Urysohn Lemma

Statement: Let X be a normal space and A, B be the disjoint closed subsets of X. Let $[a,b]$ be a closed interval in the real line. Then there exists a continuous map.

$$f : X \to [a,b]$$

Such that $f(x) = a$ for every x in A, and $f(x) = b$ for every x in B.

$\longrightarrow$

Proof: **Step 1:** Construct, using normality, a certain family U_p of open sets of X, indexed by the rational numbers.

Step 2: Using these sets, we define the continuous function f.

Step 1: Let P be the set of all rational numbers in the interval $[0,1]$.

We shall define, for each p in P, an open set U_p of X, in such a way that whenever $p<q$,

We have,
$$\bar{U}_p \subset U_q .$$

Thus, the sets U_p will be simply ordered by inclusion in the same way their subscripts are ordered in the real line.

Because, P is countable, we can use induction to define the sets U_p.

Arrange the elements of P in an infinite sequence in some way.

Let us, suppose that the numbers 1 and 0 are the first two elemets of the sequence.

Now, define the sets U_p, as follows:

First, define $U_1 = X - B$

Second, because A is closed set contained in the open set U_1.

By normality of X, choose an open set U_0 such that,

$$A \subset U_0 \quad \text{and} \quad \bar{U}_0 \subset U_1$$

In general,

Let P_n denote the set consisting of the first n rational numbers in the sequence.

Suppose that Up is defined for all rational numbers p belonging to the set P_n, satisfying the condition,

$$p < q \Rightarrow \bar{U}_p \subset U_q . \qquad \text{—— } (1)$$

Let r denote the next rational number in the sequence.

To define : U_r

Consider the set,

$$P_{n+1} = P_n \cup \{r\}.$$

It is a finite subset of the interval $[0,1]$, and as such, it has simple ordering derived from the usual order relation $<$ on the real line.

In a finite simply ordered set,

Every element (other than the smallest & largest) has an immediate predecessor and an immediate successor.

The number 0 is the smallest element,

and 1 is the largest element,

of the simply ordered set P_{n+1} & r is neither 0 nor 1.

So, r has an immediate predecessor p in P_{n+1}.

The sets U_p and U_q are already defined and $\overline{U_p} \subset U_q$ (by the induction hypothesis).

Using normality of X, we can find an open set U_r of X such that,

$$\overline{U_p} \subset U_r \quad \& \quad \overline{U_r} \subset U_q .$$

We assert that (1) now holds for every pair of elements of P_{n+1}.

If both elements lie in P_n, (1) holds by the induction hypothesis.

If one of them is r and the other is a point s of P_n, then either $s \leq p$, in which case,

$$\bar{U}_s \subset \bar{U}_p \subset U_r.$$

(or) $s \leq q$, in which case,

$$\bar{U}_r \subset U_q \subset U_s.$$

Thus, for every pair of elements of P_{n+1}, relation (1) holds.

By induction, we have U_p defined for all $p \in P$.

To illustrate:

Let us suppose, we started with the standard way of arranging the elements of P in an infinite sequence:

$$P = \left\{1, 0, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \dots\right\}$$

After defining U_0 & U_1,

We would define $U_{1/2}$ so that $\bar{U}_0 \subset U_{1/2}$ & $\bar{U}_{1/2} \subset U_1$.

Then, we would fit in $U_{1/3}$ between U_0 and $U_{1/2}$;

