$w(G-X) = n$

$n > m$.

$w(G-X) > m = |X|$.

(i.e) $w(G-X) > |X|$.

This is a contradiction to ①.

$\therefore$ G is non-hamilltonian.

$\therefore$ G is non-hamiltonian.

UNIT-III.

Matchings.

Definitions :

Let G be a graph with edge set E. A subset M of E is called a matchings in G. If its elements are links and no two elements (edges) of M are adjacent in G.

The two ends of the edge in M are said to be matched under M. Thus two vertices of a graph G are matched under M. If the edge joining the vertices is a member of the matchings M.

The number of edges |m| is called the matching m.

M- Saturated :

A vertex v of G is said to be M saturated if some edge of M is incident on v.

For the graph G,

$M = \{e_1, e_3, e_4\}$

v_5, v_6, v_4, v_3, v_1 & v_{10} are M saturated.

v_2 and v are M-saturated.

M-unsaturated:

A vertex v of G is said to M-unsaturated if no edge of M is incident on v.

perfect Matching:

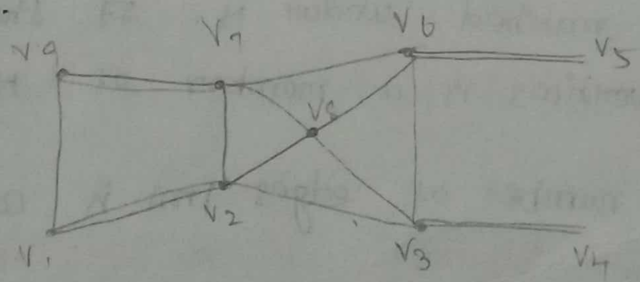
Let M be a matching in a graph G. If every vertex of G is M-saturated then the matching M is called perfect matchings.

Maximum matching:

A matching M of a graph G is called maximum matching if there is no other matching M of G with the property.

$|M|' > |M|$.

Ex:



Graph B.

M- alternating path:

Let M be a matching in G. An M alternating path in G is a path whose edge are alternating in E/M and M.

In an M- alternating path the initial edge is in M' and so the initial vertex of the path is M-saturated.

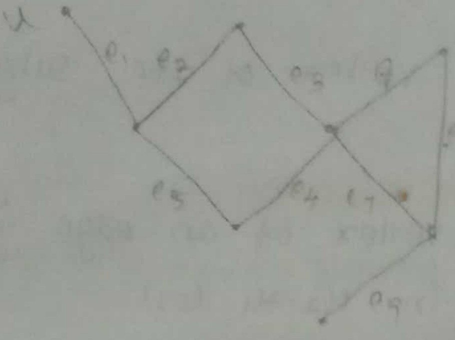
For ex:

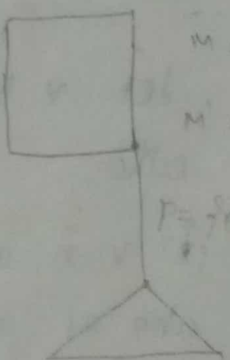
Graph(a) the path v_5, v_8, v_1, v_7, v_6 is an alternating path.

M-augumenting path:

An M-augumenting path in G is an M-alternating path whose origin and terminus are M-unsaturated.

Ex:





For the graph G, consider the matching

$M = \{e_2, e_7\}$

$M' = E/M = \{e_1, e_3, e_4, e_5, e_6, e_8, e_9\}$

The u-v path $P: e_1 e_2 e_3 e_7 e_9$ has the terminal edges $e_1, e_9 \in M$ and so the final and initial vertices of the path P are M-unsaturated.

Note:

The alternating path any two vertices in a graph need not be unique.

Result:

Let M_1 and M_2 be two matchings in a simple graph G. Let H be a subgraph of G induced by the set of edges M_1 and M_2

$\Rightarrow (M_1 - M_2) \cup (M_2 - M_1)$ is by the symmetric difference of the two matchings.

