UNIT - II.

Note :

An edge of a cycle cannot be a block because it belongs to larger subgraph without cut vertex.

2m [ An edge is a block of a graph iff it has a cut edge of G.

2.1.5 The blocks of a tree are its edges.

Every isolated vertex is itself a block.]

If a block has more than two vertices then it is 2-connected.

The blocks of a graph are its isolated vertices its cutedges and its maximal 2-connected sub-graph.

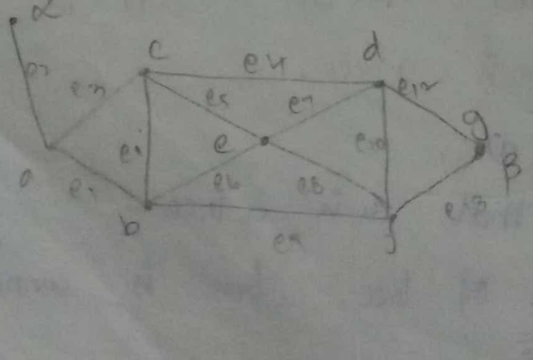
## Internally disjoint paths:

Two paths in a graph G are said to be internally disjoint if they do not have a common internal vertex.

consider a path connecting $P_1$: acdg, $P_2$: abdg, $P_3$: acbefg, $P_4$: acedg, $P_5$: acefg, $P_6$: abefg $P_7$: abefg.

$P_1$ and $P_2$, $P_1$ and $P_4$ are internaly disjoint

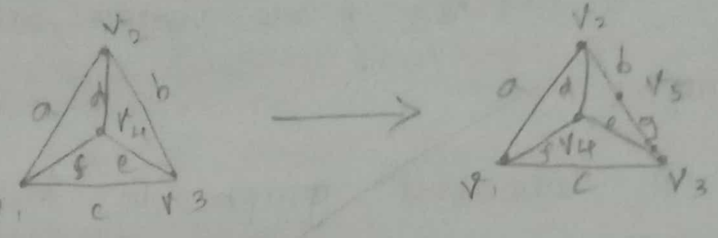
Ex:



Subdivision of an edge:

An edge e is said to be subdivided when it is deleted and replaced by a path of length 2 connectivity its ends, the internal vertex of this path being new vertex.

Ex:

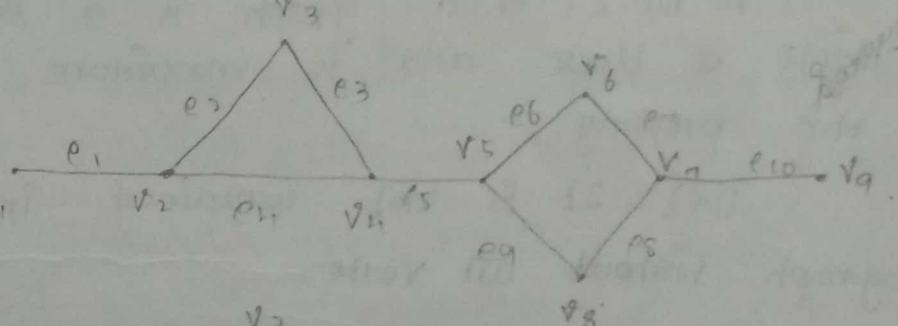


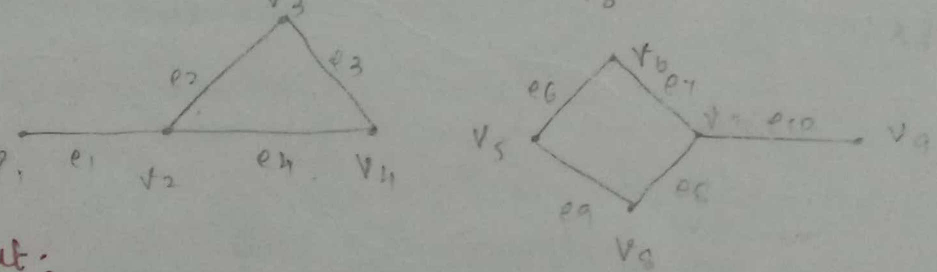

Cut edge:

An edge e of a graph G is said to be a cut edge if its removal disconnect the remaining that is G-e is disconnected graph.

The cut edge also called a bridge of a graph.

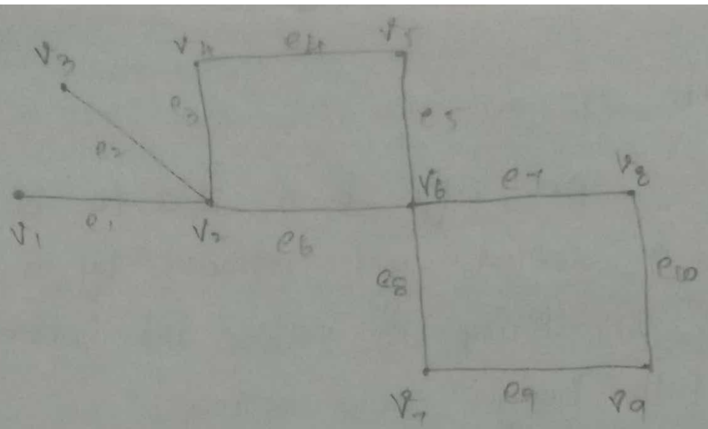
Ex:




Vertex cut:

Let G be a connected graph the vertex cut of G is subset v' its vertex set which on removal the remaining graph disconnect or trivial




2 $\{v_2\}$, $\{v_6\}$, $\{v_7, v_8\}$ are vertex cut of $G$.

**Separable graph:**

A connected graph $G$ is said to be separable if its connectivity $\kappa(G) = 1$. All other connected graph, are non-separable.

**Blocks:**

A connected graph that has no cut vertex called a block.

