which $-\infty \leq A < +\infty$

Given that $\dfrac{f'(x)}{g'(x)} \to A$ as $x \to a \to$ (1)

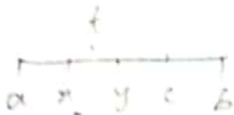
$f(x) \to 0$ and $g(x) \to 0$ as $x \to a$ $\to$ (2)

$g(x) \to +\infty$ as $x \to a \to$ (3)



Choose a real number q such that $A < q$ and Then choose r such that $A < r < q$.

Now, by eqn (1) there is a point $c \in (a,b)$ such that $\dfrac{f'(x)}{g'(x)} \to A$.



$$\Rightarrow \frac{f'(x)}{g'(x)} < r \longrightarrow (4)$$

If $a < x < y < c$, There exists $t \in (x,y)$ such that by Generalized mean value Theorem, we have

$$[f(y) - f(x)]\, g'(t) = [g(y) - g(x)]\, f'(t).$$

$$\Rightarrow \frac{-[f(x) - f(y)]}{-[g(x) - g(y)]} = \frac{f'(t)}{g'(t)} < r \longrightarrow (5) \quad (\text{by } (4))$$

Now, suppose from (5),

$f(x) \to 0,\ g(x) \to 0$ as $x \to a$, Then

taking $\dfrac{0 - f(y)}{0 - g(y)} = \dfrac{f'(t)}{g'(t)} < r < q$

$$\Rightarrow \frac{f(y)}{g(y)} < r < q \longrightarrow (6) \quad (a<y<c)$$

Suppose (3) holds.

Keeping y fixed in (5).

Choosing a point $c_1 \in (a,y)$ such That $g(x) > g(y)$ and $g(x) > 0$ if $a < x < c_1 < y$

Multiply $\frac{g(x) - g(y)}{g(x)}$ in (5), we get

$$\frac{f(x) - f(y)}{\cancel{g(x) - g(y)}} \times \frac{\cancel{g(x) - g(y)}}{g(x)} < r \cdot \left[\frac{g(x) - g(y)}{g(x)} \right]$$

$$\Rightarrow \frac{f(x) - f(y)}{g(x)} < r \cdot \frac{g(x)}{g(x)} - r \frac{g(y)}{g(x)}$$

$$\Rightarrow \frac{f(x)}{g(x)} - \frac{f(y)}{g(x)} < r - r \frac{g(y)}{g(x)}$$

$$\Rightarrow \frac{f(x)}{g(x)} < r - r \frac{g(y)}{g(x)} + \frac{f(y)}{g(x)} \longrightarrow (7)$$

Taking $\lim_{x \to a}$ in (7), we have

There exists $c_2 \in (a, c_1)$, such that

$$\frac{f(x)}{g(x)} < r - r \cdot 0 + 0 \quad [\because (3) \Rightarrow g(x) \to \infty \text{ whenever } x \to a]$$

$$\Rightarrow \frac{f(x)}{g(x)} < r$$

$$\Rightarrow \therefore \frac{f(x)}{g(x)} < q \rightarrow (8) \quad (\because r < q)$$

Similarly, If $-\infty < A \leq \infty$, we get

$$p < \frac{f(x)}{g(x)} \longrightarrow (9)$$

From (8) and (9), we get

$$p < \frac{f(x)}{g(x)} < q, \text{ whenever } -\infty \leq p < A < q \leq \infty$$

$$\frac{f(x)}{g(x)} \to A \text{ as } x \to a.$$

Higher order :

If f has a derivative f' on an interval and if f' is itself differentiable then derivative of f' is denoted by f''. Similarly derivative of f'' is f''' and continue in this manner we get sequence of functions $f, f', f'', f''', \ldots, f^{(n)}$ such that each of which is derivative of the preceeding one.

Ex: Suppose $f(x) = x^3$, then

$f'(x) = 3x^2$

$f''(x) = 6x$

$f'''(x) = 6$

$f''''(x) = 0$

Taylor's theorem:

Suppose f is the real function on $[a,b]$, n is the positive integer, $f^{(n-1)}$ is continuous on $[a,b]$ $f^n(t)$ exists $\forall t \in (a,b)$. Let α, β be distinct points of $[a,b]$ and define $p(t) = \sum_{k=0}^{n-1} \frac{f^k(\alpha)}{k!}(t-\alpha)^k$, Then P.T there exists point x between α, β such that

$$f(\beta) = P(\beta) + \frac{f^n(x)}{n!}(\beta-\alpha)^n.$$

Proof:

Let M be a real number.

