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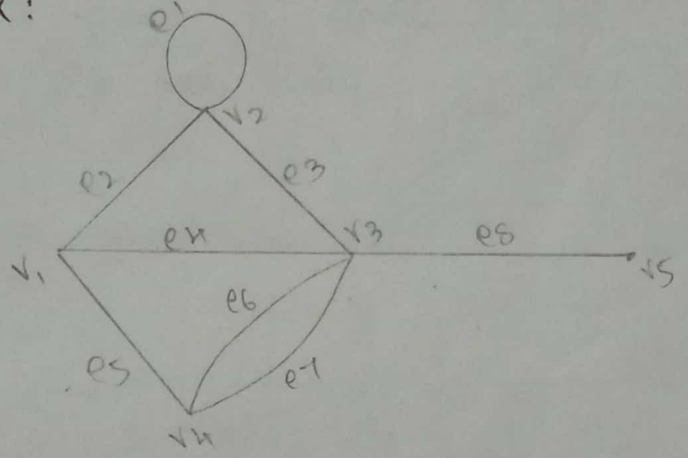
UNIT - I.

1). Graph:

A graph G is an ordered triple $(V(G), E(G), \psi_G)$ consisting of a nonempty set $V(G)$ of vertices, a set $E(G)$, disjoint from $V(G)$, of edges, and an incident function ψ_G that associates with each edge of G an unordered pair of vertices of G.

EX:



H.

$H = (V(H), E(H), \psi_H)$.

where,

$V(H) = \{v_1, v_2, v_3, v_4, v_5\}$

$E(H) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$

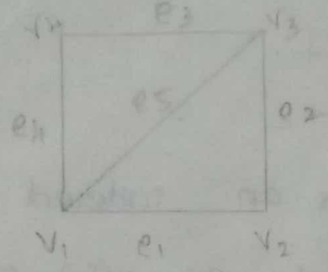
ψ_H is defined,

$\psi_H(e_1) = v_2v_2$, $\psi_H(e_2) = v_1v_2$, $\psi_H(e_3) = v_2v_3$, $\psi_H(e_4) = v_1v_3$

$\psi_H(e_5) = v_1v_4$, $\psi_H(e_6) = v_4v_3$, $\psi_H(e_7) = v_4v_3$, $\psi_H(e_8) = v_3v_5$.

2) planar graph:

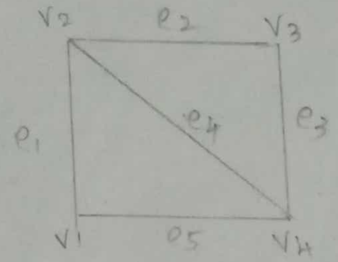
The graphs that have a diagram whose edges intersect only at their ends are called planar graph. otherwise the graph is called non-planar.



3) Incident :

The ends of an edge are said to be incident with the edge and vice versa.

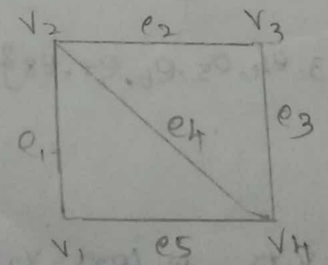
Ex :



4) Adjacent vertex :

Two vertices which are incident with a common edge are adjacent, as are two edges which are incident with a common vertex.

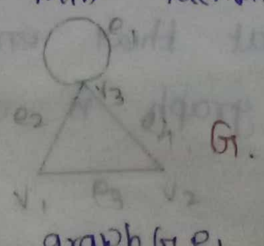
EX:



5). Loop :

An edge with identical ends is called a loop.

Ex :

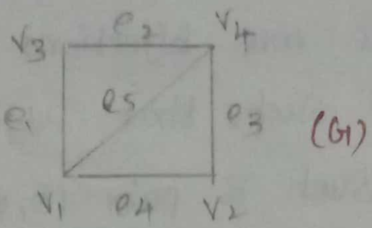


In this graph G e_1 is a loop.

6) link :

An edge with distinct ends a link.

Ex:



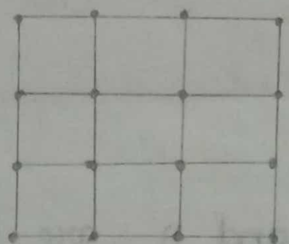
(G)

All edges of G is called link.

7) Finite graph:

A graph is finite if both its vertex set and edge set are finite.

Ex:



(H)

In this graph H is called finite graph.

8) trivial graph:

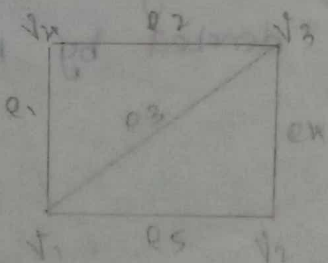
A graph have one vertex is called trivial and all other graphs non-trivial.

Ex:

9) Simple graph:

A graph is simple if it has no loops and no two of its links join the same pair of vertices.

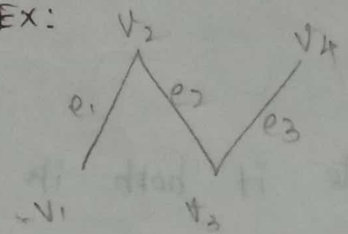
Ex:



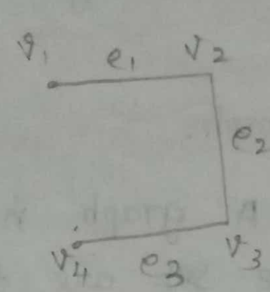
10). Isomorphic :

Two graphs G and H are said to be isomorphic if there are bijections $\theta : V(G) \to V(H)$ and $\phi : E(G) \to E(H)$ such that $\psi_G(e) = uv$ iff $\psi_H(\phi(e)) = \theta(u)\,\theta(v)$ Such a pair (θ, ϕ) of mappings is called an isomorphism between G and H.

Ex:



(G)

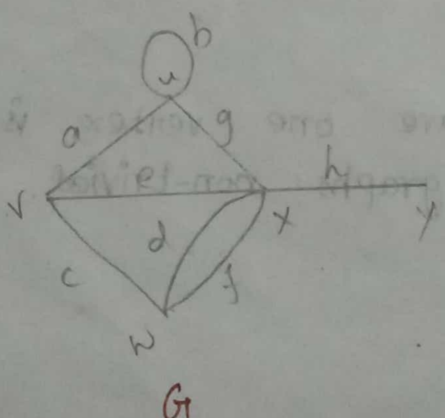


(H)

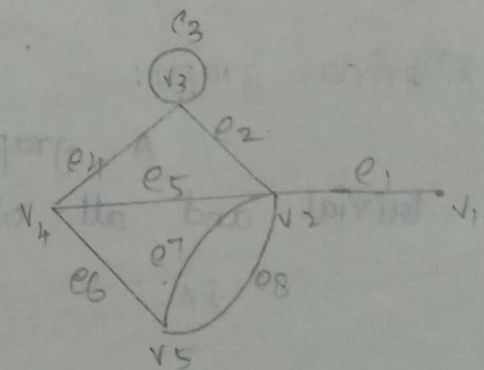
11). Identical :

Two graphs G and H are identical if $V(G) = V(H)$, $E(G) = E(H)$ and $\psi_G = \psi_H$.

Ex :



G

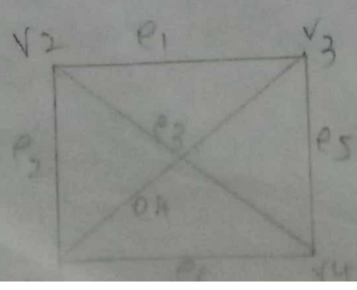


H

12) complete graph :

A simple graph in which each pair of distinct vertices is joined by an edge is called a complete graph. It is denoted by K_n.