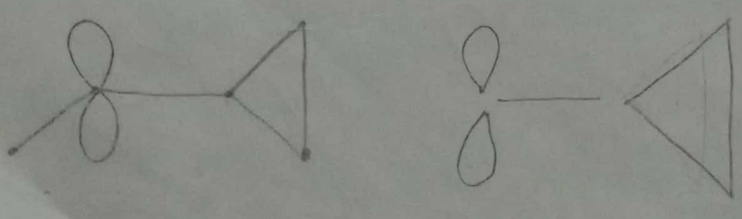
Every block with atleast ~~three~~ two vertices is two connected.

A block of a graph is a subgraph which is itself a block and is maximum with respect to this property.

(i.e) It is not contained in any other subgraph without cut vertex.

2½ hours

Ex:



Every graph is the union of its blocks.

Theorem :

For any graph $G$ $\kappa \le \kappa' \le \delta$.

Proof:

If $G$ is trivial then $\kappa' = 0 \le \delta$.

otherwise the set of links incident with a vertex of degree $\delta$.

consititudo $\delta$ edge cut of $G$.

This implies that $\kappa' \le \delta \rightarrow (1)$.

Next, we prove that $\kappa \le \kappa'$ by induction on $\kappa'$.

This result is true if $\kappa' = 0$. since then $G$ must be either trivial (or) disconnected

suppose that the result holds for all graph with edge connectivity less than $k$ and let $G$ be a graph with $\kappa'(G) = \kappa(G) > 0$.

Let $e$ be an edge in a $k$ edge cut of $G$. setting $H = G - e$. we have $\kappa'(H) = k - 1$ and so by induction hypothesis $\kappa(H) \le k - 1$.

If $H$ contains a complete graph as "o" spanning subgraph, then so does $G$ and $\kappa(G) = \kappa(H) \le k - 1$

otherwise, let $S$ be a vertex cut of $H$ with $\kappa(H)$ elements. since $G - S$ is disconnected either $G - e$ disconnected or connected.

case (i):

Suppose $G - S$ disconnected.

Then $\kappa(G) \le \kappa(H) \le k - 1$.

Since $G - S$ is disconnected $\omega(G - S) = 0$

$\therefore \kappa(G) \le \kappa(H) \le k - 1$.

$\kappa(G) \le \gamma(G) - (1) \le \kappa(H) + 1 \le k$.

(i.e) $\kappa(G) \le k \Rightarrow \kappa(H) \le \kappa'(G)$.

Case(ii):

Suppose $G-S$ is connected and $e$ is a cut edge of $G-S$.

$G-S$ has one vertex cut $\{v\}$.

$\Rightarrow S \cup \{v\}$ is a vertex cut of $G$, then

$\kappa(G) \leq \kappa(H)+1 \leq \kappa'$.

$\kappa(G) \leq \kappa'(G)$

Thus in each case we have $\kappa(G) \leq \kappa'(G) \longrightarrow (2)$.

combine the (1) & (2) we get,

$\kappa \leq \kappa' \leq \delta$.

$\therefore \kappa \leq \kappa' \leq \delta$.

Theorem:

7) A graph $G$ with $\nu \geq 3$ is 2 connected iff any 2 vertices of $G$ are connected by atleast two internally disjoint path.

Proof:

Let $G$ be a connected graph and let any 2 vertices of $G$ are connected by atleast 2 internally disjoint path.

We have to prove that $G$ is 2-connected. for this it is enough to prove that the graph $G$ has no cut vertex.

Suppose $G$ has a cut vertex $v$ then $G-v$ is disconnected.

Take any 2 vertices $u$ and $w$ in distinct component of $G-v$.

In $G$, any path connecting the vertices $u$ & $w$ will have $v$ as an internal vertex

No paths connecting u and w are internally disjoint.

This is contradiction as these are atleast 2 internally disjoint paths.

G has no cut-vertex and so G is 2-connected.

conversely,

Let us assume that G is 2-connected. Let u & v be any pair of distinct vertices of G.

We have to prove that there are atleast 2 internally disjoint path connecting u & v.

We prove the result by induction on the length d(u,v) of the path connecting u & v.

(i) suppose d(u,v) = 1, then u,v is an edge. This edge cannot be a cut edge for otherwise the end vertices u (or) v is a cut vertex.

This is a contradiction to the assumption that G is 2-connected.

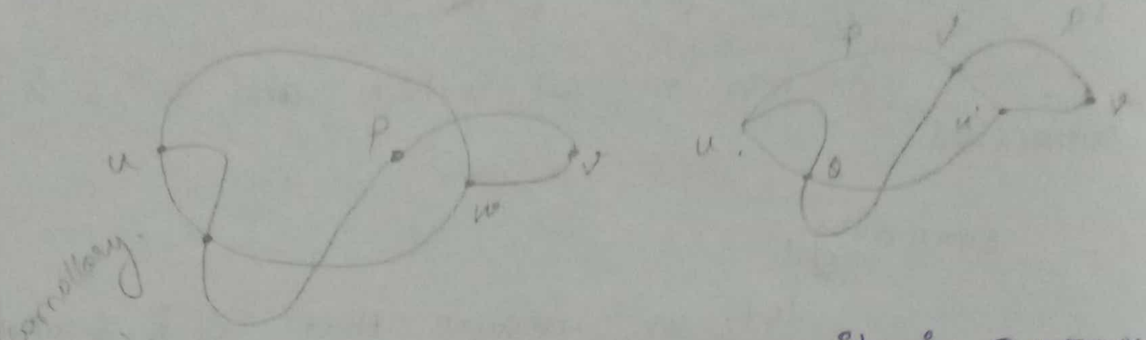
∴ uv is not a cut edge and so it belongs to a cycles.

Hence there is another path connecting u & v clearly these 2 paths are internally disjoint

(ii). Thus the results is true in this case. Now assume that the theorem hold for any 2 vertices of distance less than k and let $d(u,v) \leq k \leq 2$.

consider d(u,v) path of length k & let w be the vertex that proceeds v on its path.