$$\text{Let } f(\beta) = P(\beta) + M(\beta-\alpha)^n \longrightarrow (2)$$

$$\text{where } p(t) = \sum_{k=0}^{n-1} \frac{f^k(\alpha)}{k!}(t-\alpha)^k \longrightarrow (1)$$

$$\text{and } g(t) = f(t) - p(t) - M(t-\alpha)^n \rightarrow (3)$$

To P.T $M = \frac{f^n(x)}{n!}$ for some $x \in (\alpha, \beta)$

$① \Rightarrow p^k(t) = f^k(\alpha) \longrightarrow ④$

Also, $p^n(t) = 0 \longrightarrow ⑤$

Now, $③ \Rightarrow g(t) = f(t) - p(t) - M(t-\alpha)^n$

$$g'(t) = f'(t) - p'(t) - Mn(t-\alpha)^{n-1}$$

$$g''(t) = f''(t) - p''(t) - Mn(n-1)(t-\alpha)^{n-2}$$

$$\vdots$$

$$g^n(t) = f^n(t) - p^n(t) - Mn(n-1)\dots 1$$

$$= f^n(t) - p^n(t) - Mn!$$

$$\boxed{g^n(t) = f^n(t) - M \cdot n!} \longrightarrow ⑥$$

[by ⑤ $p^n(t)=0$]

since $p^k(\alpha) = f^k(\alpha)$ $\to ⑦$

$$p(\alpha) = f(\alpha)$$

$$p'(\alpha) = f'(\alpha) \text{ etc..}$$

Now, put $t = \alpha$ in ③, we get

$$g(\alpha) = f(\alpha) - p(\alpha) - M(\alpha-\alpha)^n \quad [\text{By } 7]$$

$$= f(\alpha) - p(\alpha) - M \cdot 0$$

$$g(\alpha) = 0.$$

Similarly,

$$g'(\alpha) = g''(\alpha) = \dots = 0$$

Also,

$$g(\beta) = [f(\beta) - p(\beta)] - M(\beta-\alpha)^n \quad [\because p(\beta) = f(\beta)]$$

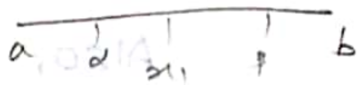
$$= f(\beta) - f(\beta) - M(\beta-\alpha)^n$$

$$= -M(\beta-\alpha)^n$$

$$= (f(\beta) - p(\beta)) = 0 \quad (\text{By } ②)$$

$$\therefore g(\alpha) = 0 ; \quad g(\beta) = 0 \quad ① $$

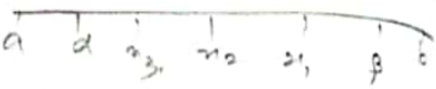
$$g(\alpha) = g(\beta)$$



Hence by Rolle's Theorem.

There exists $x_1 \in (\alpha, \beta)$ such that

$$g'(x_1) = 0$$



since $g'(\alpha) = 0$ & $g'(x_1) = 0$

There exists a point $x_2 \in (\alpha, x_1)$

such that $g''(x_2) = 0$

Similarly,

$g'''(x_3) = 0$ for some $x_3 \in (\alpha, x_2)$

After n steps, we get

$g^n(x_n) = 0$ for some $x_n \in (\alpha, x_{n-1})$

Thus $g^n(x) = 0$ for some $x \in (\alpha, \beta)$

Using this in ⑥, we get

$$0 = f^n(x) - M \cdot n!$$

$$M = \frac{f^n(x)}{n!}$$

Hence $f(\beta) = P(\beta) + \dfrac{f^n(x)}{n!}(\beta - \alpha)^n$

Differenciation of vector valued functions:

If f is the complex function defined on $[a,b]$ and if f_1 and f_2 are real and imaginary parts of f. (i.e) $f(t) = f_1(t) + i f_2(t)$ for $a \leq t \leq b$, where $f_1(t)$ and $f_2(t)$ are real.

Then $f'(x) = f_1'(x) + i f_2'(x)$. Also f is differentiable at x only if f_1 and f_2 are differentiable at x.

Theorem:

Suppose f is continuous mapping of $[a,b]$ into R^k and f is differentiable in (a,b). Then P.T there exist $x \in (a,b)$ such that $|f(b) - f(a)| \leq |b-a| \sup |f'(x)|$.

Proof:

Let $z = f(b) - f(a) \rightarrow (1)$ &

define $\phi(t) = z \cdot f(t) \rightarrow (2)$ $a \leq t \leq b$.

Then ϕ is a real valued continuous function on $[a,b]$ which is differentiable in (a,b).

By mean valued theorem, we have

$\phi(b) - \phi(a) = (b-a)\, \phi'(x)$

$= (b-a)\, z \cdot f'(x) \rightarrow (3)$ (by (2))

where $x \in (a,b)$

But

$$\phi(b) - \phi(a) = z f(b) - z f(a)$$

$$= z(f(b) - f(a))$$

$$= z.\bar{z}$$

$$= |z|^2 \longrightarrow (4)$$

using (4) in (3), we get

$$|z|^2 = |b-a \; z.f'(x)|$$

$$\leq |b-a||z||f'(x)| \quad \text{/ by Schwarz inequality}$$

$$\Rightarrow |z|^2 \leq |b-a| \, |z| \, |f'(x)|$$

$$\Rightarrow |z| \leq |b-a| \, |f'(x)|$$

$$\Rightarrow |f(b) - f(a)| \leq |b-a| \, |f'(x)|.$$

UNIT - III