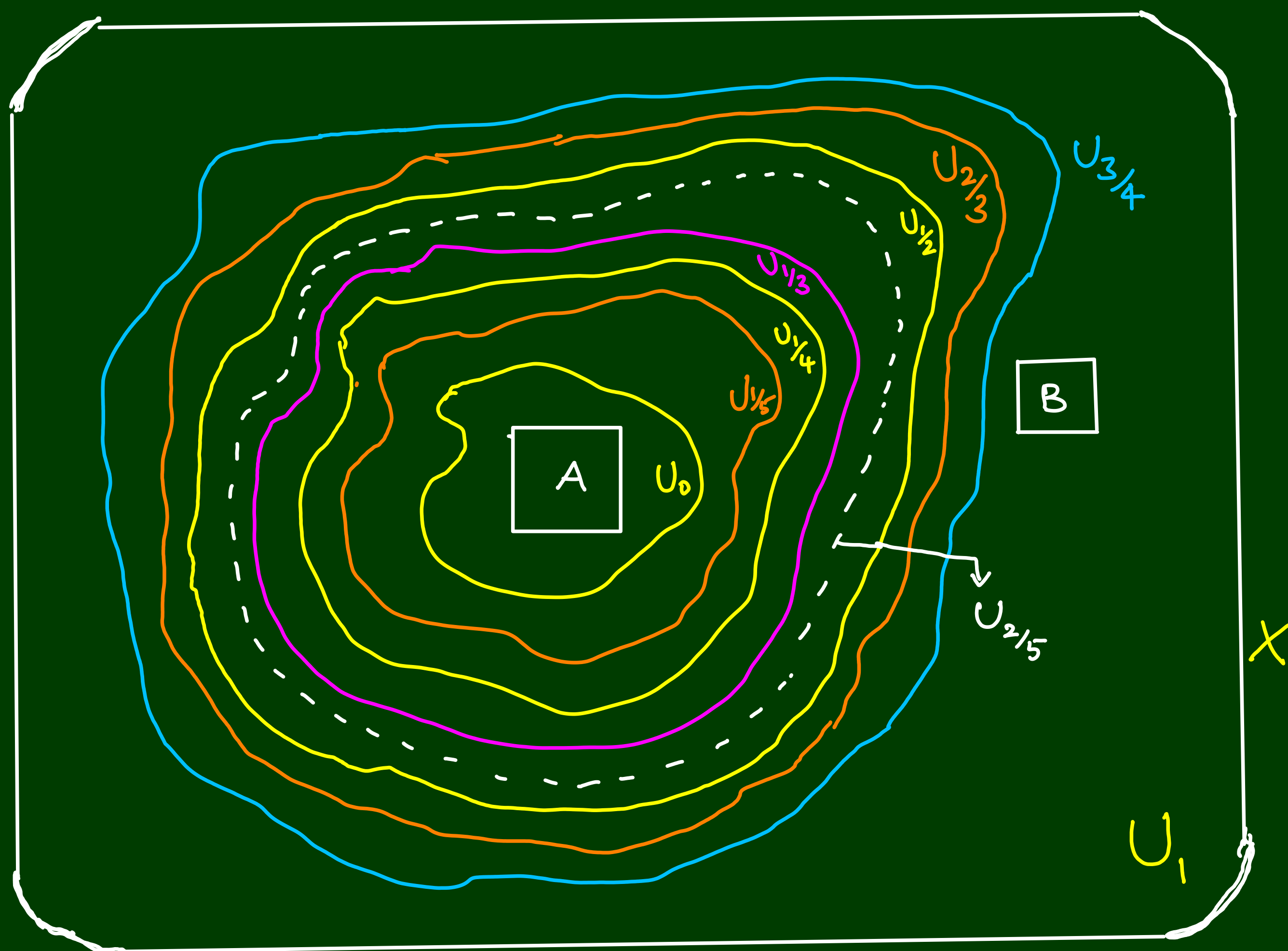
and $U_{2/3}$ between $U_{1/2}$ and U_1 ..and so on.

At the 8^{th} step of the proof,

We would have the situation pictured in Fig.

At the 9^{th} step.

Consist of choosing an open set $U_{2/5}$ to fit in between $U_{1/3}$ and $U_{1/2}$ ... and so on.



Step 2: We defined U_p for all rational numbers p in the interval $[0,1]$.

We extend this definition to all rational numbers p in $\mathbb{R}$ by defining,

$$U_p = \phi \; ; \text{ if } p < 0$$

$$U_p = X \; ; \text{ if } p > 1.$$

It is still true, that for any pair of rational numbers p and q.

$$p<q \Rightarrow \bar{U}_p \subset U_q .$$

Step 3: Given a point x of X.

Let us define $Q(x)$ to be the set of those rational numbers p such that the corresponding open sets U_p contain x:

$$Q(x) = \{ p : x \in U_p \}$$

This set contains no number less than 0.

Since,

No x is in U_p for $p>1$.

Therefore, $Q(x)$ is bounded below,

Its greatest lower bound is a point of the interval $[0,1]$.

Define,

$$f(x) = \inf Q(x) = \inf \{ p : x \in U_p \}$$

Step 4: We show that, f is the desired function.