Then each connected component of H is of one of the following two types.

A cycle of even length whose edges are alternatively in M_1 and M_2.

A path whose edges are alternately in M_1 and M_2 and whose end vertices are unsaturated in one of the two matchings.

Proof:

Let v be any vertex of the subgraph H then either.

(i) v is an end vertex of an edge in $M_1 - M_2$ and also of an edge in $M_2 - M_1$ (or)

In either case, since M_1 is a matching, there is atmost one edge in M_1 with v as one of its end points and similarly there is atmost one edge in M_2 with v as one of its end point.

(ii) v is an end vertex of an edge in one of $M_1 - M_2$ and $M_2 - M_1$ but not both.

Thus in case (i), v has degree 2 in H, while in case (ii) v has degree 1 in H.

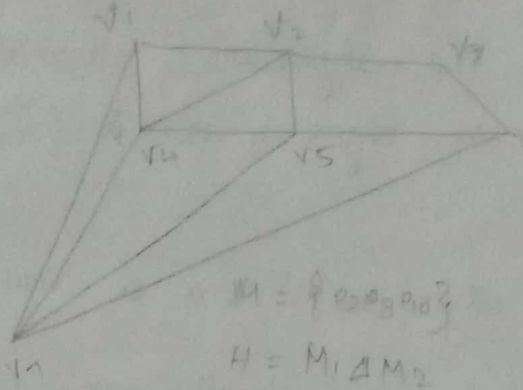
Hence every vertex of H has either degree 1 or degree 2.

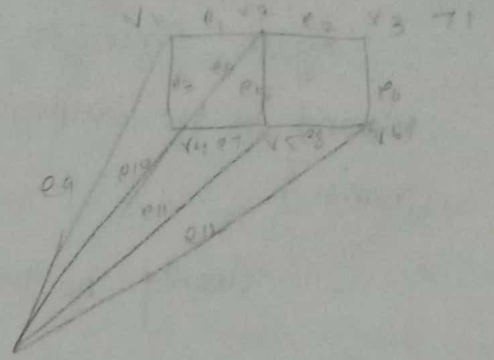
EX:

Consider the graph G. Let M_1 be the matching $\{e_2, e_8, e_{16}\}$ and take M_2 to be the matchings $\{e_1, e_7, e_{12}\}$

$M_1 \cup M_2 = \{e_2, e_8, e_{16}\} \cup \{e_1, e_7, e_{12}\}$

$= \{e_1, e_2, e_7, e_8, e_{12}, e_{16}\}$

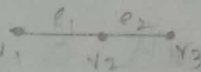


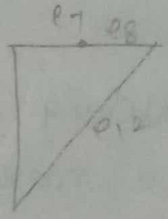


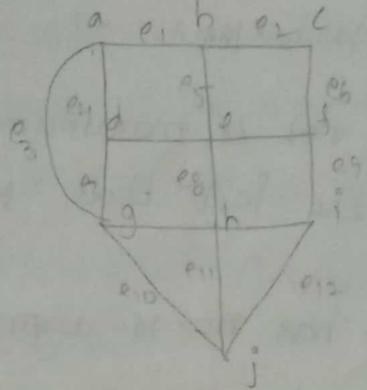
M = {e2 e8 e10}

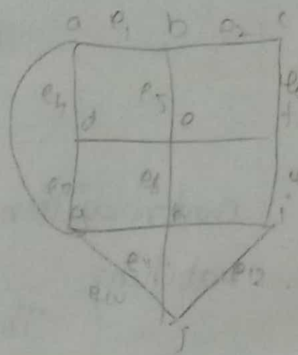
H = M1 Δ M2

e1, e2









10 mark 5M

(7) Theorem: [Berge's theorem]

A matching M in G is a maximum matching iff G contained no M-augmenting path.

V.V. IMP

U.Q Proof:

Let M be a maximum matching in G.