Since $d(u,w) = k-1$, it follows from this induction hypothesis that there are 2-internally disjoint $(u,w)$ paths $P$ and $Q$ is in $G$.




Again since $G'$ is a block it is 2-connected Then by Theorem,

It is $G$ is 2-connected then any 2-vertices of a $G$ lies on an common cycle.

The vertices $v_1$ and $v_2$ lie on a common cycle of $G$.

$e_1$ and $e_2$ lie on a common cycle of $G$.

Euler's Tours:

An open walk in a graph in which no edge is retraversed is a trail.

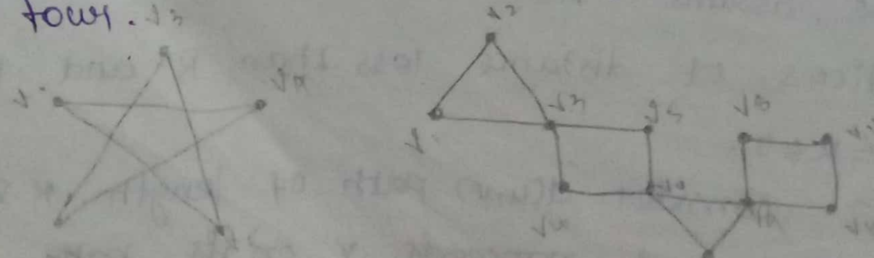
A trail that traverse every edge exactly once is called an Euler trail.

A Closed walk that traverse each edge exactly once is called an Euler tour.

(i.e) An Euler tour is a closed Euler trail.

A graph is Eulerian graph of it contains an Euler tour.

EX:



Theorem: 4.1

A non-empty connected graph is Eulerian iff it has no vertices of odd degree.

Proof:

Suppose G is an Eulerian graph.

We have to prove that all the vertices of G are of even degree.

Since G is Eulerian it contains an Euler tour. Let C be an Euler tour of G with origin and terminals.

Each time a vertex v occurs as an interval vertex of C, two of the edges incident with v are accounted for, since an Euler tour contains every edge of G.

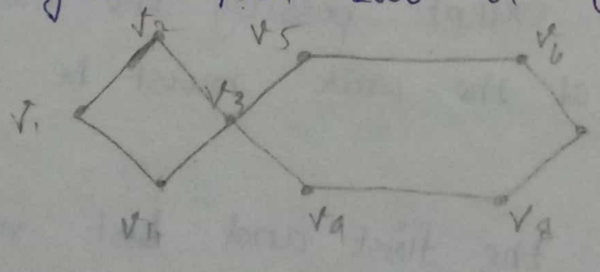
(Since C starts and ends at u. d(u) is also even.

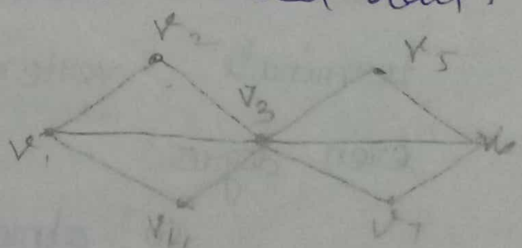
Thus G has no vertices of odd degree.

conversely,

Suppose that G is an non-Eulerian connected graph with atleast one edge and no vertices of odd degree.

Choose such a graph G with as few edge as possible, since each vertex of G, has degree after two G contains closed trail.




Let C be a closed trail of maximum possible length in G.

By assumption $C$ is not an Euler tour of $G$ and so $G-E(C)$ has some component $G'$ with $E'(G') > 0$.

Since $c$ is itself Eulerian it has no vertices of odd degree. These the connected graph $G'$ also no vertices of odd degree.

Since $E(G') < E(G)$ it follows from the choice of $G$ that $G'$ has an Euler tour $c'$.

Now because $G$ is connected, there is a vertex $v$ in $V(C) \cap V(C')$ and we have any assumption without loss of generality that $v$ in the origin and terminus of both $c$ & $c'$.

But, then $cc'$ is a closed trail of $G$ with $E(CC') > E(C)$ is contradiction the choice of $C$.

Corrollary:

A connected graph has an Euler trail iff it has atmost 2 vertices of odd degree.

proof:

Suppose that the connected graph $G$ has an Euler trail $P$. clearly $P$ is an open walk of $G$.

Again since $P$ is Eulerian every intermediate vertex, except possibly the initial and terminate vertex of the walk must be of even degree.

So atmost the first and last vertices of odd degree.

Conversely.

Let $G$ be a connected graph with atmost two vertices of odd degree.

We have to prove that the graph contains an Euler trail.

If $G$ has exactly 2 vertices and $v$ of odd degree.

In this case, let $G+E$ the denote the graph obtain from $G$, by the addition of a new edge $e'$ joining $u$ & $v$.

Clearly, each vertex of $G+e$ has even degree and so by the theorem,

"If all vertices of a graph are of even degree. Then $G$ is Eulerian".

$\therefore$ $G+e$ has an Euler tour.

$$C = v_0 e_1 v_1 e_2 \ldots \ldots v_{k+1} e_{k+1}$$

Where $e_1 = e$. The trail $v_1 e_2 v_2 \ldots e_{k+1} v_{k+1}$ is an Euler trail of $G$.

Hence the proof.

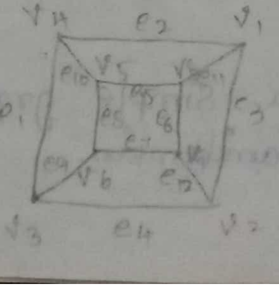
Hamilton cycles:

A path that contains every vertex of $G$ is called a hamilton path of $G$.

A hamilton cycle of $G$ is a cycle that contains every vertex of $G$.

A graph is hamiltonian if it contains a hamilton cycle.

Ex:



for the graph G.

