Section-2.

METRIC SPACES.

A set X, whose elements are known as points is said to be metric space. If with any two points p & q of X there is associated real number $d(p,q)$ called the distance from p to q. Such that.

a) $d(p,q) \geq 0$ if $p \neq q$. $d(p,p) = 0$

b) $d(p,q) = d(q,p)$

c) $d(p,q) \leq d(p,r) + d(r,q)$ for any $r \in X$. Any function with these properties is called a distance function on a metric.

Note:

If d is metric on X, then $d: X \times X \to R$ is a mapping satisfies above these condition

Example:

Set $X = R^k$ be the Euclidean space. Let $d: R^k \times R^k \to R$, be defined as $d(x,y) = |x-y|$ $x, y \in R^k$.

proof:

we shall prove d is metric on X.

Let $x = (x_1, x_2 \ldots x_k)$

$y = (y_1, y_2 \ldots y_k)$

W.K.T $|x-y| = \sqrt{\sum_{i=1}^{k} (x_i - y_i)^2}$

i) $d(x,y) = |x-y| = \sqrt{\sum_{i=1}^{k} (x_i - y_i)^2} > 0 \quad x \neq y$

$\therefore d(x,y) > 0 \; : x \neq y$

If $x = y$, $d(x,y) = |x,y| = \sqrt{\sum_{i=1}^{k} (x_i - y_i)^2} = 0$

$d(x,y) = 0 \; ; \; x = y$

ii) $d(x,y) = |x-y| = \sqrt{\sum_{i=1}^{k} (x_i - y_i)^2}$

$= \sqrt{\sum_{i=1}^{k} (y_i - x_i)^2}$

$= |y-x|$

$\therefore d(x,y) = d(y,x)$

iii) Let $z = (z_1, z_2 \dots z_k) \in R^k$.

Now $d(x,y) = |x-y|$

$= |x-z+z-y|$

$\leq |x-z| + |z-y|$

$= d(x,y) + d(z,y)$

$d(x,y) \leq d(x,z) + d(z,y)$

$\therefore d$ is metric on R^k.

$\therefore (R^k, d)$ is a metric space.

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Note :

(R^1, d) is a metric space, $d(x,y) = |x-y|$

$x, y \in R$.

Definition :

Let $a, b \in R$

$(a,b) = \{x \in R \mid a < x < b\}$ is called a segment

$[a,b] = \{x \in R \mid a \le x \le b\}$ is an interval.

Note : $[a,b) = \{x \in R \mid a \le x < b\}$ and $(a,b] = \{x \in R \mid a < x \le b\}$

are called half-open intervals.

Difinition :

Let $a_i < b_i$, $i = 1, 2 \ldots k$ be set of all points in R^k whose co-ordinates satisfy

$x = x_1, x_2 \ldots x_k$

the inequality $a_i \le x_i \le b_i$ $(1 \le i \le k) \in R^k / a_i \le x_i \le b_i$, $i = 1, 2 \ldots k\}$

is called a k-cells.

Note :

i) one cell is an interval

i.e, $A = \{x / a_i \le x_i \le b_i\}$

2) two cell is a rectangle

i.e ; $A = \{(x_1, x_2) \mid \begin{matrix} a_1 \le x \le b \\ a_2 \le x_2 \le b_2 \end{matrix}\}$.

Definition: convex.

A subset E of R^k is said to be a convex set.

if $\lambda x + (1-\lambda) y \in E$ whenever $x \in E, y \in E$ and $0 < \lambda < 1$

Eg:

Balls are convex.

Definition: (bounded).

Let (X,d) be a metric space. Let $E \subset X$. E said to be bounded if there is a real number M and a point $q \in X$ such that $d(p,q) \leq M$, $\forall p \in E$

Eg:

Let $E = [4,7] \in R$.

Let $q = 3 \in X$; Let $M = 5$

Then clearly $d(p,q) = |3-p| < 5$ $\forall p \in E$

E is bounded.

Definition: (Dense)

Let (X,d) be a metric space Let $E \subset X$ is said to be dense in X. if every pt of X is a limit point of E or a point of E or.