We have to prove that G contains no M-augumenting path.

Suppose that G contains an M-augumenting path P.

clearly P has an even number of vertices on an odd number of edges, which are alternatively from E/M and M.

Let $P : v_0 v_1 v_2 \ldots v_{2m}, v_{2m+1}$.

$\therefore$ P is an M-augumenting path the end vertices v_0 and v_{2m+1} are M-unsaturated.

So the edges v_0v_1. $v_{2m}v_{2m+1} \in M'$

Thus are 2m+1 edges in the path P. Since the first and the last are from M'. There are m+1 edge from M' and m edges ... with M and ...

Define ~~M' ⊆ E~~ by M.

$$M' = \left(M / \{v_1, v_2 \ldots v_{2m-1}, v_{2m}\} \cup \{v_0, v_1, \ldots v_{2m}, v_{2m+1}\}\right.$$

$|M| = m$ and $|M'| = m+1$, so $|M'| > |M|$

Again since in p the edges are alternatively from M' and M the sequence of edges in M' is $\{v_0v_1, v_2v_3, \ldots v_{2m}v_{2m+1}\}$.

we observe

Clearly no two edges of M' have a common vertex.

$\therefore$ The edges of M' are mutually non-adjacent and so M' is itself a matching.

~~And $|M'| = |M|+1$~~. $|M'| = |M|+1$

$\therefore$ M is not a maximum matching, which is a contradiction to the fact that M is a maximum matching.

Thus G has no M-augmenting path.

Conversely,

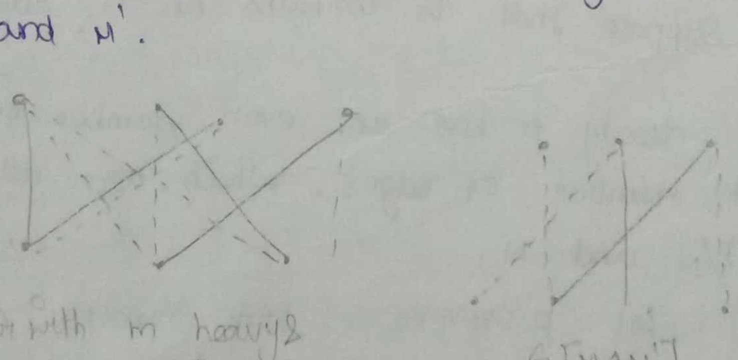
to prove

Suppose that M is not a ~~maximum~~ maximum matching, and let M' be a maximum matching in G.

Then $|M'| > |M| \longrightarrow (1)$.

consider the subgraph $H = G[M \Delta M']$.

When $M \Delta M'$ denoted the symmetric difference of M and M'.



$H = (M - M') \cup (M' - M)$

Since each vertex of H is incident with atleast one edge with M or M' and atmost two edges of one with M and another with M'. each vertex of H has degree either one or two.

we observe that each vertex of H has degree 1 or 2 for in H a vertex can be incident at most on an edge

Therefore the components of H are cycles or paths in which the edges alternate between M' and M. So the cycle component must be of even length. As M' contains more edges than M, there must be a path component p of odd length and it must start and end with edges from M'.

Thus each component of H is either an even cycle with edges alternatively in M and M' or ~~else~~ odd on path with edges alternatively in M and M'.

By (i) H contains more edges of M' than of M and therefore some path component p of H must start and end with edges of M'

Thus origin and terminus of P being M' saturated in H. So they are M-unsaturated in G.

Thus p is an M-augmenting path in G.

Hence the theorem.

Matchings and covering in bipartite graphs:

Definition : Neighbour set.

Let S be any subset of the vertex set of a graph G. Then the neighbour set of S in G is the set of all vertices adjacent to vertices in S. The set is denoted by $N_G(S)$

Theorem : [Hall's theorem].

