## INTRODUCTION:

- The study of geometrical properties and spatial relations unaffected by the continuous change of shape or size of figures.
- In mathematics, topology (from the Greek words **topo**, 'place', and **logy** 'study') is concerned with the properties of a geometric object that are preserved under continuous deformations, such as stretching, twisting, crumpling and bending, but not tearing or gluing.

- A topological space is a set endowed with a structure, called a topology, which allows defining continuous deformation of subspaces, and, more generally, all kinds of continuity. Euclidean spaces, and, more generally, metric spaces are examples of a topological space, as any distance or metric defines a topology.
- The deformations that are considered in topology are homeomorphisms and homotopies. A property that is invariant under such deformations is a topological property. Basic examples of topological properties are: the dimension, which allows distinguishing between a line and a surface; compactness, which allows distinguishing between a line and a circle; connectedness, which allows distinguishing a circle from two non-intersecting circles.

## Basis for a Topology

- The basic object of study is topological spaces, which are sets equipped with a topology, that is, a family of subsets, called open sets, which is closed under finite intersections and (finite or infinite) unions. The fundamental concepts of topology, such as continuity, compactness, and connectedness, can be defined in terms of open sets. Intuitively, continuous functions take nearby points to nearby points. Compact sets are those that can be covered by finitely many sets of arbitrarily small size. Connected sets are sets that cannot be divided into two pieces that are far apart. The words nearby, arbitrarily small, and far apart can all be made precise by using open sets. Several topologies can be defined on a given space. Changing a topology consists of changing the collection of open sets. This changes which functions are continuous and which subsets are compact or connected.
- The concept of topological space grew out of the study of the real line and euclidean space and the study of continuous

## Definition

A topology $\tau$ on a set X consists of subsets of X satisfying the following properties:

1. The empty set  and the space X are both sets in the topology.
2. The union of any collection of sets in $\tau$ is contained in $\tau$.
3. The intersection of any finitely many sets in $\tau$ is also contained in $\tau$.

**Remark.** The members of $\tau$ are called **open sets and their complements are closed sets**.

Example

Let $X = \{a, b\}$

Then the possible topologies are

$\tau_1 = \{\emptyset, X\}$ :**INDISCRETE TOPOLOGY**

$\tau_2 = \{\emptyset, \{a\}, \{a, b\}\}$

$\tau_2 = \{\emptyset, \{b\}, \{a, b\}\}$

$\tau_2 = \{\emptyset, \{a\}, \{b\}, X.\}$: **DISCRETE TOPOLOGY**

Example

Consider $X = \{a, b, c\}$
Then possible topologies are

- **INDISCRETE TOPOLOGY**
- **DISCRETE TOPOLOGY** (i.e., Collection of all subsets of X)
- $\{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$
- $\{\emptyset, X, \{a\}, \{a, b\}\}$ etc...

## Example

**Finite complement Topology**

Let $X$ be a set.

Let $\tau_f$ be the collection of all subsets of $U$ of $X$ such that $X - U$ either is finite or is all of $X$.

Then $\tau_f$ **is a topology on** $X$ **and is called finite complement topology. Proof.**

(i). If $\emptyset \in \tau_f$ then $X - \emptyset = X$. Similarly if $X \in \tau_f$ then $X - X = \emptyset$ (which is finite).

Hence Both $\emptyset$ and $X$ are in $\tau_f$.

(ii). Suppose $\{U_\alpha\} \in \tau_f$, then we have to prove that $\bigcup U_\alpha \in \tau_f$

Now $X - \bigcup U_\alpha = \bigcap(X - U_\alpha)$ ( By DeMorgan's Law)

Since each set $X - U_\alpha$ is finite, therefore $\bigcap(X - U_\alpha)$ is also finite.

Hence $X - \bigcup U_\alpha$ is finite.

Thus $\bigcup U_\alpha$ is in $\tau_f$. Hence (ii).