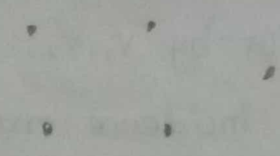
Ex:



13) empty graph:

An empty graph have only vertices with no edges.

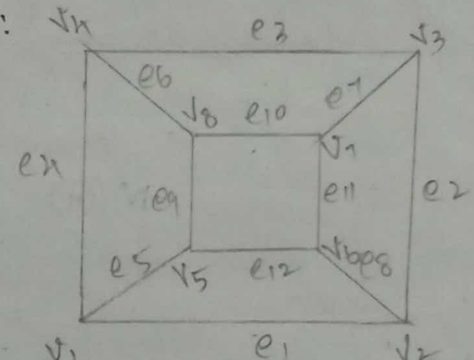
Ex:



14) Biparitite graph:

A bipartite graph is one whose vertex set can be partitioned into two subsets x and y, so that each edge has one end in x and one end in y, such a partition (x, y) is called a bipartition of the graph.

EX:



$X = \{v_1, v_2, v_3, v_4\}$

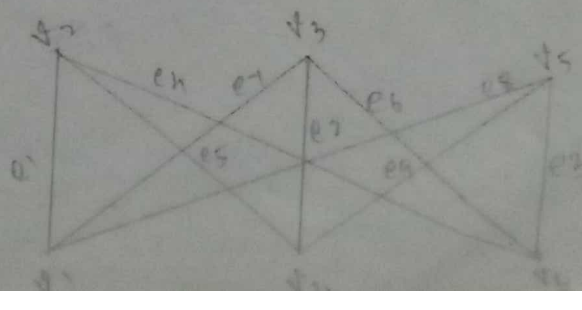
$Y = \{v_5, v_6, v_7, v_8\}$

15). complete bipartite graph:

A complete bipartite graph is a simple bipartite graph with bipartition (x, y) in which each vertex of x is joined to each vertex of y.

If $|X| = m$ and $|Y| = n$ such a graph is denoted by $K_{m,n}$.

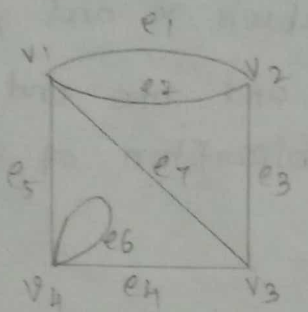
EX:



16) Incidence matrix:

Any graph G there corresponds a $V \times E$ matrix called the incidence matrix of G. Let us denote the vertices of G by $v_1, v_2, \dots v_n$ and the edges by $e_1, e_2, \dots e_n$. Then the incidence matrix of G is the matrix $M(G) = [m_{ij}]$, where m_{ij} is the number of times (0,1,2) that v_i and e_j are incident.

Ex:



G

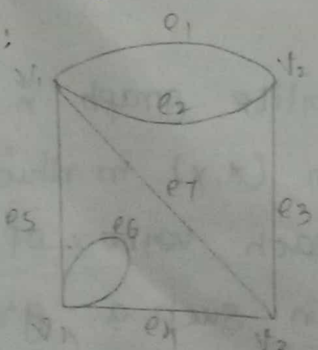
	e_1	e_2	e_3	e_4	e_5	e_6	e_7
v_1	1	1	0	0	1	0	1
v_2	1	1	1	0	0	0	0
v_3	0	0	1	1	0	0	1
v_4	0	0	0	1	1	2	0

M(G)

17) Adjacency matrix:

A matrix associated with G is the adjacency matrix, this is the $V \times V$ matrix $A(G) = [a_{ij}]$ in which a_{ij} is the number of edges joining v_i and v_j.

Ex:



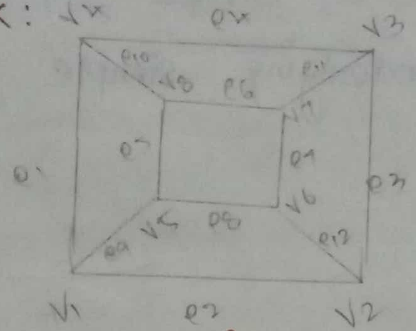
G

	v_1	v_2	v_3	v_4
v_1	0	2	1	1
v_2	2	0	1	0
v_3	1	1	0	1
v_4	1	0	1	1

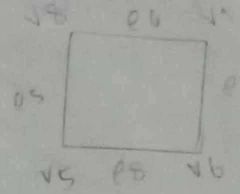
A(G)

A graph H is a subgraph of G ($H \subseteq G$) if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and ψ_H is the restriction of ψ_G to $E(H)$.

Ex:



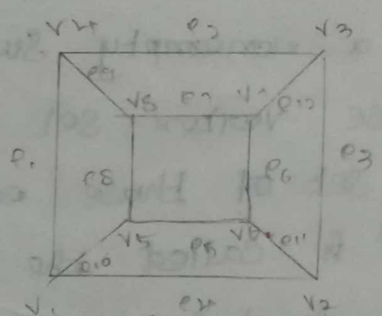
G



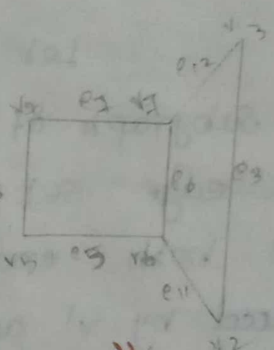
H

19) proper subgraph:

A graph H is a subgraph of G, when $H \subseteq G$ but $H \neq G$, we write $H \subset G$ and call H a proper subgraph of G.



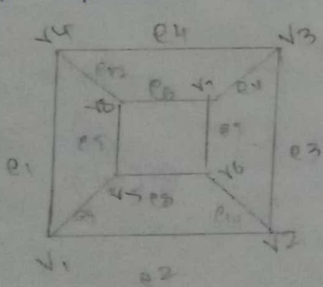
G



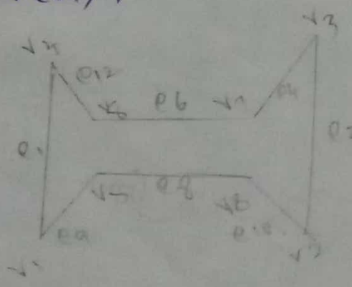
H

20) spanning subgraph:

A spanning subgraph of G is a subgraph of H with $V(H) = V(G)$.



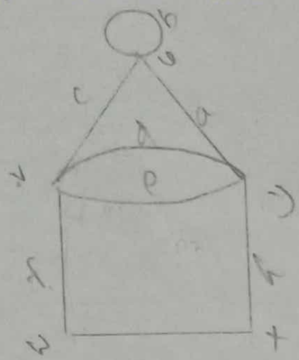
G

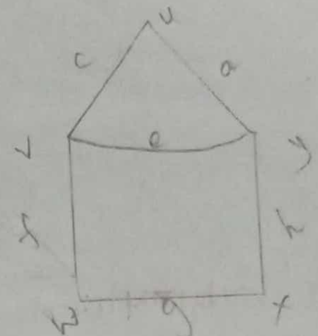


H

21) underlying simple graph:

By deleting from G all loops and for every pair of adjacent vertices but one link joining them, we obtain a simple spanning subgraph of G is called the underlying simple graph of G.