(i) $v_1 v_2 v_3 \dots v_8$.

ii) $v_1 v_8 v_5 v_6 v_7 v_2 v_3 v_4$ are hamiltonian path

(iii) $v_1 v_2 v_3 v_4 v_5 v_6 v_7 v_8 v_1$ is a hamiltonian cycle.

So the graph G is hamiltonian graph.

**Theorem:**

If G is hamiltonian then for every non-empty proper subset of V, $w(G-S) \leq |S|$.

**Proof:**

Let G be a Hamiltonian graph. Then G contains a hamiltonian cycle.

Let C be hamiltonian cycle of G. Then for every non-empty proper subset S of V.

$$w(G-S) \leq |S|$$

Also $(C-S)$ is a spanning subgraph of G-S and so,

$$w(G) \leq G-S$$

$$w(G-S) \leq w(G)$$

$$w(G-S) \leq w(C-S)$$

$$w(G-S) \leq w(C-S) \leq |S|$$

$$\therefore w(G-S) \leq |S|$$

$$\therefore w(G-S) \leq |S|$$

Hence proved.

**Theorem:**

Sufficient condition:

**Dirac's theorem:**

If G is a simple graph with $\nu \geq 3$ and $\delta \geq \nu/2$ then G is Hamiltonian.

proof :

Suppose that the theorem is not true,

let $G$ be a maximal [having maximum number of edges] non hamiltonian simple graph with $p \geq 3$. and $\delta \leq p/2$.

Now $G$ cannot be a complete graph is non-hamiltonian.

There are $u$ and $v$ vertices in $G$ which are not adjacent.

without loss of generality let us assume that $G$ has 2 non-adjacent vertices $u$ & $v$.

$\therefore G' = G + uv$ is complete and so is a hamilton graph.

Since $G$ is non-hamiltonian. each hamiltonian cycle of $G + uv$ must contain the edge $uv$.

Thus there is a hamiltonian path $v_1, v_2 \ldots v_p$ in $G$ with origin $u = v_1$ & terminus $v = v_p$.

Take $S = \{v_i / uv_{i+1} \in E(G)\}$

$T = \{v_i / vv_i \in E(G)\}$

Since $uv \notin E(G)$ we get $v \notin S$ and $u \notin T$.

Again since the graph is simple we observe $u \notin S$ and $v \notin T$.

Thus $u, v \notin S$, $u, v \notin T$.

$u, v \notin S \cup T$.

But $u, v \in G \Rightarrow S \cup T \neq V(G)$

$$\left.\begin{array}{l} |S \cup T| < p \\ |S \cap T| = 0 \end{array}\right\} \rightarrow (1).$$

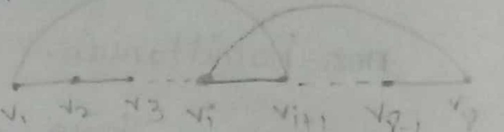
Since if $S \cap T$ contains some vertex

$v_1$ then $G$ would have the hamilton cycle $v_1, v_2, \ldots v_i$

$v_q, v_{q-1}, \ldots \ldots v_{i+1}, v_1$.

This is a contradiction to our assumption that $G$ is non-hamiltonian.

From the definitions of $S$ and $T$ are get, $d(u) = |S|$ and $d(v) = |T|$.




$$d(u) + d(v) = |S| + |T|$$

$$= |S \cup T| + |S \cap T|$$

$$< q + 0 \quad (\text{by } ①).$$

$$d(u) + d(v) < q \longrightarrow ②.$$

Given that $\delta \geq q/2$. So that the minimum degree of any vertices is $\delta \geq q/2$.

$$\therefore d(u) + d(v) \geq q/2 + q/2.$$

$$d(u) + d(v) \geq q \longrightarrow ③.$$

This is a contradiction to ②.

So our assumption that $G$ is non-hamiltonian is wrong.

Hence $G$ is hamiltonian Graph.

**Lemma:**

Let $G$ be a simple graph and let $u$ & $v$ be non adjacent vertices in $G$. such that $d(u) + d(v) \geq q$ then $G$ is H iff $G+uv$ is Hamiltonian.

**proof:**

Let $G$ be a Hamiltonian graph so it contains a Hamiltonian cycle $C$.

Given $u$ & $v$ are non-adjacent vertices of $G$

$\therefore$ $G+uv$ has also as a Hamiltonian graph.

conversely,

$G+uv$ is Hamiltonian graph where $u$ & $v$ are non-adjacent vertices of $G$ and such that,

$d(u)+d(v) \geq \nu$.

If $G$ is not hamiltonian then by Dirac's theorem,

$d(u)+d(v) < \nu$.

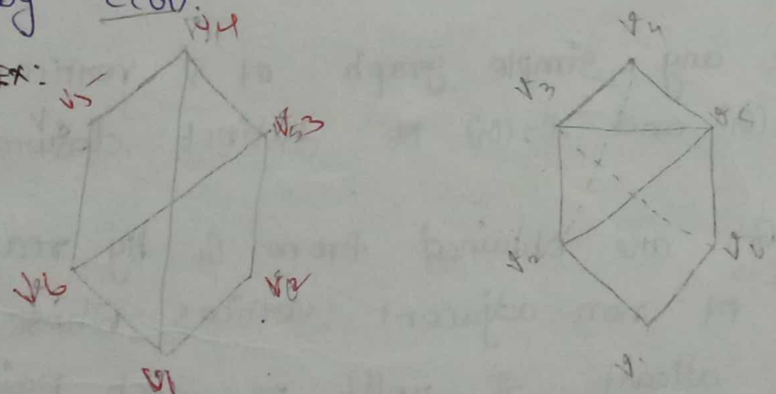
This is a contradiction

Hence $G$ must be hamiltonian.