(iii). If $U_1, U_2, ......U_n \in \tau_f$, then we have to prove that $\bigcap_{i=1}^{n} U_i$ **is**

Example

Let $X$ be a set.
Let $\tau_c$ be the countable collection of all subsets of $U$ of $X$ such that
$X - U$ either is countable or is all of $X$.
Then $\tau_c$ **is a topology on $X$ and is called countable complement topology.**

**Proof.**

(i). If $\emptyset \in \tau_f$ then $X - \emptyset = X$. Similarly if $X \in \tau_f$ then $X - X = \emptyset$ (which is countable).
Hence Both $\emptyset$ and $X$ are in $\tau_f$.
(ii). Suppose $\{U_\alpha\} \in \tau_c$, then we have to prove that $\bigcup U_\alpha \in \tau_c$
Now $X - \bigcup U_\alpha = \bigcap (X - U_\alpha)$ ( By DeMorgan's Law)
Since each set $X - U_\alpha$ is countable, therefore $\bigcap (X - U_\alpha)$ is also countable.
Hence $X - \bigcup U_\alpha$ is countable.
Thus $\bigcup U_\alpha$ is in $\tau_c$. Hence (ii).
(iii). If $U_1, U_2, ......U_n \in \tau_c$, then we have to prove that $\bigcap_{i=1}^{n} U_i$ **is in** $\tau_c$**.**

Now $X - \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n}(X - U_i)$.
As $X - U_1, X - U_2, .....X - Un$ are all are countable their Union is also countable.
That is $\bigcup_{i=1}^{n}(X - U_i)$ is countable.
Hence $X - \bigcap_{i=1}^{n} U_i$ is countable.
Thus $\bigcap_{i=1}^{n} U_i$ is in $\tau_c$. Hence (iii).
Thus $\tau_c$ is topology and this topology is called **countable complement topology**

## Example

Let $X$ be a given set.
Let $\tau$,$\tau^{'}$ be two topologies for $X$.
If $\tau \subset \tau^{'}$, then we say that $\tau^{'}$ **is finer (larger or stronger) than** $\tau$ **or** $\tau$ **is coarser(smaller or weaker) than** $\tau^{'}$**.**
If $\tau \subsetneq \tau^{'}$, then we say that $\tau^{'}$ **is strictly finer than** $\tau$**,or** $\tau$ **is strictly coarser than** $\tau^{'}$**.**
In this case, we say that $\tau$ is **Comparable** with $\tau^{'}$.

## Definition

Let $X$ be a set. A **basis** for a topology on $X$ is a collection **B** of subsets of $X$ of subsets of $X$ (**basis elements**)such that

- (1). For each $x \in X$, there is at least one basis element $B$ containing $x$.
- (2). If $x \in B_1 \bigcap B_2$, then there exists $B_3 \in \mathbf{B}$ such that $x \in B_3$ and $B_3 \subset B_1 \bigcap B_2$.

Example

Let **B** be the collection of circular regions in the plane. **B** satisfies both conditions for a basis.

Example

Let $\mathbf{B}'$ be the collection of all rectangular regions ( interiors of rectangles), in the plane, where the rectangles have sides parallel to coordinate axes. Then $\mathbf{B}'$ form a basis for a topology.

Example

If $X$ is any set then the collection of all one-point subsets of $X$ is a basis for the discrete topology on $X$.

## Definition

**Topology generated by basis**

Let $X$ be any given set. Let $\mathbf{B}$ be the basis for $X$. Then the **topology generated by B** is defined as follows:
A subset $U$ of X is said to be open in $X$ ( i.e., $U \in \tau$) if for each $x \in U$, there is a basis element $B \in \mathbf{B}$ such that $x \in B$ and $B \subset U$

**Remark.**

Each basis element is itself an element of $\tau$.

# Basis for a Topology

**Lemma 13.1** Let $X$ be a set. Let **B** be a basis for a topology $\tau$ on $X$. Then $\tau$ equals the collection of all unions of elements of **B**.
**Proof.**. Since each basis element is an element of $\tau$, The collection of elements of **B** are also elements of $\tau$.
Since $\tau$ is topology, their Union is also in $\tau$.
Conversely, Given $U \in \tau$, take $x \in U$. Then there exists an element $B_x$ of **B** such that $x \in B_x \subset U$.
Then $U = \bigcup_{x \in U} B_x$.
Hence $U$ equals a union of elements of **B**.