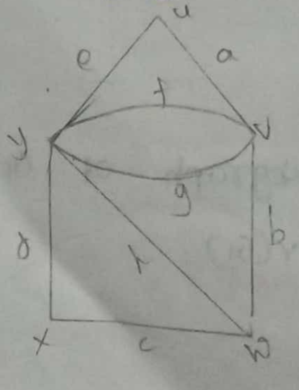
H

The graph G is a underlying simple graph.

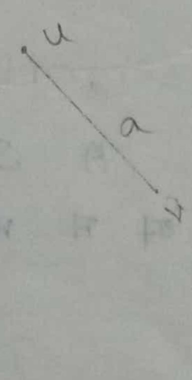
22) induced subgraph:

Let v' is a nonempty subset of v. The subgraph of G whose vertex set is v' and whose edge set is the set of those edges of G that have both ends in v' is called the subgraph of G induced by v' and it is denoted by G[v'].

G[v'] is an induced subgraph of G.



G

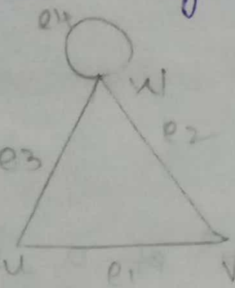


G(v')

23) vertex degrees:

The degree $d_G(v)$ of a vertex v in G is the number of edges of G incident with v, each loop counting as two edges.

We denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degrees, respectively of vertices of G.



In this graph, $d(w) = 4$,
$d(u) = 2$
$d(v) = 2$.

24) k - partite graph:

A k-partite graph is one whose vertex set can be partitioned into k subsets so that no edge has both ends in any one subset.

25) complete k-partite graph:

A complete k-partite graph is one that is simple and in which each vertex is joined to every vertex that is not in the same subset.

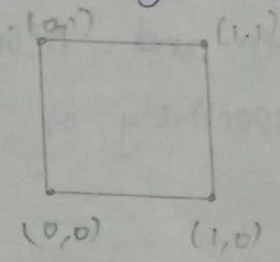
complete m-partite graph:

The complete m-partite graph on n vertices in which each part has either $[n/m]$ or $\{n/m\}$ vertices is denoted by $T_{m,n}$.

26). K-cubes:

The K-cube is the graph whose vertices are the ordered K-tuples of 0's and 1's two vertices being joined iff they differ in exactly one co-ordinate.

Ex:

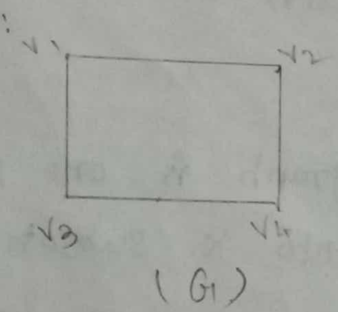


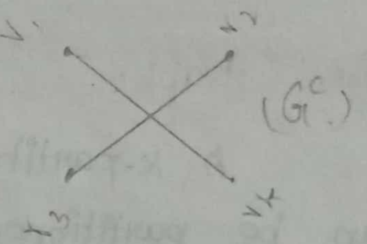
2-kcubes.

27) complement:

The complement G^c of a simple graph with vertex set V, two vertices being adjacent in G^c iff they are not adjacent in G.

Ex:





28) self-complementary:

A simple graph G is self-complementary if $G \cong G^c$.

29) Automorphism:

An automorphism of a graph is an isomorphism of the graph onto itself.

30) vertex-transitive:

A simple graph G is vertex transitive if for any two vertices u & v there is an element g in $\Gamma(G)$, such that $g(u) = g(v)$.

31) Edge- transitive :

A simple graph G is edge-transitive if, for any two edges u_1, v_1 and u_2, v_2 there is an element h in $\Gamma(G)$ such that $h(\{u_1, v_1\}) = h\{u_2, v_2\}$.

32) Degree sequence :

If G has vertices $v_1, v_2, \ldots v_v$ the sequence, $(d(v_1), d(v_2), \ldots d(v_v))$ is called a degree sequence of G.

~~If $\sum_{i=1}^{n} d_i$ is even.~~

Ex:

The sequences $(7,6,5,4,3,3,2)$ are the degree sequence.

33) Graphic :

A sequence $d = (d_1, d_2, \ldots d_n)$ is graphic if there is a simple graph with degree sequence d.

~~Ex.~~ The sequence $(7,6,5,4,3,3,2)$ are not graphic.

or $(6,6,5,4,3,3,1)$?

soln:

$(7,6,5,4,3,3,2)$ upto d_8.

$\Rightarrow (5,4,2,2,1)$ upto d_6.

$\Rightarrow (3,2,1,1,0)$ upto d_4)

$\Rightarrow (1,0,0,0)$ upto d_2

$\Rightarrow (-1,0,0)$

Degree cannot be negative therefore which is not graphic.

34) Erdos and Gallai theorem:

If d is graphic and $d_1 \geq d_2 \geq \dots \geq d_n$ then $\sum_{i=1}^{n} d_i$ is even and $\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}$ for $1 \leq k \leq n$.

35) Havel and Hakimi theorem:

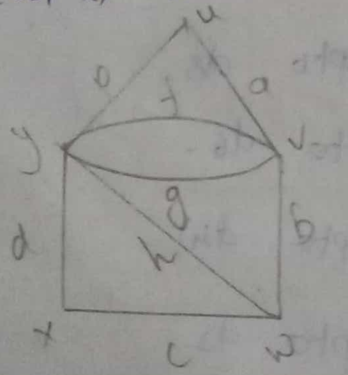
Let $d = (d_1, d_2, \dots d_n)$ be a non increasing Sequence of non-negative integers and denote the sequences $(d_2-1, d_3-1, d_{d_1+1}-1, d_{d_1+2}, \dots d_n)$ by d'.

36) walk:

A walk in G is a finite non-null sequence $W = v_0 e_1 v_1 e_2 v_2 \dots e_k v_k$ whose terms are alternately vertices and edges, such that for $1 \leq i \leq k$, the ends of e_i are v_{i-1} and v_i.

We say that w is a walk from v_0 to v_k or a (v_0, v_k) walk.

Ex:

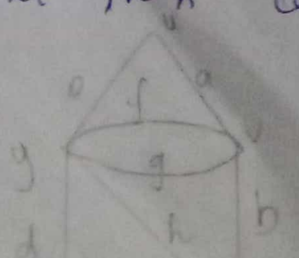


In this graph uavfyfvgyhwbv is a walk.

37) Trail:

If the edges $e_1, e_2, \dots e_k$ of a walk w are distinct, w is called a trail.

Ex:

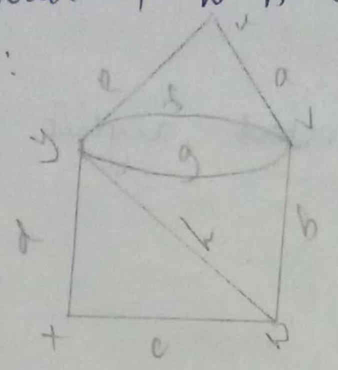


In this graph wcxdyhwbvgy is a trail of the walk

38) path:

If the vertices $v_0, v_1, \ldots v_k$ are distinct, then the walk of w is called a path.

EX:



In this graph xcwhyeuav are path of the walk.

39) Distance:

If vertices u and v are connected in G, the distance between u and v in G, denoted by $d_G(u,v)$ is the length of a shortest (u,v) path in G', if there is no path connecting u and v we define $d_G(u,v)$ to be infinite.

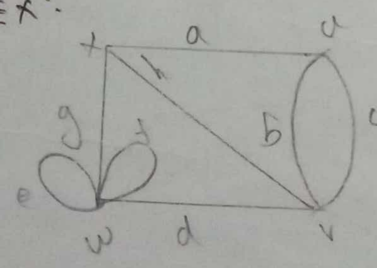
40) Diameter:

The diameter of G is the maximum distance between two vertices of G.