Let G be a bipartite graph with bipartition (X, Y) then G contain a matching that saturates every vertex in X iff $|N(S)| \geq |S|$ $\forall$ $S \subseteq X$

Proof:

Given that G is a bipartite graph with bipartition (X, Y)

Suppose that G contains a matching M that saturates every vertex of X let S be a subset of X.

Since the vertices in S are matched under the M with distinct vertices in Y, we find that the vertices in the Neighbour set N(S), cannot be less than the number of vertices in S.

$N(S) \geq |S|$ $\forall S \subseteq X$

Conversely,

Suppose that G is a bipartite graph Satisfying $|N(S)| \geq |S|$ for every subset S of X.

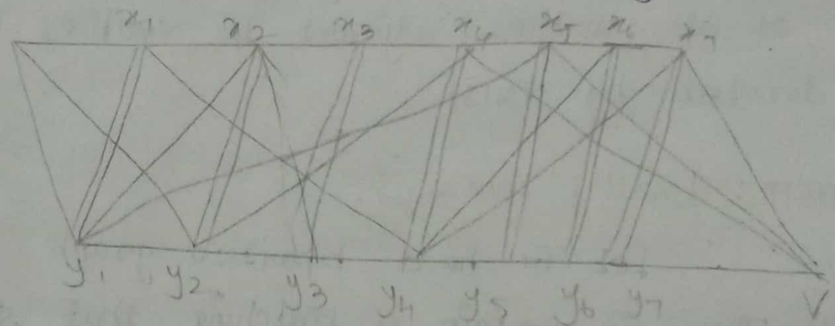
We have to prove that there is a matching which Saturates every vertex of X.

Suppose that G contains no matching saturating all the vertices of X.

Let M^* be a maximum matching in G.

By our assumption M^* does not saturate all the vertices in X.

Let u be an M^* unsaturated vertex, in X and. Let Z denote the set of all vertices which are connected to u by an M^* alternating path.



Since M^* is a maximum matching by Berge's theorem, G contains no M^*-augmenting path.

So the above M^*-alternating path is not an M^* augmenting path. So both end vertices of the path cannot be M^*-unsaturated

By our assumption, the initial vertex u is M^*-unsaturated.

$\therefore$ u is the only end vertex of the ~~path~~ path which is M^*-unsaturated. ~~v in X~~ ~~and~~ T ~~in the~~ the set of vertices ~~which is M^* unsaturated.~~

$\therefore$ S is the set of vertices in the path which are in X and T is the set of vertices in the path which are in Y

Take $S = Z \cap X$, $T = Z \cap Y$ ~~is the path which are in Y.~~

Clearly, the vertices in $S/\{u\}$ are matched under M^* ~~under~~ with the vertices in T.

$\therefore |T| = |S| - 1$.

But every vertex in $N(S)$ is connected u, by an M^* alternating path.

So $N(S) = T$.

Thus $|N(S)| = |T| = |S| - 1$

$\therefore |N(S)| < |S|$.

This is contradiction to our assumption.

Here if $|N(S)| \geq |S|$ there is a matching which saturate every vertex of X.

Hence the theorem.

Corollary:

If G is K-regular bipartite graph with $K>0$, then G has a perfect matching.

Proof:

Let G be a K-regular bipartite graph with bipartition (X,Y).

Since the graph G is K-regular every vertex is of degree K. Therefore the number of edges incident on the vertices of X in $K|X|$.

Similarly the number of edges on the vertices of Y is $K|Y|$.

$\therefore G$ is a bipartite graph with bipartition (X,Y) each edge G has one end vertex in X and the other in Y.

$\therefore K|X| = |E| = K|Y|$.

$\therefore |X| = |Y| \quad \because K>0$.

In order to prove that G has a perfect matching it is enough to prove that G has a matching M which saturates every vertex of X.

Let S be a subset of X and $N(S)$ be the neighbour set of S in G.