**Lemma 13.2** Let $X$ be a topological space. Suppose **C** is a collection of open sets of $X$ such that for each open set $U$ of $X$ and for each $x$ in $U$,
there is an element $C$ of **C** such that $x \in C \subset U$. Then $C$ is a basis for the topology of $X$.
**Proof.** First let us show that **C** is a basis.
Given $x \in X$.
since $X$ is itself a open set, There exists an element $C$ of **C** such that $x \in C \subset X$.
Hence first condition of basis is true.
Let us now check the second condition.
Let $x \in C_1 \bigcap C_2$, where $C_1$, $C_2$ are the elements of **C** .
Since $C_1$ and $C_2$ are open sets, their finite intersection $C_1 \bigcap C_2$ is also open.
Therefore, there exists an element $C_3 \in$ **C** such that
$x \in C_3 \in C_1 \bigcap C_2$.
Thus **C** is a basis for $X$.

Next to to prove that the topology $\tau$ is same as the topology generated by **C**.
Let $\tau$ be the collection of open sets in $X$.
Let $\tau^{'}$ be the topology generated by **C**.
We have to prove that $\tau = \tau^{'}$.
i.e to prove that $\tau \subset \tau^{'}$ and $\tau^{'} \subset \tau$.
If $U \in \tau$ and if $x \in U$, then by hypothesis,
there exists an element $C \in \mathbf{C}$ such that $x \in C \subset U$.
Thus $U \in \tau^{'}$
Hence $\tau \subset \tau^{'}$.

## Basis for a Topology

Conversely, if $W \in \tau^{'}$, then $W$ equal a union of elements of **C** ( by lemma 13.1)
Since each element of **C** belongs to $\tau$ and $\tau$ is topology
We have $W \in \tau$.
Thus $\tau^{'} \subset \tau$.
Hence $\tau = \tau^{'}$.
Hence**C** is a basis for the topology of $X$.

**Lemma 13.3**. Let **B** and **B**$'$ be bases for the topologies $\tau$ and $\tau'$ respectively on $X$.
Then the following are equivalent:
(1) $\tau'$ is finer than $\tau$.
(2) For each $x \in X$ and each basis element $B \in \mathbf{B}$ containing $x$, there is a basis element $B' \in \mathbf{B}'$ such that $x \in B' \subset B$.

**Proof.**. (1) $\Rightarrow$ (2).
Given $x \in X$ and $B \in \mathbf{B}$ with $x \in B$.
By definition, we have $B \in \tau$.
By (1), we have $\tau \subset \tau^{'}$.
Thus $B \in \tau \subset \tau^{'}$. This implies $B \in \tau^{'}$.
Since $\tau^{'}$ generated by $\mathbf{B}^{'}$, there is an element $B^{'} \in \mathbf{B}^{'}$ such that $x \in B^{'} \subset B$.

Next to prove that (2) $\Rightarrow$ (1).
We have to prove that $\tau \subset \tau^{'}$.
i.e. Given an element $U \in \tau$, we have to show that $U \in \tau^{'}$.
Let $x \in U$. Since $\mathbf{B}$ generates $\tau$, there is an element $B \in \mathbf{B}$ such that $x \in B \subset U$.
By condition (2), we have there is an element $B^{'} \in \mathbf{B}^{'}$ such that $x \in B^{'} \subset B$.
Thus $x \in B^{'} \subset B \subset U$.
i.e., $x \in B^{'} \subset U$.
Hence $U \in \tau^{'}$.
Thus $\tau \subset \tau^{'}$.
Hence $\tau^{'}$ is finer than $\tau$.

**Definition: Standard Topology**.
If **B** is the collection of all open intervals in the real line.
$(a, b) = \{x | a < x < b\}$.
The topology generated by **B** is called **standard topology** on the real line.