41) cycle:

A closed trail whose origin and internal vertices are distinct is a cycle.

EX:



In this graph xaubvhx is a cycle.

42) Girth:

The girth of G is the length of a shortest cycle in G if G has no cycles we define the girth of G to be infinite.

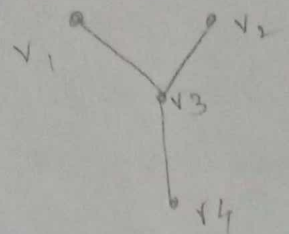
43) Acyclic graph:

An acyclic graph is one that contains no cycles

44) Tree:

A tree is a connected acyclic graph.

EX:



45) Forest:

An acyclic graph is also called a forest.

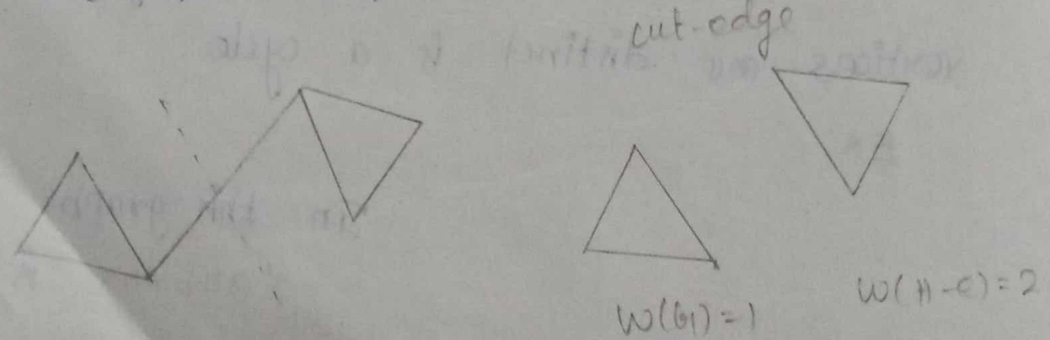
46) Centre of G.

A centre of G is a vertex u such that $\max_{v \in V} d(u,v)$ is as small as possible.

47) cut-edge:

A cut-edge of G is an edge e such that $w(G-e) > w(G)$

EX:



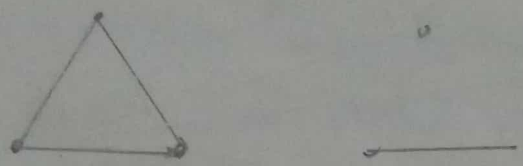
48) spanning tree:

A spanning tree of G is a spanning subgraph of G that is a tree.

49) Edge-cut :

An edge cut of G is a subset of E of the form $[S, \bar{S}]$ where S is a non-empty proper subset of V and $\bar{S} = V/S$.

EX :

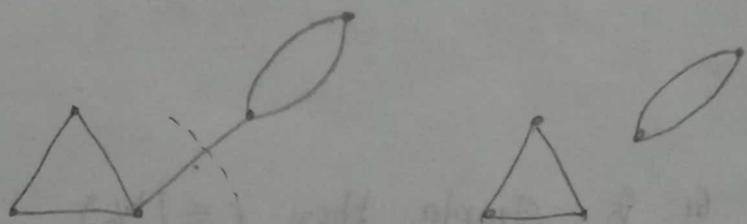


50) Bond :

A minimal edge cut of G is called a bond each cut-edge e, for instance gives rise to a bond $\{e\}$.

If G is connected, then a bond B of G is a minimal subset of E such that G-B is disconnected.

EX:

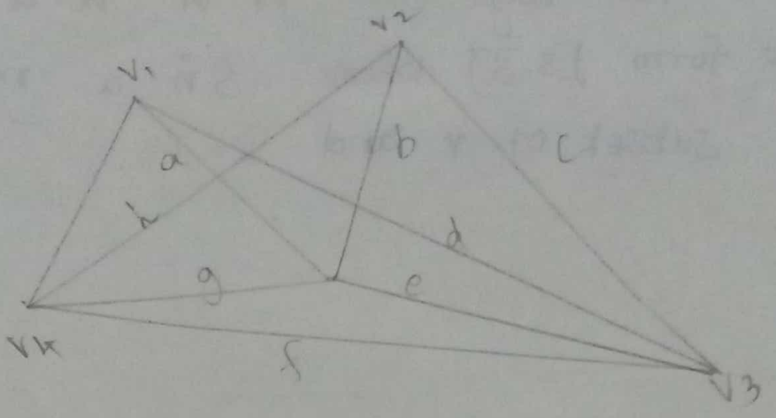


51) co-tree :

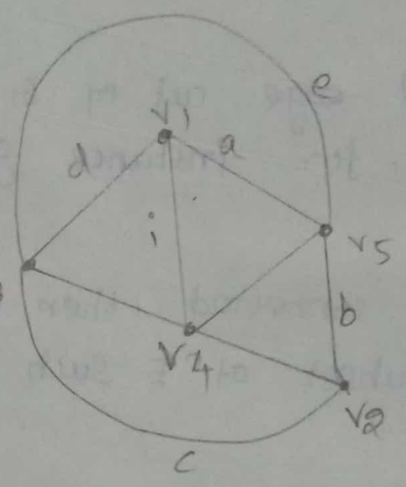
If G is connected, a subgraph of the form $\bar{T}$, where T is a spanning tree, is called a co-tree of G.

Exercise.

1.1.2:



Soln:



1.1.3. S.T if G is simple then $\epsilon \le \binom{v}{2}$.

Soln:

Let G be a graph with n vertices.

As a graph is simple. If it is has neither loop nor parallel edges.

Hence each edges incident to a two distinct elements subset $\{v_i, v_j\}$ of v.

The maximum number of two elements subset of set v in nc_2.

$$\epsilon \le \binom{v}{2}$$

Hence proved.

1.2.2. a) S.T if $G \cong H$, then $\nu(G) = \nu(H)$ and $\varepsilon(G) = \varepsilon(H)$

b) Give an example to S.T the converse is false.

proof:

Two graphs G and H are said to be isomorphic $(G \cong H)$.

If there are bijections $\theta : (V(G)) \to V(H)$ and $\phi : E(G) \to E(H)$ such that $\psi_G(e) = uv$ iff $\psi_H(\phi(e)) = \theta(u)\theta(v)$

$\psi_H(\phi(e')) = \theta(u)\theta(v)$

Such a pair (θ, ϕ) of mapping is called an isomorphism between G and H.

There is 1-1 corresponds between vertex sets and edge set.

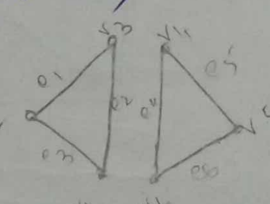
Therefore that is number of vertex of G is equal to the number of ~~edge~~ vertex in H.

$\nu(G) = \nu(H)$

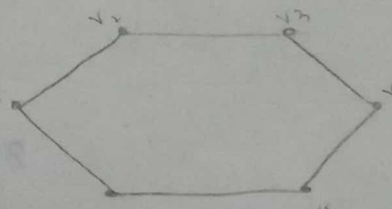
$\varepsilon(G) = \varepsilon(H)$.

Conversely,

For ex;



(G)



(H)

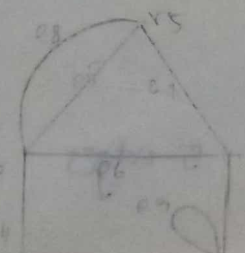
$\nu(G) = \nu(H) = 6$

$\varepsilon(G) = \varepsilon(H) = 6$.