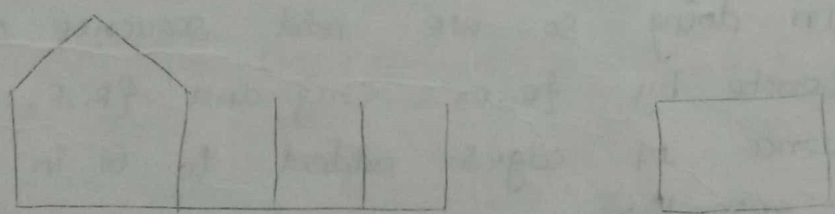
Closure of a graph:

Let $G$ be a simple graph of $\nu$ vertices. The closure of $G$ is the graph obtained from $G$ by recursively joining pairs of non-adjacent vertices whose degree sum is atleast $\nu$ until no such pairs remains. The closure of $G$ is denoted by $c(G)$.

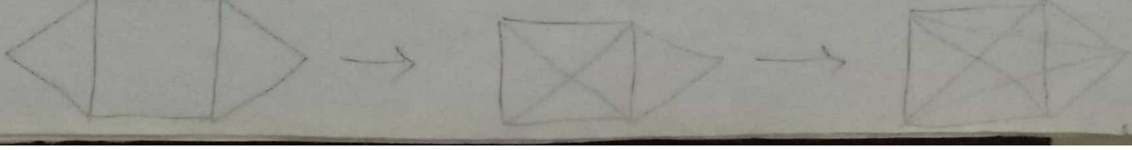
Ex:




Then,



Hence the number of vertices $\nu = 6$ $v_2, v_6$ are non-adjacent and $d(v_1)+d(v_2) = 3+3 = 6$.



join $v_2 v_6$.

$v_1 v_5$ are non-adjacent and,

$d(v_1)+d(v_5) = 4+2=6$. join $v_1, v_4$.

$v_2 v_5$ are non-adjacent.

$d(v_2)+d(v_5) = 2+5 = 7 \Rightarrow v$.

join $v_2 v_5$

$v_3, v_6$ are non adjacent and $d(v_3)+d(v_6) = 3+6=9 = v$

join $v_3 . v_5$.

$v_1 v_3$ are non-adjacent and $d(v_1)+d(v_3) = 1+3 = 4 = v$

join $v_1 , v_3$.

$v_1 v_5$ are non-adjacent and $d(v_1)+d(v_5) = 4+4 = 8 = v$

join $v_1 , v_5$

Theorem 4.42.

The closure of a graph $C(G)$ is well defined

Proof:

Let $G$ be any simple graph of $v$ vertices if possible let $C_1(G)$ and $C_2(G)$ be distinct closures of $G$.

$G_1$ and $G_2$ are obtained from $G$ by recursively joining ~~points~~ pairs of non-adjacent, vertices whose degree sum is atleast $v$ until no such pair remains.

In doing so we add sequence of edge to $G$. Denote by $\{e_1, e_2 \ldots e_m\}$ and $\{F_1, F_2 \ldots F_n\}$ the sequence of edges added to $G$ in obtaining $G_1$ & $G_2$ respectively.

We shall show that each $e_j$ is an edge of $G_2$ each $f_j$ is an edge of $G_1$.

There may be some edges common to the above sequence if possible. let $e_{k+1} = uv$, be the first edge in the sequence $e_1, e_2 \ldots e_n$. That is not an edge on $G_2$.

let $S_1 = \{e_1 \ldots e_m\}$
$S_2 = \{f_1, f_2 \ldots f_n\}$

$\therefore e_1, e_2 \ldots e_k$ are common to $G_1$ and $G_2$.

Let $H = \{e_1, e_2 \ldots e_{k-1}\} + G$.

It follows from the,

$$d_H(u) + d_H(v) \geq ? \longrightarrow ①.$$

H is sub graph of $G_1$.

It follows from the,

$$d_{G_2}(u) + d_{G_2}(v) \geq ? \quad — ②$$

Clearly
By the ~~choice of $e_{k+1}$~~, H is subgraph of ~~$G_2$~~ $C_2(u)$.

$\therefore$ The degree sum of u & v in ~~$G_2$~~ $C_2(u)$ is greater than the degree sum of u & v in H.

$$d_{G_2}(u) + d_{G_2}(v) \geq d_H(u) + d_H(v) \longrightarrow ②.$$

From ① & ② we get,

$$d_{G_2}(u) + d_{G_2}(v) \geq ?. \longrightarrow ③.$$

This is a contradiction. since u & v are non-adjacent in $G_2$.

Therefore each $e_i$ is an edge of $G_2$ and similarly each $f_i$ is an edge of $G_1$.

Thus the closure is independent of the sequence of edges added.

Hence the closure $C(G)$ of a graph G is well defined.

$S_1 = S_2$
$C_1(u) = C_2(u)$

Theorem 4.4.

A simple graph G is hamiltonian iff its closure is hamiltonian.

Proof:

Let G be any simple graph of $p$ vertices the closure $C(G)$ is obtained by recursively joining the pairs of non-adjacent vertices whose degree sum is atleast $p$.

In doing so ~~that~~ let $\{e_1, e_2, \dots e_n\}$ be the sequence of the edges added to G.

$\therefore C(G) = G + \{e_1, e_2, \dots e_n\}$

We know that G is Hamiltonian iff $G+uv$ is Hamiltonian ~~whose~~ where u & v are non-adjacent vertices of G. Whose degree sum is atleast $p$.

$\therefore$ G is hamiltonian $\Leftrightarrow$ $G+\{e_1\}$ is hamiltonian.

$\Leftrightarrow$ $G+\{e_1, e_2\}$ is Hamiltonian.

$\Leftrightarrow$ $G+\{e_1, e_2, e_3\}$ is "

$\Leftrightarrow$ $G+\{e_1 \dots e_n\}$ " "

$\Leftrightarrow$ $C(G)$ is "

Hence G is Hamiltonian iff the closure $C(G)$ is Hamiltonian.