Eg

Let R^1 neighbourhood are segment. In R^2 neighbourhoods are interiors of circles

Eg:

Let $E = \{1/n . n = 1, 2 \ldots\}$

Then 0 is a limit point of E. But $0 \notin E$ also if no point of E is a limit point of E.

E is not closed also E is not open.

Let $E = R^2$

Then if R is a limit point of E then $P \in E$

If $P \in E$ then P is an interior point of E

$\therefore$ E is both open and closed.

Theorem: 2.19

Every ~~bounded~~ neighbourhood is an open set.

Proof:

Let (x, d) be a metric space. Let $p \in X$ and $\delta > 0$. Let $E = N_r(P)$ be a neighbourhood of P.

we have to prove E is open.

For that we have to prove each point of E is an interior point of E

$\therefore E = N_r(P) = \{x \in E \mid d(P, x) < r\}$

Let $q \in E$

$d(P, q) < r$

There exist a positive real number 'h' such that. $d(P, q) = r - h$.

now, Let us consider $N_h(q) = \{x \in E / d(P, q) < h\}$

Let $S \in N_h(q)$ then $f(q, s) < h$.

Now $d(p,s) \le d(p,q) + d(q,s)$

$< r-h+h$

$d(p,s) < r$

$\therefore s \in N_r(p) = E \quad \forall s \in N_h(q)$

Since q is arbitrary every point of E is an interior point of E.

$\therefore E$ is open.

Theorem: 2.20

If p is a ~~point~~ limit point of a set E. Then every neighbourhood of p contains infinitely many point of E.

Proof:

Let (X,d) be a metric space

Let $E \subset X$. Let p be a limit point of E

we have to prove every neighbourhood of p contains infinitely many points of E.

Suppose not there exist a neighbourhood N of p which contains only a finite number of point of E

let. $q_1, q_2 \dots q_n$ for these point of $N \cap E$

i.e. $N \cap E = \{q_1, q_2, \dots q_n \mid q_i \ne p\}$.

Let $r = \min d(p_1, q_m) = \min \{d(p, q_1), \dots d(p, q_n)\}$

$1 \le m \le n$ clearly $r > 0$.

Let us consider a neighbourhood $N_r(p)$ of q

$q \in N_r(p)$.

$d(p, q) < r \le d(p, q_i)$

$\therefore d(p, q) < d(p, q_i)$

$\therefore q \notin E$

$N_r p$ contains no points of E other than p.

This is a contradiction.

$\therefore$ Every neighbourhood of p contains infinitely many point of E.

Corollary:

A finite point set has no limit point.

proof:

Let (X, d) be a metric space.

Let E be a finite subset of X. we have to prove E has no limit points. suppose $p \in X$ is a limit point of E

Then by above theorem Every neighbourhood of p contains infinitely many point of E.

this is a contradiction to the fact that E is finite :

E has no limit points.

Hence the proof.

Theorem: 2.22

Let $\{E_\alpha\}$ be a finite or infinite collection of sets E_α then $(\bigcup_\alpha E_\alpha)^c = \bigcap_\alpha (E_\alpha)^c$

Proof:

Let $A = (\bigcup_\alpha E_\alpha)^c$ and

$B = \bigcap_\alpha (E_\alpha)^c$

we have to prove $A=B$

$\therefore x \in (\bigcup_\alpha E_\alpha)^c$

$\therefore x \notin \bigcup_\alpha E_\alpha$

$x \notin E_\alpha \ \forall \alpha.$

$x \in E_\alpha^c \ \forall \alpha.$

$x \in \bigcap_\alpha (E_\alpha)^c$

$\therefore x \in B$

$\therefore A \subset B \rightarrow (1)$

conversely

Let $x \in B.$

$\therefore x \in \bigcap_\alpha (E_\alpha)^c$

$x \in E_\alpha^c \ \forall \alpha.$

$x \notin E_\alpha \ \forall \alpha.$

$x \notin \bigcup_\alpha E_\alpha.$

$x \in (\bigcup_\alpha E_\alpha)^c$

$x \in A. \Rightarrow B \subset A \rightarrow (2)$

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From ① & ② we get,

$A \subset B$, $B \subset A \Rightarrow A = B$.