**Definition:Lower limit Topology**.
If $\mathbf{B}^{'}$ is the collection of all half- open intervals in the real line.
$[a, b) = \{x | a \leq x < b\}$.
The topology generated by $\mathbf{B}^{'}$ is called **lower limit topology** on the real line. When $\mathbb{R}$ is given the lower limit topology then it is denoted by $\mathbb{R}_{\mathbf{l}}$

**Definition: K- topology**.
Let $K = \{1/n, n \in \mathbb{Z}_+\}$.
Let $\mathbf{B}^{''}$ be the collection of all open intervals $(a, b)$ along with the sets of the form $(a, b) - K$.
The topology generated by $\mathbf{B}^{''}$ is called **K- topology** on $\mathbb{R}$.
When $\mathbb{R}$ is given, this topology is denoted by $\mathbb{R}_K$.

**Lemma 13.4** The topologies of $\mathbb{R}_l$ and $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$,
but are not comparable with one another.
**Proof.** Let $\tau, \tau^{'}, \tau^{"}$ be the topologies of $\mathbb{R}_l$ , $\mathbb{R}_K$,$\mathbb{R}$ respectively.
Given a basis element $(a, b)$ for $\tau$, Let $x \in (a, b)$ be a point.
Now the basis element $[x, b)$ for $\tau^{'}$ contains $x$ and $[x, b)$ lies in $(a, b)$.
Thus $\tau^{'}$ is finer than $\tau$.
Next to prove that $\tau^{'}$ is strictly finer than $\tau$.
Now, Given basis element $[x, d)$ for $\tau^{'}$, there is **no open interval** $(a, b)$ that contains$x$ and lies in $[x, d)$.
Hence $\tau^{'}$ is strictly finer than $\tau$.

Next to prove that $\tau^{''}$ is strictly finer than $\tau$.
Given a basis element $(a, b)$ for $\tau$, Let $x \in (a, b)$ be a point.
Now the basis element of $\tau^{''}$ contains $x$ and lies in $(a, b)$.
Thus $\tau^{''}$ is finer than $\tau$.
On the other hand, Given a basis element $B = (-1, 1) - K$ for $\tau^{"}$
and the point 0 of $R$
there is no open interval that contains 0 and lies in $B$
Hence $\tau^{"}$ is strictly finer than $\tau$.
**Clearly $\mathbb{R}_l$ and $\mathbb{R}_K$ are not comparable**.

**Definition. Subbasis**.

A **Subbasis** $\mathbb{S}$ for a topology on $X$ is a collection of subsets of $X$ whose union equals $X$.

**Definition. Topology generated by Subbasis**.

The **Topology generated by Subbasis** $\mathbb{S}$ is defined to be the collection $\tau$ of all unions of finite intersections of elements of $\mathbb{S}$.

**Remark.** We have to check $\tau$ generated by subbasis is a topology.
**Proof.** Let $\mathfrak{B}$ be the collection of all finite intersection of $\mathbb{S}$.
Next to prove that $\mathfrak{B}$ is a basis.
Given $x \in X$ , $x$ belongs to an element of $\mathbb{S}$.
Hence $x$ belongs to an element of $\mathfrak{B}$.
First condition of basis is verified.

To prove second condition,
Let $B_1 = S_1 \bigcap S_2 \bigcap ...... \bigcap S_m$ and $B_2 = S_1^{'} \bigcap S_2^{'} \bigcap ...... \bigcap S_n^{'}$ be two elements of $\mathfrak{B}$.
Their intersection
$B_1 \bigcap B_2 = (S_1 \bigcap S_2 \bigcap ...... \bigcap S_m) \bigcap (S_1^{'} \bigcap S_2^{'} \bigcap ...... \bigcap S_n^{'})$ is also finite intersection of elements of $\mathbb{S}$.
Hence $B_1 \bigcap B_2$ belongs to $\mathfrak{B}$.
Thus $\mathfrak{B}$ is a basis. By **Lemma.13.1**, we have the collection of union of elements of $\mathfrak{B}$ is a topology.
Thus $\tau$ generated by subbasis is a topology.