Theorem: Chvatal theorem

Let G be a simple graph of $p$ vertices with degree sequence $(d_1, d_2, \dots d_p)$ where $d_1 \le d_2 \le d_3 \le \dots \le d_p$ and $p \ge 3$. Suppose that there is no value of $m$ less than $p/2$ for which $d_m \le m$ and $d_{p-m} \le p-m$ then G is Hamiltonian.

$d_{p-m} \le p-m$

Hamiltonian

Proof:

Given that G is simple graph with $\gamma$ degree sequence $\{d_1, d_2, \dots d_\gamma\}$ where $d_1 \le d_2 \le \dots \le d_\gamma$ and $\gamma \ge 3$

It is also given that there is no value $m \le \gamma/2$ for which $d_m \le m$ and $d_{\gamma-m} \le \gamma - n$.

We have to prove that G is Hamiltonian. For this it is enough to prove that the closure of G is Hamiltonian

Claim:

The closure $C(G)$ is a complete graph.

Let us assume that $C(G)$ is not complete.

$\therefore$ There exist atleast 2-non adjacent vertices u and v in $C(G)$

By definition of $C(G)$, we observe that the degree sum of $\gamma$ ~~and~~ in $C(G)$ ~~and d'~~ is less than $\gamma$

$d'(u) + d'(v) < \gamma$

Without loss of generality, we take,

$d'(u) \le d'(v)$ and u, v are such that $d'(u) + d'(v)$ is as large as possible.

$V(G) - \{u\}$ ~~Denote by~~ let S be the set of vertices in $V'$. $V/(v) \Rightarrow V - \{v\}$ which are non-adjacent to $V(G) - \{u\}$

and by T the set of vertices in $V/(u)$ which are non adjacent to u in $C(G)$. Clearly $u \in S$ and $v \in T$ G.

Again by the choice of u and v the degree of each vertex in S does not exceed $d'(u)$ and the ~~closure~~ degree of each vertex in T does not exceed $d'(v)$

Now the number of vertices in S and T are,

$|S| = \gamma - 1 - d'(v) \longrightarrow ①$

$|T| = \gamma - 1 - d'(u) \longrightarrow ②$

Since degree of any vertex of S does not exceed $d'(v)$, we find that there are $|S| = \gamma - 1 - d'(v)$ vertices whose degrees do not exceed $d'(u)$

In $T\cup\{u\}$ there are $|T|+1$ vertices not adjacent to u.

$|T|+1$ vertices have degrees not exceeding $d'(v)$

Take $d'(v) = m$

$|T| = \gamma - 1 - d'(v)$

$= \gamma - 1 - m$

$|T|+1 = \gamma - m \rightarrow (2)$

Thus there are $\gamma - m$ vertices in $C(G)$ whose degree do not exceed $d'(v)$

From (1) and (2) in $C(G)$, there are atleast m vertices of degree atmost m and atleast $\gamma - m$ vertices of degree $< \gamma - m$

$d'_m < d'(v)$

i.e., $d'_m < m$

Also, we have $d'(u) + d'(v) < \gamma$ and

$d'(u) \leq d'(v)$

$d'(u) + d'(v) < \gamma$

$2d'(u) < \gamma$

$2m < \gamma$

$m < \gamma/2$

Thus for $C(G)$ there exist an integer m such that $d'(m) < m$, $d'_{\gamma - m} < \gamma - m$ and $m < \gamma/2$

Since the graph G is a spanning subgraph of $C(G)$ the above result is ~~four~~ true for G also.

That is there exist an integer $m \leq \gamma/2$ such that $d_m < m$ and $d_{\gamma - m} \leq \gamma - m$. This is a contradiction to the given data. So our assumption that $C(G)$ is not complete is among.

Hence our claim that $C(G)$ is complete is among

$\therefore C(G)$ is Hamiltonian and hence G is hamiltonian

## Degree majorised sequence:-

A sequence of real numbers $(p_1, p_2, \dots p_n)$ is said to be majorised by another such sequence $(q_1, q_2, \dots q_n)$ $p_i \le q_i$ for $i \le j \le n$

A graph G is degree majorised by a graph H is $\delta(G) = \delta(H)$ and the non-decreasing degree sequence of G is majorised by that of H.

Ex: $\varepsilon(G) < \varepsilon(G')$

The 5 cycle is degree majorised by $K_{2,3}$ because $(2,2,2,2,2)$ is majorised by $(2,2,2,3,3)$

## Chvatal's graph:

Let m and n be two positive integer such that $1 \le m < n/2$. The chvatal graph is denoted by $C_{m,n}$ is defined as

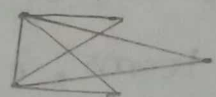
$$C_{m,n} = K_m \vee [K_m^c + K_{n-2m}]$$

$$\Rightarrow K_m \vee [K_m^c + K_{n-2m}]$$

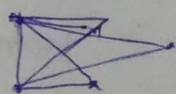
Ex:

$$C_{m,n} = K_m \vee [K_m^c + K_{n-2m}]$$

$$C_{2,5} = K_2 \vee [K_2^c + K_1]$$



$$C_2 = | \vee [\because]$$



$$C_{1,5} = K_1 \vee [K_1^c + K_3]$$

$$= \cdot \vee [\triangle]$$

G.

$$C_{1,5} = \cdot\!\!\triangle \qquad K_3 = \triangle$$

## Degree maximal graph:

A non Hamiltonian graph is said to be degree maximal it cannot be degree majorised by any other non-Hamiltonian simple graph having the same number of vertices.

Ex:

The chvatal's graph $C_{m,n}$ is degree maximal.

Theorem:

If $G$ is a non-Hamiltonian simple graph with $\nu \geq 3$ then $G$ is degree majorised by some $C_{m,\nu}$.

Proof:

Let $G$ be a non-Hamiltonian simple graph with degree sequence $(d_1, d_2, \ldots d_\nu)$ where $d_1 \leq d_2 \leq \ldots \leq d_\nu$ and $\nu \geq 3$.

