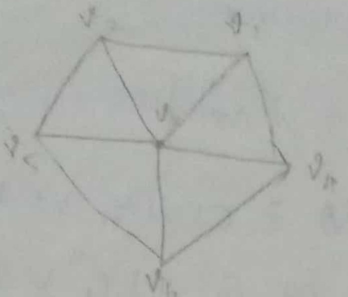
Independent set :

Let G be a graph with vertex V.
A subset S of V is called an independent set of G if no two vertices of S are adjacent in G.

Ex :



Independent sets of G

$S = \{v_2\}$, $S_1 = \{v_2, v_4\}$

$S_2 = \{v_1, v_5\}$

Maximum Independent set :

An independent set is maximum if G has no independent set S' with $|S'| > |S|$.

Ex :

For the above graph G, S_1 & S_2 are minimum independent sets.

Theorem :

A set of $S \subseteq V$ is an independent set of $G \Leftrightarrow V/S$ is a covering of G.

Proof :

Let S be an independent set.

$\therefore$ But the definition of S. no edge of G has both ends in S.

$\therefore$ Each edge of G has atleast one end in V/S

$\therefore$ V/S is a covering of G.

conversely,

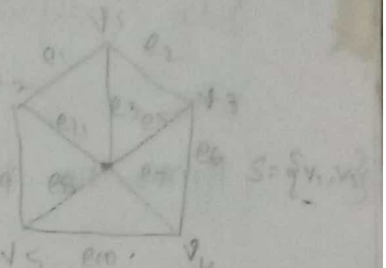
Let V/S be a covering of G.

Then each edge of G has atleast one end in V/S.

(i.e) No edge has both ends in S.

Thus, S is an independent set.

Hence the theorem.



$V-S = \{v_1, v_2, v_3, v_4\}$ → covering

Note :

2m The number of vertices in a maximum independent set of G is called the independent number of G and is denoted by, $\alpha(G)$ (or) α.

2m The number of vertices in a minimum covering of G is the covering number of G and is denoted by $\beta(G)$ or β.

Corrollary : 57.

For any graph G of p vertices $\alpha + \beta = p$.

Proof:

Let V be the vertices set of the graph G.

$\therefore |V| = p$.

Let S be a maximum independent set of G and let K be a minimum covering of G.

$\therefore |S| = \alpha$ and $|K| = \beta$.

$\therefore$ S is an independent set, then W.K.T V/S is a covering.

$\therefore |K| \leq |V/S|$. $\beta \leq p/\alpha$ $\beta \leq p - \alpha$

(i.e) $\beta \leq p - \alpha \Rightarrow \alpha + \beta \leq p \longrightarrow ①$.

Further, since 'K' is a covering V/K is an independent set of G.

Again, since S is a maximum independent set, we get $|S| \geq |V/K|$

(i.e) $\alpha \geq p - \beta \Rightarrow \alpha + \beta \geq p \longrightarrow ②$

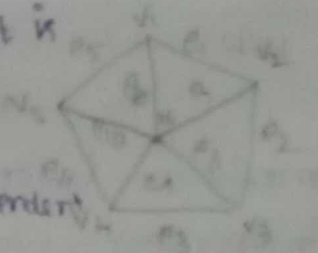
From ① & ② we get,

$\alpha + \beta = p$.

Edge independent set:

An edge independent set is a set of links no two of which are adjacent

An edge independent set is a matchings.



A maximum edge independent set is a maximum matching.

Edge independent number.

The number of edges in a maximum edge independent set is called the edge independent number and is denoted by $\alpha'(G)$.

Edge covering:

An edge covering of a graph G is a Subset L of E. Such that vertex of G is an end of some edge in L.

Minimum edge covering:

An edge covering L of a graph G is said to be the minimum edge covering if there does not exist another covering if there does not exist as $L' \Rightarrow |L'| < |L|$ the number of edges in a minimum edge covering is called the edge covering number and is denoted by $\beta'(G)$.

note:

The edge covering need not always exists it exists only, when $\delta > 0$

Theorem : 52 :

If $\delta > 0$ then $\alpha' + \beta' = \gamma$.

Proof:

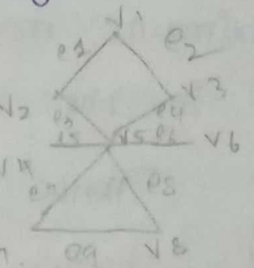
Since $\delta > 0$, there is atleast one edge incident on every vertex and so no edge is isolated.

Let M be a maximum matchings in G.

Let U be the set of all M-unsaturated vertices.

Since $\delta > 0$ and M is maximum there exists a set E' of $|U|$ edges are incident with each vertices in U.

Clearly, $M \cup E'$ is an edge covering of G and so.