If $x \in A$, then $x \in U_p$ for every $p \geq 0$, so that $Q(x)$ equals the set of all non-negative rationals, and

$$f(x) = \inf Q(x) = 0 .$$

Similarly,

if $x \in B$, then $x \in U_p$ for no $p \leq 1$,

So that, $Q(x)$ consists of all rational numbers greater that 1 and $f(x) = 1$

To show: f is Continuous

For that,
(i) $x \in \bar{U}_r \Rightarrow f(x) \leq r$.

(ii) $x \notin U_r \Rightarrow f(x) \geq r$.

To prove (i): If $x \in \bar{U}_r$, then $x \in U_s$ for every $s > r$.

Therefore, $Q(x)$ Contains all rational numbers greater than r.

So that, By definition we have,

$$f(x) = \inf Q(x) \leq r.$$

To prove (ii): If $x \notin U_r$, then $x \notin U_s$, for any $s < r$.

Therefore $Q(x)$ Contains no rational numbers less than r,

So that,

$$f(x) = \inf Q(x) \geq r.$$

To prove: Continuity of f.

Given a point x_0 of X, and an open interval (c, d) in $\mathbb{R}$ containing the point $f(x_0)$.

To find a neighbourhood U of x_0 Such that

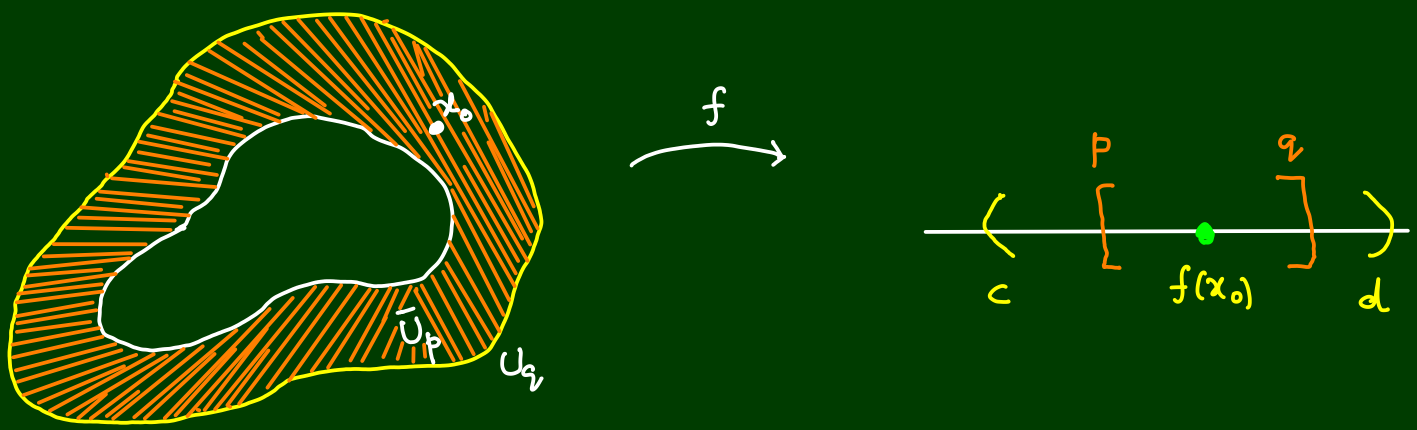
$$f(U) \subset (c, d)$$

Choose rational numbers p and q such that

$$c < p < f(x_0) < q < d.$$

We assert that, the open set

$$U = U_q - \bar{U}_p$$ is the neighbourhood of x_0.



First, we note that $x_0 \in U$.

For the fact that $f(x_0) < q$ implies by the condition (ii) that $x_0 \in U_q$, while the fact that $f(x_0) > p$ implies by (i) that $x_0 \notin \bar{U}_p$.

Second, we show that $f(U) \subset (c,d)$.

Let, $x \in U$, then $x \in U_q \subset \bar{U}_q$,

So that,
$$f(x) \leq q, \quad (\text{by (i)}).$$

And, $x \notin \bar{U}_p$, so that $x \notin U_p$ and $f(x) \geq p$ (by(ii)).

Thus,
$$f(x) \in [p,q] \subset (c,d)$$ as desired.