But it is not isomorphic.

1.2.3 S.T the following graphs are not isomorphic:

(G)



(H)

Soln:

$$v(G) = v(H) = 5$$

$$E(G) = E(H) = 9.$$

Graph G having 2 vertex of degree 4 and they are adjecent.

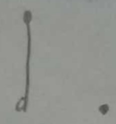
But in graph H two vertex of degree 4 and they are not adjecent vertices.

So the graphs are not isomorphic.

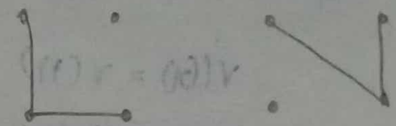
1.2.4. S.T there are eleven non-isomorphic simple graphs on four vertices.

Proof:

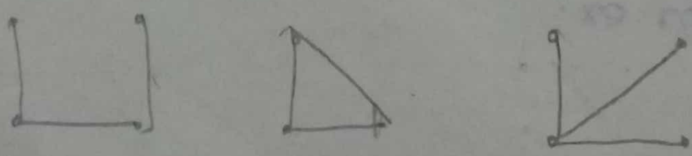
0-edge



1-edge

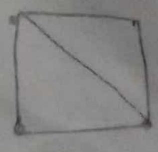


2-edges.

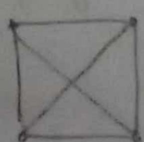


3-edges

4-edges.



5-edges



6-edges.

one graph with 0 edges

one graph with 1 edges

Two non-isomorphic with 2 edges.

Three " " " 3 "

Two " " " 4 "

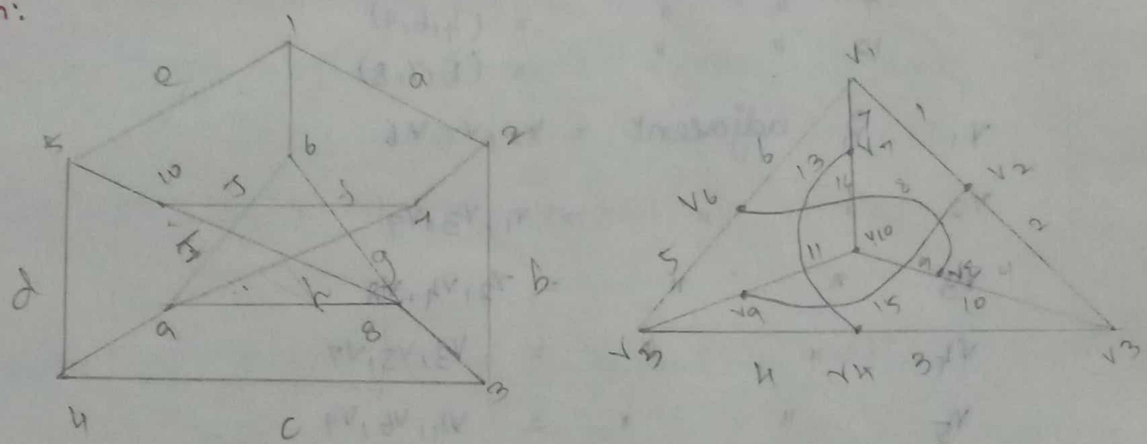
one " " " 5 "

one " " " 6 "

Therefore there are eleven non-isomorphic graph on 4 vertices.

1.2.5- S.T the following graphs are isomorphic.

soln:



(i) $v(G_1) = v(G_2) = 10$.

(ii) $E(G_1) = E(G_2) = 15$

(iii) $d(1) = 3$ $d(v_1) = 3$

$d(2) = 3$ $d(v_2) = 3$

$d(3) = 3$ $d(v_3) = 3$

$d(4) = 3$ $d(v_4) = 3$

$d(5) = 3$ $d(v_5) = 3$.

$d(6) = 3$

$d(7) = 3$

$d(8) = 3$

$d(9) = 3$

$d(10) = 3$

(iv) $\phi(1) = v_1$ $\phi(6) = v_6$
$\phi(2) = v_2$ $\phi(7) = v_7$
$\phi(3) = v_3$ $\phi(8) = v_8$
$\phi(4) = v_4$ $\phi(9) = v_9$
$\phi(5) = v_5$ $\phi(10) = v_{10}$.

v) 1 is adjacent = (2, 6, 5)
2 " " = (1, 3, 7)
3 " " = (2, 8, 4)
4 " " = (3, 9, 5)
5 " " = (1, 4, 10)
6 " " = (1, 9, 8)
7 " " = (2, 10, 9)
8 " " = (3, 6, 10)
9 " " = (4, 6, 7)
10 " " = (5, 7, 8)

v_1 is adjacent = v_2, v_7, v_6.
v_2 " " = v_1, v_3, v_9
v_3 " " = v_2, v_4, v_8
v_4 " " = v_3, v_5, v_7
v_5 " " = v_4, v_6, v_9
v_6 " " = v_1, v_5, v_8
v_7 " " = v_1, v_4, v_{10}
v_8 " " = v_3, v_6, v_{10}
v_9 " " = v_{10}, v_5, v_2
v_{10} " " = v_7, v_8, v_9.

1.2.7 Let G be simple s.t $E = \binom{V}{2}$ iff G is complete.

proof:

Let G be a graph with n vertices.
As a graph is simple.
If it has neither loop nor parallel edges.

Hence each edges incident to a two distinct element subset $\{v_i, v_j\}$ of v.

The possible number of two element subset of set v is nC_2.

$$E = \binom{n}{2}$$

1.2.8. S.T a) $E(K_{m,n}) = mn$.

The vertex set of $K_{m,n}$ consists of two disjoint set x & y.

Such that x contains m vertices and y contains n vertices.

No two vertices in x or in y are adjacent.

Hence degree of each vertex in $x = n$.

The degree of each vertex in $y = m$.

Sum of degree in $x = n+n+\cdots+m$ times.

$= mn$.

Sum of degree in $y = m+m+\cdots+n$ times

$= mn$.

Sum of degrees in $(K_{m,n}) = nm + mn$

$= 2mn$.

W.K.T sum of degree of vertices in $K_{m,n} = 2E$

$$2mn = 2E$$

$$E = mn$$

$$E(K_{m,n}) = mn.$$

There are mn edges

$$\therefore E(K_{m,n}) = mn.$$

b) If G is simple and bipartite, then $E \leq v^2/4$.

Soln:

Let G be a bipartite graph with $m+n$ vertices.

(ie) $v = m+n$.

The number of edges is maximum when G is complete.

The maximum number of edges of $G \leq mn$.

(i.e) $E \leq mn$

$mn \geq E$.

W.K.T arithmetic mean $\geq$ Geometric mean.

$A.M \geq G.M.$

$$\frac{m+n}{2} \geq \sqrt{mn}.$$

Squaring on both sides,

$$\frac{(m+n)^2}{4} \geq mn.$$

$$\frac{v^2}{4} \geq mn \geq E.$$

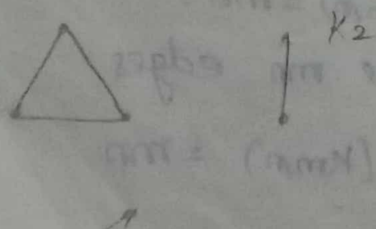
$$\frac{v^2}{4} \geq E.$$

(ie) $E \leq v^2/4$. $E \leq v^2/4$.

1.2.11 S.T the K-cube has K_n^c & $K_{m,n}^c$

a) The complement of a complete graph with n vertices is a empty graph of n vertices with no edges.