Let E_1 be the set of edges incident with vertices in S and E_2 be the set of edges incident with vertices in $N(S)$.

Then since $N(S)$ is the set of vertices which are joined by edges to S

We have, $E_1 \subseteq E_2$.

Thus, $|E_1| \leq |E_2|$

$k|S| \leq k|N(S)|$

$|S| \leq |N(S)|$

$|N(S)| \geq |S|$.

∴ The Hall's theorem there is a matching M which satuartes every vertex of X.

Again, since $|X| = |Y|$ every vertex of Y is also M saturated.

Thus, the matching M is a perfect matching in G.

Hence the corrollary.

Conversely,

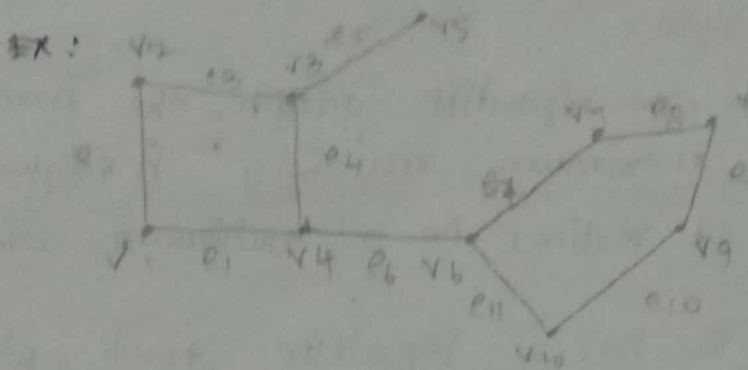
* Covering:

A covering of a graph G is a subset K of the vertex set V such that every edge of G has atleast one end ~~in K~~ vertex in K.

Minimum covering:

A covering K of a graph G is a minimum covering if there is no covering K' with the property $|K'| < |K|$.

Ex:



Lemma:

Let M be a matching and K be a covering such that $|M| = |K|$, then M is a maximum matching and K is a minimum covering

proof:

Given that M is a matching and K is a covering and $|M| = |K|$.

We have to p.t M is a maximum matching and K is minimum covering

Suppose let M^* be a maximum matching

$\therefore |M| \leq |M^*|$.

And let $\bar{K}$ be a minimum covering G.

$|\bar{K}| \leq |K|$.

For any graph G,

$|M^*| \leq |\bar{K}|$.

$|M| \leq |M^*| \leq |\bar{K}| \leq |K|$

$|M| = |K|$ it follows that,

$|M| = |M^*| = |\bar{K}| = |K|$.

Since $|M| = |M^*|$, M is a maximum matching.

Again since $|\bar{K}| = K$ K is a minimum covering

Thus if $|M| = |K|$, then M is a maximum matching and K is a minimum covering.

Hence the theorem.

KONIG's Theorem:

In a bipartite graph, the number of edges in a maximum matching is equal to the number of vertices in a minimum covering.

Proof:

Let G be a bipartite graph with bipartition (X, Y).

Let M^* be a maximum matching in G.

Let U be the set of M^* unsaturated vertices in X.

Let Z be the set of vertices ~~in~~ connected by an M^*, alternating paths, vertices of U take $S = Z \cap X$, & $T = Z \cap Y$.

Consider any vertex $u \in U$ we observe that the vertices in $S - \{u\}$ ~~subset~~ matched under M^* with the vertices of T.

$\therefore |T| = |S| - 1$

But every vertex in T is connected to u by an M^* alternating path and so $N(S) = T$.

Let us define $\bar{K} = (X/S) \cup T$. $(X - S) \cup T$

Every edge of a G must have atleast one of it's ends in $\bar{K}$. Clearly $\bar{K}$ is a covering of G.

For otherwise, there will be an edge with one end vertex in S and the other end vertex in Y/T.

This is a contradiction as $N(S) = T$.

If $\chi'(G) = k$, then G is said to be k edge chromatic.