Theorem: 2.2.3

A set E is open if and only if it complement is closed

proof:

Let (x, d) be a metric space. let $E \subset X$

Assume that E^c is closed we have to prove E is open.

Let $x \in E$

we have to prove x is an interior point of E.

$\therefore x \in E$. $x \notin E^c$

$\because E^c$ is closed. x is not a limit point of E^c by definition of closed set.

$\exists$. a neighbourhood N of x such that

$E^c \cap N = \phi$

$E^c \cap N = \phi$

$\therefore N \subset E$. where N is a neighbourhood of x.

$\therefore x$ is an interior point of E.

E is open.

conversly,

Assume that E is open.

we have to prove E^c is closed.

Let x be a limit point of E^c. every neighbourhood of x contains a point of E^c.

we can't find a neighbourhood of x which is a subset of E.

$\therefore$ x is not an interior point of E.

$\therefore$ $x \notin E$ ($\because$ E is open)

$\therefore$ $x \in E^c$

E^c is closed

Hence the theorem.

corollary:

A set F is closed if its complement is open.

Proof:

Let (x.d) be a metric space.

Let $F \subset X$.

Assume that F^c is open

By above theorem $(F^c)^c = F$

F is closed.

conversly.

Assume that F is closed.

$F = (F^c)^c$

By above result F^c is open.

Hence the theorem.

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NOTE :

$$\left(\bigcap_{\alpha} E_\alpha\right)^c = \bigcup_{\alpha} (F_\alpha^c)$$

Theorem : 2.24

a) For any collection $\{G_\alpha\}$ of open sets $\bigcup_\alpha G_\alpha$ is open.

b) For any collection $\{F_\alpha\}$ of closed sets, $\bigcap_\alpha F_\alpha$ is closed.

c) For any finite collection $G_1, \dots G_n$ of open sets $\bigcap_{i=1}^{n} G_i$ is open. ($\bigcap_{i=1}^{n} G_i$ is also open)

d) For any finite collection $F_1, F_2 \dots F_n$ of closed sets $\bigcup_{i=1}^{n} F$ is closed.

Proof :

Let (X,d) be a metric space.

Let $\{G_\alpha\}$ be a collection of open sets.

Let $G = \bigcup_\alpha G_\alpha$

we have to prove G is open.

Let $x \in G$.

$\therefore x \in \bigcup_\alpha G_\alpha$

$x \in G_\alpha$ for some α.

Each G_α is open, x is an interior point for G_α ($\because$ By definition of open set)

There exist a neighbourhood of x.

$\therefore x$ is an interior point of G.

$\therefore G$ is open.

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b) W.K.T

$$\left(\bigcap_\alpha F_\alpha\right)^c = \bigcup_\alpha (F_\alpha)^c$$

Let $\{F_\alpha\}$ be a collection of closed set.

∴ F_α^c is open. (By known theorem).

∴ By ⓐ, $\bigcup_\alpha (F_\alpha)^c$ is open.

$\left(\bigcap_\alpha F_\alpha\right)^c$ is open.

∴ $\bigcap_\alpha F_\alpha$ is closed

c) Let $\{G_1, \ldots G_m\}$ be a finite collection of open set.

Let $H = \bigcap_{i=1}^{n} G_i$.

We have to prove H is open.

Let $x \in H$.

∴ $x \in G_i \; \forall i$

Each G_i is open, x_i is an interior point of G_i, $i = 1, \ldots n$. There exist neighbourhood N_i of x with radius r_i such that $N_i \subset G_i$.

Let $r = \min (r_1, r_2, \ldots r_n)$

Let us consider a neighbourhood $N_r(x)$

$N_r(x) = \{y \in X \mid d(x, y) < r\}$

Let $q \in N_r(x)$

$\therefore d(x, q) < r \leq r_i$

$d(x, q) < r_i$

$\therefore q \in N_i \quad \forall i$

$N_r(x) \subset N_i \subset G_i \quad (\because N_i \subset G_i)$

$\therefore N_r(x) \subset G_i \qquad (i = 1 \ldots n)$

$\therefore N_r(x) \subset H \qquad \because H = \cup G_i$

$\therefore x$ is an interior point of H.