K_3:



The complement of $K_{m,n}$ is $K_m \cup K_n$ which is a disjoint union of 2 complete graphs.

b) A simple graph G is self-complementary if $G \cong G^c$.
S.T if G is self-complementary then $v = 0, 1 \pmod 4$
(or) Suppose that G is a graph on n vertices.
* S.T G is isomorphic to its own complement G^c (or $\bar{G}$)
P.T $v = 0 \pmod 4$ (or) $v = 1 \pmod 4$.

Since G is isomorphic it must be simple graph.

If G is simple $E \leq \binom{v}{2} = \frac{v(v-1)}{2}$.

Every pair of vertices in v is an edge in exactly one of the graphs G & G^c.

Hence the no of edges E(G) of G and the no of edges $E(G^c)$ of G^c.

satisfy:

$$E(G) + E(G^c) = \binom{v}{2} \longrightarrow ①.$$

$G \cong G^c$ they must have same no. of edges.

$$E(G) = E(G^c)$$

$$① \Rightarrow \quad 2E(G) = \binom{v}{2}$$

$$= \frac{v(v-1)}{2}$$

$$E(G) = \frac{v(v-1)}{4}.$$

$v(v-1) = 4E(G)$ must be an integers.

p is a multiple of 4 (or) v = 1 is a mod of 4.

(i.e) $v = 0 \pmod 4$ (or)

$v - 1 = 0 \pmod 4$.

Exactly one of the number n and $n-1$ is even.

So either 4 divides n or 4 divides $n-1$.

(i.e) $v = 0 \pmod 4$

$v = 1 \pmod 4$.

1.2.10 S.T the k-cube has 2^k vertices, $k2^{k-1}$ edges and is bipartite.

Another Question: How many edges does Q_k have?

In Q_k vertices correspond to the sequence $(a_1, a_2, \ldots a_k)$ where each $a_i = 0$ (or) 1.

So the number of vertices is equal to the number of such sequences.

Since the sequence is of the length k, and there are two choice or each place in the sequence.

No. of. Sequences $= 2 \cdot 2 \ldots 2$ (k times)

$= 2^k$.

Hence 2^k vertices.

No. of. vertices in $Q_k = 2^k$.

(ii) An edge joings vertices and if the sequences corresponding to the vertices differ in exactly one and place.

consider the sequences corresponding to the vertices differ in exactly one place.

Since it is of length k there are k sequence which differ from it is exactly one place.

So the degree of any vertices is k degree of any vertices is k-cube $= k \rightarrow (2)$.

Sum of degree of vertices $= 2E$.

$$E = \frac{\text{sum of degree of vertices}}{2}$$

$$= \frac{k + k + \cdots + 2^k \text{ (times)}}{2}$$

$$= \frac{k \cdot 2^k}{2} = k 2^{k-1}$$

(i.e) No. of edges in k-cube $= k \cdot 2^{k-1}$.

(iii) Defined subset of V as X and Y such that,

$X = \{$vertices whose bit string contains an even no of 1's$\}$.

$Y = \{$vertices whose bit string contains an odd no. of 1's$\}$

If u and v are connected by a edge in k-cube. Then the bit string u and v differ in exactly one bit.

So u as exactly one more or less 1 or v.

So if v is even u must be odd and vice versa.

Thus all adjacent are between vertices of opposite parity.

If V partition the vertices by parity we have no adjacenties which in a part.

So k-cube is bipartite.

1.5.1 Show that $\delta \leq \frac{2\epsilon}{v} \leq \Delta$.

Proof:

$$\sum_{v \in V} d(v) = 2\epsilon$$

$$\Rightarrow \sum_{v \in V} d(v) = 2|E(G)|$$

$\delta(G)$ is the minimum degree in graph.

$$\sum \delta(G) \leq \sum_{v \in V} d(v)$$

$$|V(G)|\,\delta(G) \leq 2|E(G)|$$

$$\delta(G) \leq \frac{2|E(G)|}{|V(G)|} \rightarrow (1).$$

$\Delta(G)$ is maximum degree in graph.

$$\sum(\Delta(G)) \geq \sum d(v)$$

$$|V(G)|\,\Delta(G) \geq 2|E(G)|$$

$$\Delta(G) \geq \frac{2|E(G)|}{|V(G)|} \rightarrow (2).$$

From (1) and (2) we get,

$$\delta(G) \leq \frac{2|E(G)|}{|V(G)|} \leq \Delta(G)$$

$$\delta \leq \frac{2\epsilon}{v} \leq \Delta.$$

1.5.3 S.T if a k-regular bipartite graph with $k>0$ ha[s] bipartition (X, Y) then $|X| = |Y|$.

Proof:

Let G be a k-regular bipartite graph with the bipartition (X, Y).

W.K.T, each edge of G has one end in X and another end in Y.

Since x is regular, we have

$$K|x| = K|y|$$

Since, $K>0$

$$|x| = |y|.$$

1.5.4 S.T in any group of two or more people, there are always two with exactly, the same number of friends inside the group.

proof:

Let the people we represent vertices of a graph G with an edge indicating that two people corresponding to its end point are friends.

Then this statement translated the follows.

Let G is any simple graph with 2 (or) more vertices then x must has two vertices with same degree.

Suppose G has n vertices where $n>2$.

The smallest possible degree is 0. (A person with no friends). and largest degree is $n-1$. (A person whose is everyone is friends).

Thus there are n possible values $0, 1, 2, \ldots n-1$ for n degree of G.

But 0 and $n-1$ cannot both the degree, because if one vertices has no neighbours. Then no vertices is adjacent to every other vertex and vice versa.

Thus the possible value of the degree are either $0, 1, 2 \ldots n-2$ (or) $1, 2, \ldots n-2, n-1$.

In either case there are only n-1 values but that must be assigns to n vertices. Some values must be used twice.

1.6.1 Show that if there is a (u,v) - walk in G then there is also a (u,v) - path in G.

We prove the statement by induction on the length l of the u-v walk.

Basic step:

l=0 having no edge w consist of a singe vertices u=v.

This vertices is a u,v path of length 0.

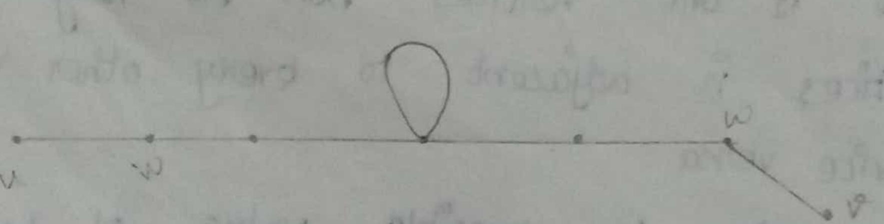
Induction Step:

l≥1 we suppose that the claim walk of length.

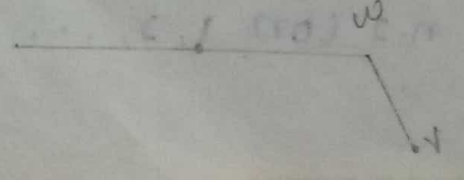
If w has no repeated vertices then its vertices and edges form a u-v path.

If w has repeated vertices has length l. then delecting the edges and vertices between appearness of w. (leaving one copy of) yields the shortest uv walks w' contained in w.

w:



w':



By induction hypothesis w' contained $u-v$ path P and this path P contained in w.

This proves the theorem.

1.2.12 The set of all automorphism of a simple graph G forms ~~from~~ a group ~~and a~~ under the composition of maps.

Proof:

Let G be a simple graph and $\Gamma(G)$ be a set of all automorphism of G.