Lemma:

Let G be a connected graph that is not an odd cyclic than G has a 2-edge clouring in which both represented at each vertex of degree atleast two.

Proof:

Assume that G is non-trivial. The given connected graph G is either Eulerian (or) non-Eulerian.

Case (i):

Suppose G is Eulerian.

a) Suppose G is a cycle. Since G is given to be not an odd cycle. Clearly G must an even cycle.

Therefore, there are an even number of edges which can be alternatively assigned the colours 1 & 2.

$\therefore$ G is 2-edge colourable.

So, the theorem is true in this case.

b) Suppose G is not a even cycle.

$\therefore$ G is Eulerian, every vertex is of even degree. Again G has a vertex v_0 of degree atleast four.

Let $v_0 e_1 v_1 \ldots e_k v_0$ be an Euler tours of G.

Let $E_1 = \{ e_i / i$ is odd $\}$ &

$E_2 = \{ e_i / i$ is even $\}$

Now, the edges of the graph can be coloured with two colours 1 & 2. Such that the edges in E_1 receive colour 1 and the edge in E_2 receive colour 2.

Thus (E_1, E_2) is a 2 edge. colouring of G.

Every vertex has edge of the form e_{2k+1} and e_{2k}.

So the colours 1 and 2 are represented at each vertex of the graph.

So, the theorem is true in this case also.

Case(ii) :

Suppose G is a non-Eulerian.

Since G is a non-Eulerian there are some odd degree vertices and the number of odd degree in G. vertices is even.

clearly. G^* is Eulerian

$\therefore$ As before the edge set of G^* can be partitioned into E_1^* and E_2^* such that (E_1^*, E_2^*) is a 2-edge clouring. of G^* in which both colours are represented at each vetex.

[Construct a new Graph G^* by adding a new vertex v_0 and joining it to each vertex to odd degree in G.]

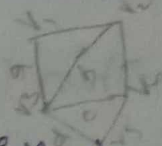
Now, $E \cap E_1^*$ and $E \cap E_2^*$ form a partition of the edge set E of the given graph G.

$\therefore$ $(E \cap E_1^*, E \cap E_2^*)$ is a 2-edge colouring of G having the required property.

Hence the theorem.

Improvement of a colouring:

[consider the k-edge colouring f' of G.
Then the number of distinct colours, represented at v is denoted by $c(v)$.



Ex: In G, a,b,c,d are the distinct colours represent at v then $c(v)=4$.)

In the edge colouring is proper then all the edges incident on v get different colours and so the number of colours represented at v is same as the number of edges incident on v.

$$c(v) = d(v)$$

Let f be an edge colouring of a graph G, so that $c(v)$ is the number of colours represented at v. Find the sum $\Sigma(cv)$ of the colours represented at all the vertices of G.

Let f' be an another colouring and find the sum of $\Sigma c'(v)$ of the colours represented at the vertices of G.

If $\Sigma c'(v) > \Sigma c(v)$ then the colouring b' is said to be an <u>improvement of the colouring</u> of b.

[An edge colouring b is said to be an <u>optimal edge colouring</u> if there is no improvement on b.

Clearly, a <u>proper edge colouring</u> is an optimal edge colouring.

Lemma :

Let $b = (E_1 E_2 \dots E_k)$ be an optimal k edges colouring of G, if there is a vertex colouring u in G. and colours i & j such that i is not represented at u & j is represented atleast twice at 'u' then the component of G $(E_i \cup E_j)$ that contains u is an odd cycle.

Proof :

Let $b = \{E_1 E_2 \dots E_k\}$ be an optimal k edge colouring of G.

Let u be a vertex of G. Such that the colour i is not represented at 'u' and the colour j is represented atleast twice at u.

E_i is the set of Edges of G getting the colours i and E_j is the set of edges of G getting the colour j.

H is given to be the component $E_i \cup E_j$ of G.