H is open.

d) W.K.T.

$$\left(\bigcup_{i=1}^{n} F_i\right)^c = \bigcap_{i=1}^{n} (F_i)^c$$

let $\{F_1 \ldots F_n\}$ be a finite collection of closed sets.

$\therefore F_i^c$ is open. ($\because$ By known theorem)

By ⓒ $\bigcap_{i=1}^{n} (F_i)^c$ is open

$\therefore \left(\bigcup_{i=1}^{n} F_i\right)^c$ is open

$\bigcup_{i=1}^{n} F_i$ is closed.

Note:

The intersection of an infinite collection of open sets and need not be open.

Proof:

Let $G_n = \{-1/n, 1/n \mid n = 1, 2 \ldots\}$ clearly

Each G_n is an open subset of R

Let $G = \bigcap_{i=1}^{\infty} G_n$; then $G = \{0\}$.

$\therefore$ G is not open.

$|||^{ly}$ the union of an infinite collection of closed sets need not be closed.

Definition :

closure :

Let (x,d) be a metric space.

let $E \subset X$

let E' denote the set of all limit point of E in x. Then the closure of E denoted by $\bar{E}$ is defind on $\bar{E} = E \cup E'$

Theorem: 2.27

a) $\bar{E}$ is closed

b) $E = \bar{E}$ iff E is closed

c) $\bar{F} \subset F$ for every closed sets $F \subset X$ such that $E \subset F$.

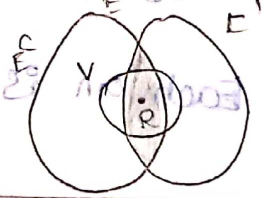
proof :

Let (x,d) be a metric space

For that we shall prove $(\bar{E})^c$ is open

let $P \in \bar{E}^c$

$\therefore$ $P \notin \bar{E} = E \cup E'$



$p \notin E$ & $p \notin E'$

$\therefore$ p is not a limit point of E.

There exist as neighbourhood N of p such that

$N \cap E = \phi$ $(\because p \notin E)$.

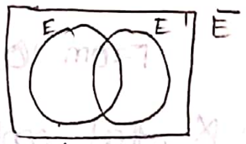
$\therefore$ $N \subset \bar{E}^c$ where N is a neighbourhood of p.

$\therefore$ p is an interior point of $\bar{E}^c$

$\therefore$ $\bar{E}^c$ is open

$\bar{E}$ is closed.

b) assume that $E = \bar{E}$



By (a) $\bar{E}$ is closed

conversly assume that E is closed.

$\therefore$ $E' \subset E$

we have to prove $E = \bar{E}$

since E is closed its contain all its (limit point)

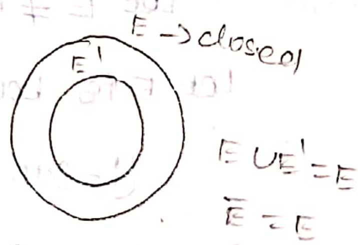
$\therefore$ $E' \subset E$ $\quad [\because E \cup E = E \cup E'$

$\therefore$ $E \cup E' = E$ $\quad \bar{E} = E$]

i.e, $\bar{E} = E'$

$E = \bar{E}$

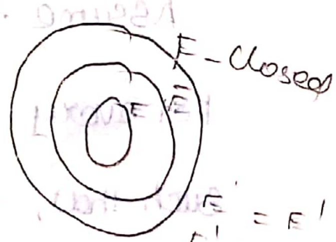
Hence the result.



c) Let $F \subset X$ be a closed set such that $E \subset F$

i.e, $E \subset F \subset X$

We have to prove $\bar{E} = F$.



given that $E \subset F \longrightarrow (1)$.

Since F is closed.

$F' \subset F \rightarrow ②$

$① \Rightarrow E' \subset F' \Rightarrow E \subset F' \subset F$

$\therefore E' \subset F \rightarrow ③$

From ① & ② $E \cup E' \subset F$

$\Rightarrow \bar{E} \subset F.$

Hence the proof.