Let $\alpha: G \to G$ be an automorphism so α is a bijective map which preserves adjacencies of the vertices.

If $e = uv \in G$ then $\alpha(e) = \alpha(u)\alpha(v)$ and so

$\alpha(uv) = \alpha(u).\alpha(v)$

Now let us verify the axium of a group

Vertification of Group axiom:

(i) Closure:

Let $\alpha, \beta \in \Gamma(G)$.

So that α and β are automorphism on G.

Since α and β are bijective maps their composition $\alpha\beta$ is also bijective.

Let $u, v \in G$ be adjacent vertices.

Since β is an automorphism.

we find $\Rightarrow \beta(u)$ and $\beta(v)$ are adjacent in G and

$\beta(uv) = \beta(u).\beta(v)$

Again since α is an automorphism

$\alpha(\beta(u))\ \alpha(\beta(v)) \in G$ are adjacent.

$\alpha(\beta(u))\alpha(\beta(v)) = \alpha(\beta(u)).\alpha(\beta(v))$

$\alpha(\beta(uv)) = \alpha(\beta(u))\ \alpha(\beta(v))$

$(\alpha\beta)(uv) = (\alpha\beta)u\ (\alpha\beta)v$

$(\alpha \circ \beta)\, v = (\alpha \circ \beta) u \; (\alpha . \beta) v$

$\alpha \circ \beta \in \Gamma_G$.

So Γ_G is closed under compositive.

(ii) Associative:

we know that

Since composition of maps is associative

$(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma) \;\; \forall \; \alpha, \beta, \gamma \in \Gamma_G$.

(iii) Identity:

consider the identity map given by

$I(u) = u \quad \forall u \in G$.

We have $I(u, v) = u . v$ ~~$\cdot$ ie~~ $I(uv) = uv$

(i.e) $I(uv) = I(u) . I(v)$

clearly I is a bijective map preserving adjacency

Now, $(I \circ \alpha) u = I(\alpha(u))$

$= \alpha(u)$

$(I \circ \alpha)u = \alpha$ $(\alpha \circ I)u = \alpha$

$I \circ \alpha = \alpha$.

$(\alpha . I) u = \alpha [I(u)]$

$= \alpha u$.

$\alpha \circ I = \alpha$.

This $I \circ \alpha = \alpha \circ I$

$= \alpha \;\; \forall \; \alpha \in \Gamma_G$.

I is the identity element in the set of automorphism Γ_G.

(iv) Inverse.

Again take $\alpha \in \Gamma_G$

since α is bijective α^{-1} exist and is also bijective.

Also, if $u, v \in G$ are adjacent then,

$\alpha(uv) = \alpha(u)\alpha(v)$

put $\alpha(u) = u_1$, $\alpha(v) = v_1$

$u = \alpha^{-1}(u_1)$, $v = \alpha^{-1}(v_1)$

Now,

$\alpha(uv) = \alpha(u), \alpha(v)$

$\alpha(uv) = u_1 v_1$

$\Rightarrow \alpha^{-1}(u_1, v_1) = u \cdot v$

$= \alpha^{-1}(u_1) . \alpha^{-1}(v_1)$

$\alpha^{-1} \in \Gamma G$.

$\alpha^{-1}(\alpha(uv)) = \alpha^{-1}(\alpha(u)) . (\alpha(v))$

$= \alpha^{-1}(\alpha(u)) . \alpha^{-1}(\alpha(v))$

$= (\alpha^{-1}\alpha) u \; (\alpha^{-1}\alpha) v$.

$= Iu . Iv$.

$(\alpha^{-1}\alpha)(uv) = I(uv)$

$\alpha^{-1}\alpha = I$

Also, $\alpha(\alpha^{-1}(u_1, v_1)) = \alpha(\alpha^{-1}(u_1)) . (\alpha^{-1}(v_1))$

$= \alpha(\alpha^{-1}(u_1)) . \alpha(\alpha^{-1}(v_1))$

$= (\alpha\alpha^{-1}) u_1 . (\alpha\alpha^{-1}) v_1$

$= I(u_1) . I(v_1)$

$\alpha\alpha^{-1}(u_1, v_1) = I u_1 v_1$.

$\therefore \alpha\alpha^{-1} = I$.

Thus $\alpha\alpha^{-1} = \alpha^{-1}\alpha = I \quad \forall \alpha \in \Gamma G$.

Thus all axioms of a group are satisfied.

$\therefore \Gamma G$ is a group under the composition of maps.

A graph is bipartite iff if it contains no odd cycle.

Proof:

Let G be a bipartite graph.

We have to p.T G contains no odd cycles.

Since G is bipartite the vertex set is partitioned in to two non empty set x and y. Such that every edge of G has one vertex in x and other vertex in y.

consider the cycles.

$C = v_1 v_2 v_3 \dots v_n v_1$

Let $v_1 \in X$, so that $v_2 \in Y$, $v_3 \in X$, $v_4 \in Y$.

In general $v_k \in X$, if k is odd $v_k \in Y$ if k is even.

For the cycle C the last vertex $v_1 \in X$ and so the proceeding vertex $v_n \in Y$.

Therefore n is even and so the cycle c has an even no. of vertices and consequently it has an even no of edges.

The cycle c is an even cycle since the cycle is an arbitary. it follows that every cycle is even.

Hence the graph G has no ~~of~~ odd cycle.

conversely,

Let the graph G be without odd cycles.

W.K.T.P.T G is bipartition.

Let v be any vertex of G.

Let $X = \{x \in G / d(x, v) = \text{even}\}$

$Y = \{y \in G / d(y, v) = \text{odd}\}$.

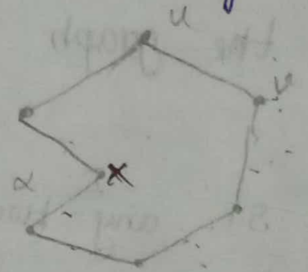
So X and Y are user manner. Clearly X & Y are non-empty disjoint subset of the vertex set of G.

we now ~~W.K.~~ P.T no two vertices of X are adjacent.

Suppose the vertex $u, v \in X$ be adjacent.

So there uv is an edge

Now, the x-u path & x-v path and the edge uv together form a cycle.



Length of cycle.

$\Rightarrow d(x,u) + d(x,v) + 1$

$=$ even $+$ even $+ 1$

$=$ odd.

This cycle is an odd cycle.

This is a contradiction to the assume that the graph is without odd cycles.

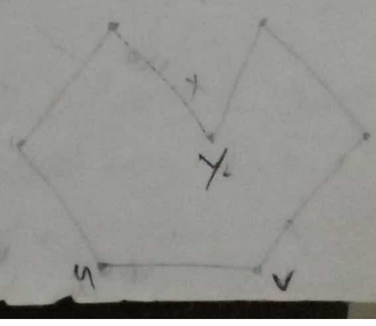
No two vertices of X are adjacent.

Again we P.T no two vertices $u, v \in Y$ are adjacent.

Suppose two vertices $u, v \in Y$ be adjacent

So that uv is an edge

Now y-u path and y-v path and the edge uv together form a cycle.



Length of cycles.

$\Rightarrow d(y,u) + d(y,v) + 1$

$=$ odd $+$ odd $+ 1$

$=$ even $+ 1$ $=$ odd.

This cycle is an odd cycles.

This is a contradiction to the assume that the graph is without odd cycles.

Therefore no two vertices x & y are adjacent.

Thus the vertices set of graph G is divided into non-empty disjoint subset X, Y such that no two vertices of X are adjacent and no two vertices of Y are adjacent.

∴ (X, Y) is a bipartition of G and so the graph G is a bipartite graph.

