UNIT - I

WEIERSTRASS THEOREM.

Statement:

Every bounded infinite subset of R^k has a limit point in R^k

Proof:

Let us assume that E is every bounded infinite subset of R^k

we have show that E has a limit point in R^k

By known theorem we have

" Every k cell is compact".

E is compact => E has infinite subset of E has a limit point is E

By known theorem we have

" If E is an infinite subset of a compact of E then E has a limit point of K'

E has a infinite subset of E has a limit point of E

Hence the theorem.

2. Heine-Borel Theorem:

Statement:

If a set E in R^k has one of the following three properties, then it has the other two.

a) E is closed and bounded

b) E is compact

c) Every infinite subset of E has a limit point in E

Proof :

Let E be a set in R^k

Let us assume that E is closed and bounded. every infinite subset of E has a limit point in E

$\Rightarrow$ E is compact

By known theorem (closed subset of compact set are compact)

$\Rightarrow$ E is compact.

Let us assume that E is not bounded

then E contains points x_n with $|x| < n$. $n = 1, 2 \ldots$

the set S consisting of these points x_n is infinite and clearly has no limit point in R^k, hence has none in E

which is a contradiction

thus implies that E is bounded.

If E is not closed then there is a point $x_0 \in R^k$ which is not closed then,

there is a point $x_0 \in R^k$ which is a limit point of E but not a point E

for $n = 1, 2, \ldots$ there are points $x_n \in E$

$\ni$: $|x_n - x_0| \leq 1/n$

let S be the set of these points x_n

Then S is infinite otherwise $|x_n - x_0|$ would have a constant +ve value for infinitely many n. S has no x_0 as a limit point in R^k

for if $y \in R^k$ $y \neq x_n$ then

$$|x_n - y| \geq |x_0 - y| - |x_n - x_0|$$

$$\geq |x_0 - y| - 1/n$$

$$\geq 1/n |x_0 - y|$$

V but infinitely many n, this shows that y is not a limit point of S.

Thus S has no limit point in E.

this is a contradiction.

Every infinite subset of E has a limit point in E

Theorem: 2.38.

If $\{I_n\}$ is a sequence of intervals in R^1, such that $I_n \supset I_{n+1}$ $(n=1,2,3,\dots)$ then $\bigcap_{n=1}^{\infty} I_n$ is non empty

Proof:

(R,d) be a metric space

Let $\{I_n\}$ be a sequences of intervals. Such that

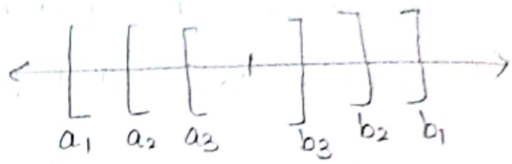
$I_n \supset I_{n+1}$ $(n=1,2,\dots)$

Let $I_n = [a_n, b_n]$

Let E be the set of all 'a_n'

clearly $E \neq \phi$

Also, $a_n \leq b$, $\forall n$



b is an upper bound of E

E has a Supremum

Let $x = \sup E$

If m & n are +ve integers then $\cancel{a_n \leq a_{m+n}}$

$a_n \leq a_{m+n} \leq b_{m+n} \leq b_m$

$\therefore$ $a_n \leq b_m$ $\forall n$

$\therefore$ b_m is an upper bound of E

since x is Supremum of E

$x_n \leq b_m$

Also $a_m \leq x$ [x is upper bound of E]

$\therefore$ $a_m \leq x \leq b_m$.

$x \in [a_m, b_m]$

$x \in I_m \quad m = 1, 2 \ldots$

$x \in \cap^{\infty} I_n$

$\bigcap_{n=1}^{\infty} I_n$ is not empty.

Theorem: 2.39

Let k be a +ve integer. If $\{I_n\}$ is a sequence of k-cells such that $I_n \supset I_{n+1}$ $(n = 1, 2 \ldots)$ then $\left(\bigcap_{n=1}^{\infty} I_n\right)$ is not empty.

proof:

Let k be a +ve integer.

Let $\{I_n\}$ be sequence of k-cells.