Note:

From ⓐ & ⓒ $\bar{E}$ is a smallest closed subset of X, that contains E.

Theorem: 2.28

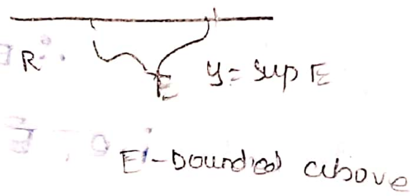
Let E be a non-empty set of real numbers which is bounded above.

Let $y = \sup E$. then $y \in \bar{E}$ hence $y \in E$. If E is closed.

Proof:

Let $E \neq \emptyset \subset R$

Let E is bounded above



$y = \sup E$

Suppose $y \in E$ then $y \in E \cup E' = \bar{E}$

$y \in \bar{E}$

Assume that $y \notin E$

For every $h > 0$ there exist then a point $x \in E$ such that,

$y-h < x < y$ (if $x < y-h$ then) $y-h$ is an upper bound of E]

$y = \sup E$.

$\therefore$ For every $h > 0$ there exist a point $x \in E$

Such that $y-h < x < y+h$ ($\because y < y+h$)

$\therefore$ Every neighbourhood of y contains a point of E other than y.

$\therefore$ y is a limit point of E

$\therefore$ $y \in E'$

$\therefore$ $y \in E \cup E'$

$\therefore$ $y \in \bar{E}$

Suppose E is closed then $E = \bar{E}$

$\therefore y \in \bar{E} = E$

$y \in E$.

Hen the theorem.

Note:

Let (X, d) be a metric space $Y \subset X$ (Y, d) is also a metric space.

Let $E \subset X$. E is closed said to be open relative to Y, if for each $p \in E$ there is associated an $r > 0$ such that $d(p, q) < r$ and $q \in Y \Rightarrow q \in E$

Theorem: 2.30.

Suppose $Y \subset X$. A subset E of Y is open relative to Y iff $E = Y \cap G$ for some open subset G of X.

Proof:

Let (X, d) be a metric space.

Let $Y \subset X$

Let $E \subset Y$

X relative to Y

$E = Y \cap G$ — open in X

Assume that E is open relative to Y.

$\therefore$ To each $p \in E$ there is a +ve number r_p such that the condition $d(p,q) < r_p$,

$$q \in Y \Rightarrow q \in E \rightarrow (1)$$

Let V_p be the set of all $q \in X$ such that $d(p,q) < r_p$ and define

$$G = \bigcup_{p \in E} V_p.$$

V_p is a neighbourhood of p in X.

V_p is neighbourhood of p.

$\therefore$ V_p is an open subset of X.

$\therefore$ G is open subset of X.

[$\because$ arbitrarily union of open sets is open]

$p \in E \Rightarrow p \in V_p$, $\forall\, p \in E \Rightarrow p \in V_p$.

$\therefore E \subset \bigcup_{p \in E} V_p$. $\qquad \Rightarrow p \subset \bigcup_{p \in E} V_p$

$\therefore E \subset G$ Already $E \subset Y$

$\therefore E \subset G \cap Y \longrightarrow$ (2)

Next to prove that $G \cap Y \subset E$

let $q \in G \cap Y$

$\therefore q \in G$ and $q \in Y$

$q \in \bigcup_{p \in E} V_p$.

$\therefore q \in V_p$ for some p.

$\therefore d(p,q) < r_p$ and $q \in Y$

$\Rightarrow q \in E$ $\quad$ ($\because$ By (1))

$\therefore G \cap Y \subset E \longrightarrow$ (3).

From (2) & (3) we have

$E = E \cap Y$, where G is an open subset of X,

Conversely

Assume that $E = G \cap Y$ for some open subset G of X.

let $p \in E$

$\therefore p \in G \cap Y$

$\therefore p \in G$.

G is open. p has a neighbourhood $V_p \subset G$.

$\therefore V_p \cap Y \subset G \cap Y$

$\therefore V_p \cap Y \subset E$ where $V_p \cap Y$ is a neighbourhood of p in Y

$\therefore E$ is open relative to Y.