1.6.10 S.T any two longest paths in a connected graph have a vertex in common.

proof:

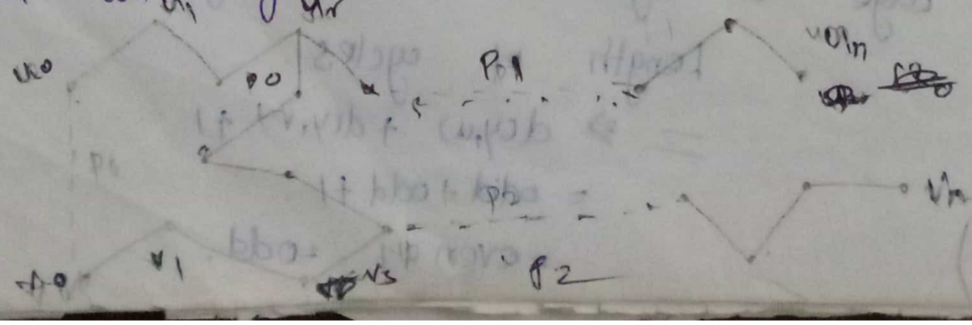
Let n be the length of any ~~the~~ longest path in a connected graph G.

Let $P_1 = u_0 u_1 u_2 \dots u_n$ and ~~the~~ another path $P_2 = v_0 v_1 v_2 \dots v_n$ be two ~~P₂ be a~~ longest paths so that each is of length n.

Let us assume that P_1 and P_2 have no common vertex

So $u_0 \neq v_0$.

Since G is a connected graph. there is a path connecting u_0 and v_0.



Case i) :

Let the path be P_0. If P_0 has no vertices other than u_0 and v_0 common with ~~path~~ P_1 and P_2 respectively, then P_0 is the edge u_0v_0.

In this case $P_1 \cup P_0 \cup P_2$ is a path ~~length~~ longer than P_1 and P_2.

which is a contradiction to the hypothesis.

So there must be some vertices u_r and v_s of P_1 and P_2 respectively common with P_0.

Case ii) :

Let No vertex of P_0 between u_r and v_s be common with ~~P_1 and P_2~~ P_1 or P_2

since the length of the ~~longest~~ path P_1 ~~and~~ is of length n, we observe that the either the (u_0-u_r) section or the ~~section~~ (u_r-u_n) section of P_1 is of length $\geq n/2$.

Similarly, either the section (v_0-v_s) or the section (v_s-v_n) of P_2. is of the length $\geq n/2$.

without loss of generality we may assume that the (u_0-u_r) section of P_1 and the (v_s-v_n) section of P_2 are of length $\geq n/2$. also the length of (u_r, v_s) section of P_0 is ≥ 1

Also (u_r-v_s) of P_0 is ≥ 1.

$$(u_0-u_r) \cup (u_r-v_s) \cup (v_s-v_n) \geq n/2+1+n/2.$$

$$\geq n+1.$$

This is a contradiction as the length of a longest path is n.

So ~~the~~ our assume that P_1 and P_2 have no common vertex is wrong. Hence ~~proved~~ any two longest paths have a common vertex

1.7.2 S.T if $\delta \geq 2$ then G contains a cycle.

proof:

Let $P = \{v_1, v_2 \dots v_k\}$ be the path of maximum length.

Since v_1 as degree 2 it is adjacent to a vertex, $w_1 \neq v_2$.

The vertex w is in P. otherwise $p = (w_1, v_1 \dots v_k)$ would be a longest path than p.

Thus $w = v_j$ for some $j > 2$. and so $(v_1 v_2 \dots v_j, v_1\}$ is a cycle $\{\because v_1$ is adjacent with v_2 and $w = v_j\}$.

1.7.3 S.T if G is a simple and $\delta \geq 2$ then G contains a cycle of length at least $\delta + 1$.

proof:

Given that G has minimum degree $\delta \geq 2$, we have to P.T G contains a cycle of the length atleast the $\delta + 1$.

Let $P: v_1 v_2 v_3 \dots v_n$ be a path of maximum length of G and $A = \{v_1 v_2 \dots v_n\}$

Let $d(v_1) = r + 1$,

$\delta(G) \geq 2$

We find $r \geq 1$.

So that there are $r+1$ edges incident on v_1.

(ie) v_1 is adjacent to r vertices $u_1, u_2, \dots u_r$ say and v_2.

Let $B = \{u_1, u_2, \dots u_r\}$

We prove that $B \subseteq A$.

Suppose not then there exist $u_k \in B$ but $u_k \notin A$.

In this case u_k is adjacent to v_1 so that we get a path $u_k v_1 v_2 \ldots v_n$ which is of greater than B contradiction our assume that P is a path of maximum length.

Hence $B \subseteq A$

So the vertices $u_1, u_2, \ldots u_r$ coincides with some ~~are~~ vertices of the set $A = \{v_1, v_2 \ldots v_n\}$

let v_i be a forthest vertex in A with coincides with a vertex u_s of B.

Now $v_1 v_2 \ldots v_i (= u_s) \ldots v_1$

form a cycle of length $\geq r+2$

$$\geq r+1+1$$
$$= d(v_1)+1$$
$$\geq \delta+1.$$

Hence there exist a path of length atleast $\delta+1$

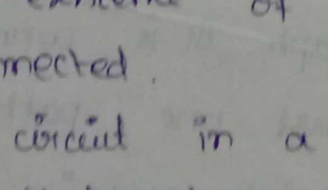
A graph without any cycle is called an acyclic graph
A connected acyclic graph is called a tree

2.1.1 S.T if any two vertices of a loopless graph of G are connected by a unique path, then G is tree.

Proof:

The existence of the path u, v between the path are connected.

A circuit in a graph (with two are more vertices) implies that there is atleast one pair of vertices a, b such that there is atleast or are two distinct path between a & b.



Since G has one and only path between every pair of vertices.

∴ G have no circuits

∴ G is a tree.

2.1.5 Let G be graph with v-1 edges. S.T the following three statement are equivalent

a) G is connected.

b) G is acyclic.

c) G is a tree.

Proof:

(a) ⇒ (b).

Given G is connected, graph of p vertices and p-1 vertices

To prove that G is acyclic.

Suppose G is connected of ~~v~~ p vertices and ~~v~~-1 edges.

Suppose G is not acyclic.

Then its contained a cycle.

Let c be a cyclic of all vertices and so there are ~~v~~ r edges in a cycle.

Now there are v-r vertices not in a cycle c.

Since the graph G is connected.

There is atleast one path connecting u & v.

Any such path contains atleast one edge not in a cycle c.

Since there are v-r vertices not in the cycle there must be atleast v-r edges not in a cycle.

∴ The number of edges of the in a graph G must be atleast ~~v-r+r~~. $p-r+r=p$

r edges in a graph.

$q=p-1$ This is a contradiction as there are only v-1 edges.

So our assumption G is not acyclic is wrong.

∴ G must be acyclic.

(b) ⇒ (c).

Given that G is an acyclic of v p vertices and v-1 p-1 edges.

To prove that G is a tree.

It is enough to prove that there is a unique path connecting every pair of vertices u & v.

Suppose there are two distinct paths p_1 and p_2 between the vertices u ~~&~~ and v.

We know that $p_1 \cup p_2$ contains a cycle and so the graph G is cyclic.

This is a contradiction as a graph is given to be acyclic.

So there is only a unique path between any pair of vertices and so the graph is a tree.

∴ G is a tree.

(c) ⇒ (a)

Given that G is a tree.

By definition of tree, ~~tree~~ we know G is connected.