## Urysohn Metrization theorem:

Every regular space $X$ with a countable basis is metrizable.

**Proof:**

Let $X$ be a regular space with countable basis.

To P.T $X$ is metrizable.

For proving this, ~~we have to construct~~

Let us imbedding $X$ in a metrizable space $Y$.

For that, we have to show that $X$ is homeomorphic with a subspace of $Y$.

We have, $R^{\omega}$ in the product topology is metrizable.

Take $Y = R^{\omega}$.

**Step: 1.**

We prove the following:

There exists a countable collection of continuous functions $f_n : X \to [0,1]$ having the property that given any point $x_0$ of $X$ and any neighbourhood $U$ of $x_0$, there exists an index $n$ such that $f_n(x_0) > 0$ and ~~$f_n(U) = 0$~~ $f_n(X-U) = 0$

Let $\{B_n\}$ be a countable basis for $X$.

For each pair $n, m$ of indices for which $\overline{B_n} \subset B_m$, apply the Urysohn lemma, Choose a continuous function,

$$g_{n,m} : X \to [0,1] \text{ such that}$$

$g_{n,m}(\overline{B_n}) = \{1\}$ and $g_{n,m}(X - B_m) = \{0\}$.

Given $x_0$ and given a neighborhood $U$ of $x_0$, One can choose a basis element $B_m$, such that $x_0 \in B_m \subset U$.

Since $X$ is regular, choose $B_n$ so that $x_0 \in B_n$ & $\overline{B_n} \subset B_m$.

Then, we get a function $g_{n,m}$, for $n,m$ Pair of indices, such that

$g_{n,m}(x_0) > 0$ & $g_{n,m}$ vanishes outside $U$.

Since the collection $\{g_{n,m}\}$ is indexed with a subset of $Z_+ \times Z_+$. is countable.
Therefore, it can be reindexed with positive integers. $\{f_n\}$.

## Step: 2

Given the functions $f_n$ of step 1, take $R^\omega$ in the product topology, and define a map $F: X \to R^\omega$ by the rule,

$$F(x) = (f_1(x), f_2(x), \dots).$$

We claim that F is an imbedding.

Since $R^\omega$ has the product topology, and each $f_n$ is continuous, hence F is continuous.

Next to p.T F is 1-1.

$x \neq y \Rightarrow$ there is an index n such that $f_n(x) > 0$ and $f_n(y) = 0$.

$\therefore (f_1(x), f_2(x) \dots f_n(x), \dots) \neq (f_1(y), f_2(y) \dots f_n(y) \dots)$

$\Rightarrow F(x) \neq F(y)$.

Thus F is injective.

Finally, we have to p.T F is a homeomorphism of X onto its image, the subspace $Z = F(X)$ of $R^\omega$.

We know that $F$ defines a continuous bijection of $X$ with $Z$.

We need only show that for each open set $U$ in $X$, the set $F(U)$ is open in $Z$.

Let $z_0 \in F(U)$.

We shall find an open set $W$ of $Z$ such that $z_0 \in W \subset F(U)$.

Let $x_0 \in U$ such that $F(x_0) = z_0$.

Choose an index $N$ for which $f_N(x_0) > 0$ and $f_N(X-U) = \{0\}$.

Take the open ray $(0, \infty)$ in $\mathbb{R}$, let $V$ be the open set,

$$V = \pi_N^{-1}((0, +\infty)) \text{ of } \mathbb{R}^{\omega}.$$

Let $W = V \cap Z$.

Then ~~W~~ $W$ is open in $Z$.

Then $z_0 \in W \subset F(U)$.

We assert that $z_0 \in W \subset F(U)$.

Since, $\pi_N(z_0) = \pi_N(F(x_0)) = f_N(x_0) > 0$.

$\Rightarrow z_0 \in W$.

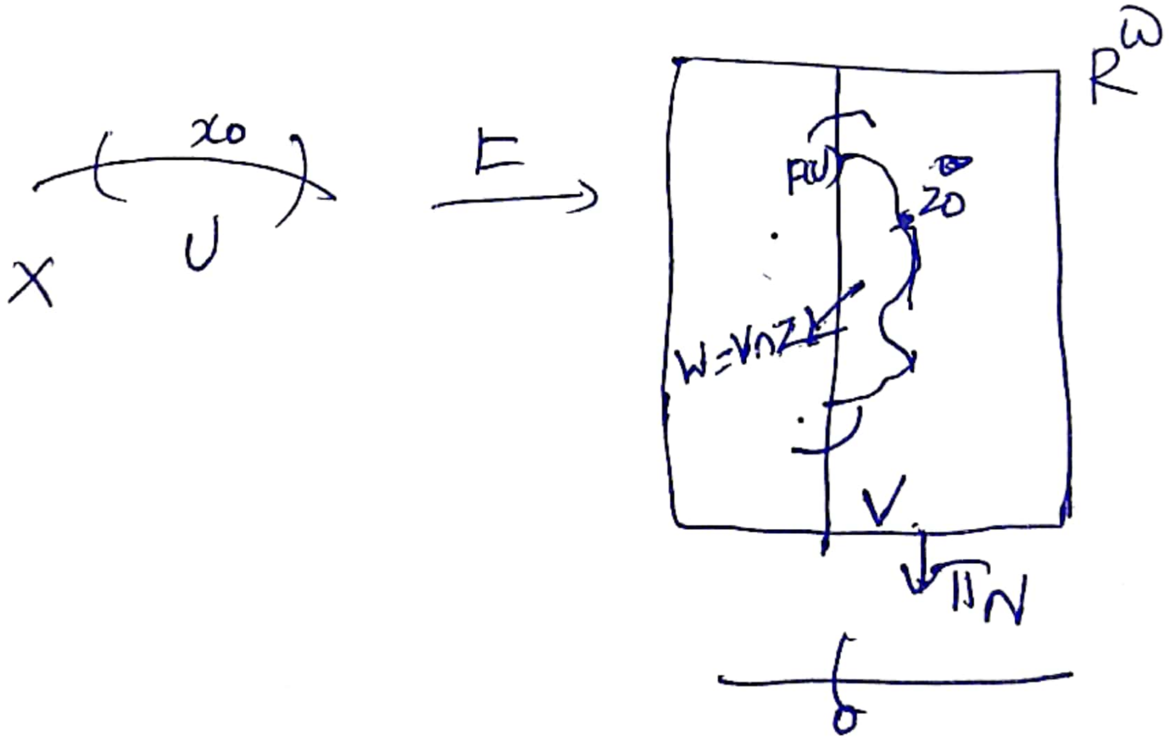
Next to P.T $W \subset F(U)$.

If $z \in W$ then $z = F(x)$ for some $x \in X$.
& $\pi_N(z) \in (0, \infty)$.
Since $\pi_N(z) = \pi_N(F(x)) = f_N(x)$
& $f_N$ vanishes outside $U$.
$\therefore$ the point $x \in U$.
Thus $z = F(x) \in F(U)$.

Hence $z_0 \in W \subset F(U)$.
$\Rightarrow$ $z_0$ is interior point of $F(U)$.
Thus $F(U)$ is open in $Z$.

Thus $F$ is an imbedding of $X$ in $R^{\omega}$.

Hence $X$ is metrizable.

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