Since $G$ is non-Hamiltonian by the theorem, there exists $m < \nu/2$ such that $d_m \leq m$ and $d_{\nu-m} < \nu - m$.

Now the first degree, sequence of $G$ can be taken as $(d_1, d_2, \ldots d_m, d_{m+1}, \ldots d_{\nu-m}, d_\nu)$

Here the first $m$ terms are each less than or equal ~~m for $d_m \leq m$~~ than $m$ for $d_m \leq m$. The next $\nu-2m$ terms are each $\leq \nu - m - 1$ for $d_{\nu-m} < \nu - m$

The last $m$ terms are each $\leq n-1$, for the maximum degree of any vertex is $\nu - 1$.

Thus the above degree sequence of $G$ is majorised by the degree sequence of $G$ is $(m, m, \ldots m)$ terms, $\nu-m-1, \nu-m-1, \ldots \nu-2m$ terms, $\nu-1, \nu-1, \ldots m$ terms of $C_{m,\nu}$.

Hence the given non-hamiltonian graph $G$ is degree majorised by $C_{m,\nu}$.

Corrollary:

If $G$ is a simple graph with $\nu \geq 3$ & $\epsilon > \binom{\nu-1}{2} + 1$ $\left(\frac{(\nu-1)(\nu-2)}{2}+1\right)$, then $G$ is H more over the only non-Hamiltonian simple graph with $\nu$ vertices and $\binom{\nu-1}{2} + 1$ edges are $C_{1,\nu}$ and for $\nu = 5$, $C_{2,5}$.

Proof:

Given that $G$ is a simple graph of $q$ vertices $q \geq 3$ and the number of edges $\varepsilon > \frac{(q-1)(q-2)}{2} + 1$

We have to prove that $G$ is hamiltonian

Suppose $G$ is non-hamiltonian then there exit $m < q/2$ such that $G$ is degree majorised by the chvatal graph $C_{m,q}$

$$\varepsilon(G) \geq \varepsilon(C_{m,q})$$

But,

$$\varepsilon(m,q) \leq \frac{(q-1)(q-2)}{2} + 1$$

$$\varepsilon(G) \leq \frac{(q-1)(q-2)}{2} + 1.$$

This is a contradiction to the hypothesis.

So our assumption that $G$ is non-hamiltonian is wrong.

Hence $G$ must be Hamiltonian.

(ii) Let $G$ be a non-hamiltonian graph of $q$ vertices $q \geq 3$ and $\binom{q-1}{2} + 1$ edges

$$\varepsilon(G) = \frac{(q-1)(q-2)}{2} + 1.$$

Since $G$ is non-hamiltonian there exit $m < q/2$, such that $G$ is degree majorised by $C_{m,q}$

$$\varepsilon(G) \leq \varepsilon(C_{m,q})$$

(i.e) $\frac{(q-1)(q-2)}{2} + 1 \leq \varepsilon(C_{m,q})$

But,

$$\varepsilon(C_{m,q}) \leq \frac{(q-1)(q-2)}{2} + 1$$

$$\therefore \varepsilon(C_{m,q}) = \frac{(q-1)(q-2)}{2} + 1$$

$$= \frac{q^2 - 3q + 4}{2}$$

We know that,

$$E(C_{m,r}) = \frac{1}{2}\left[m(r-1)+m^2+(r-2m)(r-m+1)\right]$$

$$\frac{1}{2}\, m(r-1)+m^2+(r-2m)(r-m+1) = \frac{r^2-3r+4}{2}$$

$$3m^2-2mr+m+2r-4=0.$$

$$(3m^2-3)+(2r-2mr)+(m-1)=0.$$

$$3(m+1)(m-1)-2r(m-1)+(m-1)=0$$

$$(m-1)(3m-2r+4)=0.$$

Either $m=1$ or $3m-2r+4=0$.

Take $m=1$ in this case the graph is $a, r$

Now consider,

$$3m-2r+4=0.$$

$$2r=3m+4.$$

$2m < \nu$

But $m < \frac{1}{2}$ $\boxed{\text{By } m < r/2}$ $[\because r \geq m]$

$$4m \leq 2r.$$

(i.e) $4m < 3m+4$

$$m<4.$$

$m < \nu/2$

$2m < \nu$

$4m < 3m+4$

We have, $3m+4=2r$

$3m+4 = \text{even}.$

$3m+4$ is even, and $m<4$ given,

$$m=2.$$

$$\therefore 3m+4=2r \Rightarrow r=5.$$

Thus for the graph $C_{m,r}$ we have $m=2$ & $r=5$ and the graph is $C_{2,5}$.

Hence the only non-hamiltonian graph with $n$ vertices are $C_{m,r}$ and $C_{2,5}$.

Theorem:

If a graph $G$ has a hamiltonian path then for every proper subsets of $v$, $W(G-S) \leq |S|+1$.

proof:

Let $P$ be a given hamiltonian path in the graph $G$ of $r$-vertices.

Let $u, v$ be the initial and terminal vertices of $p$. Take a new vertex $w$ and join it with $u$ and $v$.

Let $G'$ be the new graph, Thus obtain clearly $G$ has $r+1$ vertices and $PU(uw, wv)$ is an cycle.

So $G'$ is a hamiltonian graph of vertices.

Let $S' = [SU(w)]$ so that $|S|+1$.

Since $G'$ is a hamiltonian graph and $S'$ in a non-empty of the vertex set of $G'$ we have $W(G'-S') \leq |S'|$

Now, remove the new vertex $w$ so that new edges $u,w$ and $wv$ are also removed and we get the original graph $G$.

$$\therefore G'-S' = G-S.$$

$$W(G-S) = W(G'-S') \leq |S'| \leq |S|+1.$$

$$W(G-S) \leq |S'| \leq |S|+1$$

$$W(G-S) \leq |S|+1.$$

Hence the theorem.