(i.e) $H = G[E_i \cup E_j]$.

We have to p.t H is an odd cycle containing u.

Suppose that H is not an odd cycle,

then by known lemma,

"H has a 2-edge colouring in which both colours are represented at each vertex of degree atleast two in H"!

~~when~~ we can recolour the edges of H with the colours i & j this way, we obtain a new k-edge colouring.

$b' = (E_1', E_2' \dots E_k')$ of G.

Denote by $c'(u)$ the number of distinct colours at u in the colouring b' then we have,

$c'(u) = c(u) + 1$.

Since now both the colours $i \& j$ are represented at u in $\mathcal{C}'$ and also $c'(v) \geq c(v)$ for $v \Rightarrow u$.

$\therefore \sum_{v \in V} c'(v) > \sum_{v \in V} c(v)$.

This shows that the colouring c' is an improvement of the colouring "c" of the graph G.

This is a contradiction as the colouring c is an optimal edge colouring of G.

So, our assumption that H is not an odd cycle is wrong.

Hence H must be an odd cycle.

Again for any vertex v in H of degree greater than or equal to 2, the two colours i and j are represented at v in c'' while atmost two colours i and j are represented at v in c'. $\therefore \sum c'(v) > \sum c(v)$, $c'(v) > c(v)$ $\forall v \in H$

Theorem 4.8:

If G is a bipartite, then $\chi' = \Delta$.

proof:

Let G be a bipartite graph with $\chi' > \Delta$

Let $b = \{E_1 \dots E_\Delta\}$ be an optimal Δ edge colouring of G. Then there must exist two adjacent ends of G which receive the same colour in b for otherwise b will be a proper edge colouring of G.

So, there exists a vertex u such that, the number of colour represented at u is less than the number of edges incident on it,

$\therefore c(u) < d(u)$.

Clearly, u satisfies the hypothesis that if there are two colours $i \& j$ in which 'i' is not represented at $u \& j$ is represented twice at u.

$\therefore$ G contains an odd cycle. [$\because$ By known theorem]

This is contradiction as G is a bipartite graph.

So our assumption that $\cancel{\varphi'} \chi' > \Delta$ is wrong.

$\therefore \cancel{\varphi'} \chi' \leq \Delta$. $\therefore \chi' \leq \Delta$

Again for any graph G,

$\cancel{\varphi'} \chi' \geq \Delta$, $\therefore \cancel{\varphi' = \Delta}$ $\chi' = \Delta$

$\chi' = \Delta$.

VIZING'S theorem:

If G is simple, then either $\cancel{\varphi'} \chi' = \Delta$ (or) $\cancel{\varphi' = \Delta + 1}$ $\chi' = \Delta + 1$.

proof:

By known theorem if G is bipartite then $\chi' = \Delta$

Let G be a simple graph and we know that $\cancel{\varphi' \geq \Delta}$ $\chi' \geq \Delta$.

We need only to S.T $\varphi' \not\leq \Delta + 1$. $\chi' \leq \Delta$

suppose $\cancel{\varphi'} \chi' > \Delta + 1$.

Let $\mathcal{C} = (E_1, E_2 \ldots E_{\Delta+1})$ be an optimal $(\Delta+1)$ ~~degree~~ edge colouring of G, and let u be vertex such that,

$$c(u) < d(u).$$

Then there exists colours i_0 and i_1 $\ni$: i_0 is not represented as u and i_1 is represented atleast twice at u.

Let uv_1 have colours i_1 as in the figure

Since, $d(v_1) < \Delta + 1$.

Some colour i_2 is not represented at v_1, Now i_2 must be represented at u.

Since otherwise by re-colouring at (u, v_1) with i_2 we would obtain an improvement on $\mathcal{C}$.

Thus some edge uv_2 has colour i_2.

Again since $d(v_2) < \Delta + 1$ some colour i_3 is not represented at v_2 & is must be represented at u.

