Theorem 32.2

Every metrizable space is normal.

Proof:

Let (X, d) be metrizable space.

We have to P.T X is normal.

Let A and B be disjoint closed subsets of X.

For each $a \in A$, choose ϵ_a so that $B(a, \epsilon_a)$ does not intersect B.

Similarly, For each $b \in B$, choose ϵ_b such that the ball $B(b, \epsilon_b)$ does not intersect A.

Let
$$U = \bigcup_{a \in A} B(a, \epsilon_a/2)$$
&
$$V = \bigcup_{b \in B} B(b, \epsilon_b/2).$$

Then U and V are open sets containing A and B respectively.

We have to P.T $U \cap V = \phi$.

If possible, Take $U \cap V \neq \phi$.

Let $z \in U \cap V$.

Then $z \in U$ and $z \in V$

$\Rightarrow z \in \bigcup_{a \in A} B(a, \epsilon_{a/2})$ & $z \in \bigcup_{b \in B} B(b, \epsilon_{b/2})$

$\Rightarrow z \in B(a, \epsilon_{a/2})$ & $z \in B(b, \epsilon_{b/2})$

for some $a \in A$ & some $b \in B$.

$\therefore z \in B(a, \epsilon_{a/2}) \cap B(b, \epsilon_{b/2})$.

~~Now~~ $\Rightarrow d(a, z) < \epsilon_{a/2}$ & $d(z, b) < \epsilon_{b/2}$ $\rightarrow (1)$

Now

$$d(a,b) \leq d(a, z) + d(z, b)$$

$$d(a,b) < \frac{\epsilon_a}{2} + \frac{\epsilon_b}{2} \rightarrow (2)$$

Case i) : if $\epsilon_a < \epsilon_b$

$\therefore (2) \Rightarrow d(a,b) < \frac{\epsilon_b}{2} + \frac{\epsilon_b}{2}$

$\therefore d(a,b) < \epsilon_b$

$\therefore a \in B(b, \epsilon_b)$

This is not possible.

Case (ii): ~~If $\epsilon_b < \epsilon_a$~~

Case :(ii) Similarly, if $\epsilon_b < \epsilon_a$,

Then we have, $b \in B(a, \epsilon_a)$.

This situation is also not

Possible.

Hence $z \notin U \cap V$.

Thus $U \cap V = \phi$.

Hence, we have two disjoint

open sets U and V of A and B

respectively.

Thus X is Normal.

Hence $\boxed{\text{Every metrizable space is normal}}$.

Theorem : 32.3

Every Compact Hausdorff Space is normal.

<u>Proof:</u>

Let X be Compact Hausdorff Space.

We have to P.T X is normal.

Let A and B disjoint closed sets of X.

"Result:

If $x \in X$, & B is a closed set such that $x \notin B$, then B is compact.

By Lemma (26.4), there exists distinct open sets of x and B respectively).

(ie) X is regular".

~~Now, Choose $\forall a \in$~~ A.

Hence for each point $a \in A$, we have disjoint open sets U_a of a and V_a of B.

Now the collection $\{U_\alpha\}$ covers A.

Since A is closed subset of compact space X, we have A is compact.

Thus $\{U_\alpha\}$ has finite subcover of A.

Say $U_{\alpha_1}, U_{\alpha_2}, \ldots U_{\alpha_m}$.

(ie) $U A = U_{\alpha_1} \cup U_{\alpha_2} \ldots U_{\alpha_m}$ which covers A.

Take $V = V_{\alpha_1} \cap V_{\alpha_2} \cap \ldots V_{\alpha_m}$.

Clearly $B \subset V$. $(\because B \subset V_{\alpha_i} \; \forall \alpha_i)$

Also $U \cap V = \phi$.

Thus, we have two disjoint open ~~closed~~ sets U and V of A and B respectively..

Hence X is normal.

Thus compact Hausdorff space is normal.

Theorem : 32.4 :

Every well-ordered set X is normal in the order topology.

<u>Proof:</u>

Let X be a well-ordered set.

First we have to prove that every interval of the form $(x, y]$ is open in X.

If X has a largest element ~~of~~ y, then $(x, y]$ is a basis for X and $y \in (x, y]$.

Thus $(x, y]$ is open.

If y is not the largest element of X, then $(x, y]$ equals the open set (x, y') (where y' is immediate successor of y).

In any cases, $(x, y]$ is open.

<u>Next to P.T X is Normal.</u>

Let A and B be given disjoint Closed sets of X.

Case : (1)

Let a_0 be the smallest element of X.

Assume that $a_0 \notin A$ and $a_0 \notin B$.

Now, for each $a \in A$, there exists a basis element $(x, a]$ about "a" disjoint from B.

Choose for each $a \in A$, $(x_a, a]$ disjoint from B.

Similarly, for each $b \in B$, choose an interval $(y_b, b]$ disjoint from A.

Take $$U = \bigcup_{a \in A} (x_a, a] \quad \& \quad V = \bigcup_{b \in B} (x_b, b]$$

Then U and V are open sets containing A and B respectively.

Now To P.T $U \cap V = \phi$.

Suppose, $z \in U \cap V$.

Then $z \in (x_a, a] \cap (y_b, b)$ for some $a \in A, \& b \in B$.

Case (i) Assume that $a < b$.

Then if $a \leq y_b$, then $(x_a, a] \cap (y_b, b] = \phi$.

$\therefore z \notin (x_a, a] \cap (y_b, b]$. $\forall$ $a \in A$ & $b \in B$.

$\therefore U \cap V = \phi$.

On the other hand, if $a > y_b$, we have $a \in (y_b, b]$.

This is contradicts that $(y_b, b]$ disjoint from A.

Hence $z \notin (x_a, a] \cap (y_b, b]$ $\forall$ $a \in A$ & $b \in B$.

Case: (ii). Assume that $b < a$.

Then if $b < x_a$, then $(x_a, a] \cap (y_b, b] = \phi$

$\therefore z \notin (x_a, a] \cap (y_b, b]$ $\forall$ $a \in A$ & $b \in B$.

III^ly if $b > x_a$, then $b \in (x_a, a]$ which is not possible.

Thus in any cases $U \cap V = \phi$.

Case: 2

Assume that A and B are disjoint closed sets in X. and A contains smallest element a_0 of X.

Then the set $\{a_0\}$ is both open and ~~B~~. closed.

$\therefore$ By the ~~result~~ result of Case (1), there exists disjoint open sets U and V containing the closed sets $A - \{a_0\}$ and B respectively.

Then, $U \cup \{a_0\}$ is open set containing A and V is open set containing B such that $[U \cup \{a_0\}] \cap V = \phi$.

Hence X is normal.