Such that $I_n \supset I_{n+1}$. $(n = 1, 2, 3 \ldots)$

we have to prove $\left(\bigcap_{n=1}^{\infty} I_n\right)$ is not empty

W.K.T a kcell $\cdot x = \{(x_1, x_2 \ldots x_k) / a_j \leq x_j \leq b_j\}$ $j = 1, 2, 3 \ldots$

Let $I_n = \{(x_1, x_2, \ldots x_k) / a_{n,j} \leq x_j \leq b_{n,j} \; j = 1, 2 \ldots k\}$

where $n = 1, 2, 3 \ldots$

Let $I_{n,j} = [a_{n,j}, b_{n,j}]$

$I_{n+1,j} = [a_{n+1,j}, b_{n+1,j}]$

Since $I_n \supset I_{n+1}$, $I_{n,j} \supset I_{n+1,j}$.

Now, $\{I_{n,j}\}$ is a collection of intervals in R.

Such that $I_{n,j} \supset I_{n+1,j}$

By a known theorem $\bigcap_{n=1}^{\infty} I_{n,j} \neq \phi$

$x_j^* \in I_{n,j} = [a_{n,j}, b_{n,j}] \ \forall n$

$x_j^* \in [a_{n,j}, b_{n,j}]$

$a_{n,j} \le x_j^* \le b_{n,j}$ $\quad n = 1, 2 \dots$
$\quad j = 1, 2 \dots$

Put $x^* = (x_1^*, x_2^* \dots x_k^*)$

$x^* \in I_n \quad n = 1, 2, 3 \dots$

$x^* \in \bigcap_{i=1}^{\infty} I_n$

$\bigcap_{n=1}^{\infty} I_n$ is not empty.

Hence the theorem.

Theorem : 2.40.

Every k-cell is compact

Proof:

Let I be a k-cell

i.e: $I = \{(x_1, x_2, \dots x_k) \cdot a_j \le x_j \le b_j . \ j = 1, 2 \dots k\}$

we have to prove that

I is compact.

Suppose not,

There exist an open cover $\{G_\alpha\}$ of I. which contains no finite subcover of I.

$I \subset \bigcup_{\alpha} G_\alpha$.

Put $\delta = \sqrt{\sum_{j=1}^{k} (b_j - a_j)^2}$

Let $x = (x_1, x_2, \ldots, x_k)$

$y = (y_1, y_2, \ldots, y_k) \in I$

$$|x-y| = \sqrt{\sum_{j=1}^{k} (x_j - y_j)^2} \leq \sqrt{\sum_{j=1}^{k} (b_j - a_j)^2} = \delta \text{ (say)}$$

$(\because a_j \leq x_j \leq b_j, \forall j)$

If $x, y \in I$ then $|x-y| \leq \delta$

Put $c_j = \frac{a_j + b_j}{2}$

then $[a_j, c_j]$ & $[c_j, b_j]$ determine 2^k k-cells, Q_i whose union is I.

[for eg: if $k=1$, then $I = [a,b]$ is a 1-cell

Let $c = \frac{a+b}{2}$, then $[a,c]$ & $[c,b]$ are two-cells.

Also, $[a,c] \cup [c,b] \Rightarrow [a,b]$

Atleast one of the cells Q_i is cannot be covered by any finite sub collection of $\{G_\alpha\}$

(otherwise I would be so covered.

if be I_1 clearly $I \supset I_1$.

then $|x-y| \leq \delta/2$ Now subdevided. I.

Also if $x, y \in I_1$ & continue the same process then we obtain a sequence of $\{I_n\}$ with the following properties.

a) $I \supset I_1 \supset I_2 \supset I_3 \supset \ldots$

b) I_n is not covered by any finite sub collection of $\{G_\alpha\}$

c) If $x \in I_n$ & $y \in I_n$ then $|x-y| < 2^{-n}\delta$

By ⓐ above theorem we have $\bigcap_{n=1}^{\infty} I_n \neq \phi$

There is a point $x^* \in I_n, \forall n$

for some α, $x^* \in G_\alpha$ ∴ since G_α is open

$\exists r > 0 \ni |y - x^*| < r$

If n is so large, such that $2^{-n}\delta < r$

by ⓒ if $x, y \in I_n$ we have.

$|x-y| < 2^{-n}\delta < r$.

$\Rightarrow x, y \in G_\alpha$.

this is contradiction.

Every open cover of k-cell has a finite subcover.

Hence k-cell is compact

suppose.

$2^{-n}\delta \geq r$

$\delta/2^n \geq r$

$\delta/r \geq 2^n$

$\Rightarrow 2^n \leq \delta/r$

This contradicts archemidean property