Since otherwise, by recolouring uv_1 with i_2 and uv_2 with i_3 we could obtain an improved $(\Delta+1)$ edge colouring.

Thus some edge uv_3 has colour is containing this procedure we construct a sequence $v_1, v_2 \ldots$ of vertices and a sequence $i_1, i_2 \ldots$ of colour such that i, j.

(i) uv_j has colour i_j and,

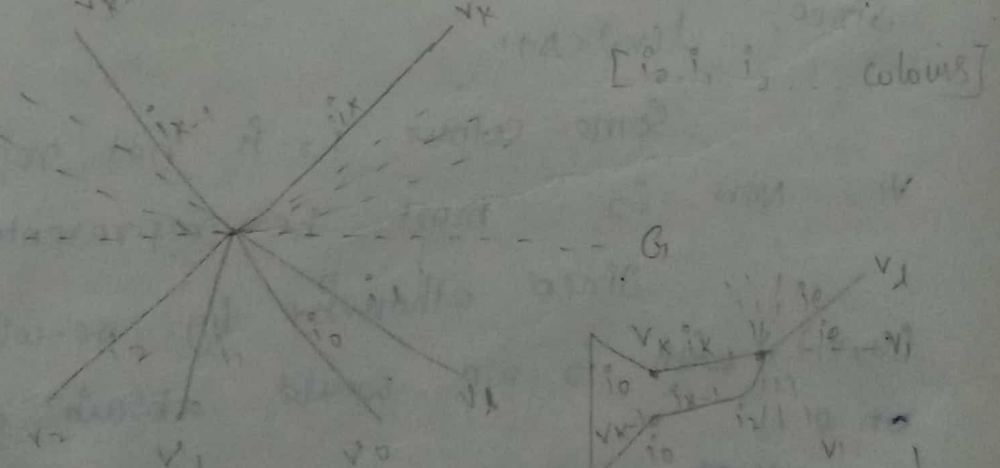
(ii) i_{j+1} is not represented at v_j.

Since the degree of u is finite there exists G the smallest integer l such that for some $k<l$.

(iii) $i^{e}_{\Delta+1} = i_k$.

The situation is repeated in the above figure we now recolour G as follows.

for $i \leq j \leq k-1$ recolours uv_j with colour i_{j+1} yielding a new $(\Delta+1)$ edge colouring $G = (E'_1 \cdots E'_{\Delta+1})$



Clearly $c'(v) \geq c(v)$ $\forall v \in v_1$ and therefore $\varphi^{e'}$ is also an optimal $(\Delta+1)$ edge colouring of G.

By known theorem, the component $H = G\{E_{i_0}' \cup E_{i_k}'\}$ that contains u is an odd cycle.

Now, in addition recolour uv_j with colour i_{j+1} for $k \le j < l-1$ and colour uv_l by i_k to obtain a $(\Delta+1)$-edge colouring

$\mathcal{C}'' \quad G'' = [E_1'', E_2'' \dots E_{\Delta+1}'']$

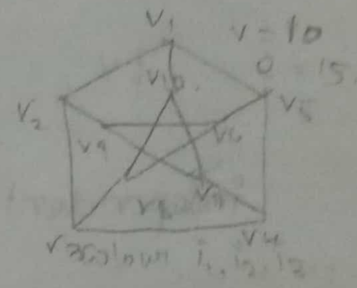
As above $c''(v) \ge c(v) \forall v \in V$

and the component H' of $G(E_{i_1}'' \cup E_{i_k}'')$ that contains u is an odd cycle.

But since v_k has degree two in H'.

v_k clearly v_k has degree one in H''.

which is a contradiction.



$\chi' \le \Delta+1$.

Then we have $\Delta \le \chi' \le \Delta+1$.

Hence, if G is simple, then either $\chi' = \Delta$ (or) $\chi' = \Delta+1$.

$\chi' \le \Delta+1$