Theorem:

Let $G$ be a non-trivial simple with degree sequence $(d_1, \ldots d_r)$ where $(d_1 \leq d_2 \leq \ldots \leq d_r)$ S.T if there is no value of $m \leq \left(\frac{r+1}{2}\right)$ for which $d_m < m$ and $d_{r-m+1} < r-m$ then $G$ has a hamiltonian path.

proof:

Given simple graph $G$ of $r$ vertices has the degree sequence $(d_1, d_2, \dots d_r)$ where $d_1 \le d_2 \le \dots \le d_r$

consider the graph $G'$ get by taking a new vertex $v$ and joining it to the vertices of $G$.

clearly $G$ has $r+1$ vertices and the degree sequence of $G'$ is $(d_1', d_2', d_3' \dots d_r', d_{r+1}')$

where,
$$d_1' = d_1+1$$
$$d_2' = d_2+1$$
$$\vdots$$
$$d_r' = d_r+1$$
$$d_{r+1}' = d_r.$$

clearly, $d_1' \le d_2' \le \dots \le d_r' \le d_{r+1}'$.

Given that there is no $m$ such that $m < \frac{r+1}{2}$ for which $d_m < m$ and $d_{r-m+1} < r-m$.

So if $m < \frac{r+1}{2}$. Then we have,
$$d_m \ge m$$
$$d_m + 1 \ge m+1$$
$$d_m' \ge m+1$$
$$d_m' > m.$$

Also,
$$d_{r-m+1} \ge r-m$$
$$d_{r-m+1} \ge r-m+1$$
$$d'_{(r+1)-m} \ge (r+1)-m$$

Thus there does not exist a value $m < \frac{r+1}{2}$ for which $d_m' \le m$ and $d'_{r+1} < m \le (r+1)$

(i.e) the simple graph $G'$ with $\gamma+1$ vertices and having degree sequences.

$(d_1', d_2' \ldots d_{\gamma+1}')$ with $d_1' \le d_2' \le \ldots \le d_{\gamma+1}'$ is such that there does not exist $m < \frac{\gamma+1}{2}$ for which $d_m' \le m$ and $d'_{(\gamma+1)-m} - m < (\gamma+1) - m$.

$\therefore$ By chvatal theorem $G'$ is a hamiltonian path.

Hence $G$ has a hamiltonian path.

Corrollary:

If $G$ is 2-connected, then any two vertices of $G$ lie on a common cycle.

proof:

Let $u$ & $v$ be any two vertices of a 2 connected graph $G$.

Since $G$ is 2-connected there are atleast two internal disjoint paths $P$ & $Q$ connecting $u$ & $v$.

$\therefore P \cup Q$ is a cycle in the graph $G$ and the vertices $u$ & $v$ lie on this cycle.

Hence the theorem.

Corrollary:

If $G$ is a block with $\gamma \ge 3$, then any two edges of $G$ lie on a common cycle.

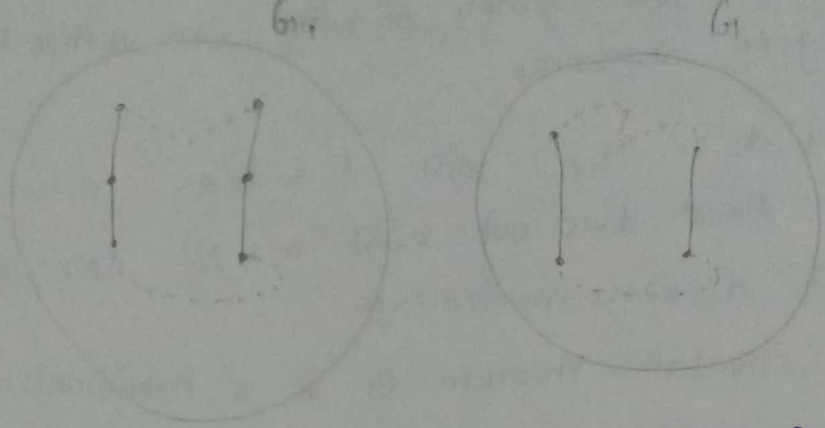
proof:

Let $G$ be a block with $\gamma \ge 3$ and let $e_1$ & $e_2$ be two edges of $G$.

Form a new graph $G'$ by subdividing $e_1$ & $e_2$ are denote the new vertices by $v_1$ & $v_2$.

Since $G$ is a block, $G'$ is also a block.

Again since $G$ has more than three vertices $G'$ has atleast five vertices.




Again since G' is a block it is 2-connected.

Then by theorem.

"If G is 2-connected, then any two vertices of G lie on a common cycle".

$\therefore$ The vertices $v_1$ & $v_2$ lie on a common cycle of G'.

$\therefore$ $e_1$ & $e_2$ lie on a common cycle of G.

**Theorem:**

If G is bipartite with bipartition (X, Y) where $|X| \neq |Y|$ then G is non-hamiltonian.

**Proof:**

If G is given (X, Y) is the partition of G so that $|X| = m$, $|Y| = n$.

Without loss of generality, we may assume $m < n$, if $K_{m,n}$ is hamiltonian.

Then,

$$w(G-X) \leq |X| \rightarrow (1)$$

Now,

$K_{m,n} - X$ is totally disconnected graph of n vertices

$w(G-X) = n$

$n > m$.

$w(G-X) > m = |X|$.

(i.e) $w(G-X) > |X|$.

This is a contradiction to (1).

$\therefore$ G is non-hamiltonian.

$\therefore$ G is non-hamiltonian.

## UNIT-III.

## Matchings.

Definitions :

Let G be a graph with edge set E. A subset M of E is called a matchings in G. If its elements are links and no two elements (edges) of M are adjacent in G.

The two ends of the edge in M are said to be matched under M. Thus two vertices of a graph G are matched under M. If the edge joining the vertices is a member of the matchings M.

The number of edges |M| is called the matching m.

M- Saturated :

A vertex v of G is said to be M saturated if some edge of M is incident on v.