$M = \{e_1, e_6, e_9\}$

$U = \{v_4, v_3\}$

$E' = \{e_3, e_5\}$

$M \cup E' = \{e_1, e_3, e_5, e_6, e_9\}$.

$$\beta' \leq |M \cup E'|$$
$$= \alpha' + (\gamma - 2\alpha')$$
$$= \gamma - \alpha'.$$

$$\beta' \leq \gamma - \alpha' \Rightarrow \alpha' + \beta' \leq \gamma \rightarrow (1).$$

Let L be a minimum edge covering of G set $H = G(L)$ and let M be a maximum-matching of H. Denote the set of M-unsaturated vertices in H by U. Since M is maximum $H(U)$ has no links and therefore $|L| - |M| \Rightarrow |L| + |M| \geq \gamma. \rightarrow (2)$.

Since H is a subgraph of G, and M is a matching in G we have,

$$\alpha' + \beta' \geq |M| + |L| \geq \gamma \rightarrow (3). \ [\because \text{using } (2)]$$

comparing (2) & (3) we get,

$$H = G(L)$$
$$|L| + |M| = |V|$$
$$|L| - |M| = \gamma - 2|M|$$
$$|L| - |M| + 2|M| = \gamma$$
$$|L| + |M| = \alpha' + \beta'$$
$$\alpha' + \beta' \geq |M| + |L| \geq \gamma$$
$$\alpha' + \beta' = \gamma.$$

Theorem 5.3:

In a bipartite graph G with $\delta > 0$, the number of vertices in a maximum independent set is equal to the number of edges in a minimum edge covering.

Proof:

Let G be a bipartite graph with $\delta > 0$

Since $\delta > 0$ we have from known theorem that,

$$\alpha + \beta = \alpha' + \beta' = \nu$$

$$\alpha + \beta = \alpha' + \beta'.$$

Since G is bipartite, the number of edges in maximum matching is equal to the number of vertices in the minimum covering.

Hence the theorem.

RAMSEY'S Theorem:

Definition: clique:

Let G be a simple graph and S be a subset of the vertex set V.

If G(S) is complete, then S is called a clique.

Note:

S is a clique of G iff S is an independent set of G^c, and so the two concepts are complement

Ramsey's Numbers:

For any two positive integers k & l, there exists a smallest integer $\gamma(k,l)$ such that every graph on $\gamma(k,l)$ vertices contains either a clique of k vertices or an independent set of l vertices. The number $\gamma(k,l)$ is defined to be the Ramsey's numbers.

Note :

It is evident that,

$r(1,l) \Rightarrow r(K,l) = 1.$

and $r(2,l) = (l), \ldots \quad r(K,2) = K.$

Ramsey's theorem : 54

For any two integers $K \geq 2$ and $l \geq 2$

$r(K,l) \leq r(K,l-1) + r(K-1,l) \quad \longrightarrow (1).$

Further more if $r(K,l-1)$ and $(K-1,l)$ are both even, then strictly inequality holds.

Proof :

If G is a simple graph of r vertices containing either a clique of K-vertices or an independent set of l vertices then the Ramsey numbers $r(K,l) = r$.

To prove the theorem, it is enough to prove a simple graph with

$r = r(K-1,l) + r(K,l-1)$ vertices contains either a clique of K-vertices or an independent set of "l" vertices.

Let G be a simple graph of r.

$r = r(K,l-1) + r(K-1,l)$ vertices contains either a clique of K-vertices or an independent set of "l" vertices.

Let G be a simple graph of r.

$r = r(K,l-1) + r(K-1,l)$ vertices, let v be any vertex of G.

Let us partition the vertices of G as S & T.

where S is the set of vertices non-adjacent to v, and T is the set of vertices adjacent to v.

Clearly $|S| + |T| = r-1$.

We claim that, either,

$|S| \geq r(k, l-1)$ (or) $|T| \geq r(k-1, l)$

Suppose not, then $|S| < r(k, l-1)$ &

$|T| < r(k-1, l)$

$\therefore |S| \leq r(k, l-1) - 1$ and $|T| \leq r(k-1, l) - 1$

$|S| + |T| \leq r(k-1, l) + r(k, l-1) - 1 - 1$

$\leq r - 1 - 1$

$|S| + |T| \leq r - 2.$

This is a contradiction as $|S| + |T| = r - 1$

Hence our claim that either

$|S| \geq r(k, l-1)$ (or) $|T| \geq r(k-1, l)$ hold

We distinguish two cases.

(i) v is non-adjacent to a set S of atleast $r(k, l-1)$ vertices.

(ii) v is adjacent to a set T of atleast $r(k-1, l)$ vertices.

Either, case (i) or case (ii) must hold because $|S| + |T| = r - 1$.

In case (i), $G(S)$ contains either a clique of k-vertices or an independent set of $(l-1)$ vertices and therefore $G\{S \cup \{u\}\}$ contains either a clique of k-vertices or an independent set of l-vertices.

In case (ii) $G(T)$ contains either a clique of $(k-1)$ vertices or an independent set of l-vertices and therefore $G\{T \cup \{v\}\}$ contains either a clique of k-vertices or an independent set of l-vertices.

This proves first part of the theorem.

Now, suppose that $r(k, l-1)$ and $r(k-1, l)$ are both even and let G be a graph on $r(k,l-1)+r(k-1,l)-1$ vertices.

Since G has an odd number of vertices it follows from a known corollary that some vertex v is of even degree.

In particular v cannot be adjacent to precisely $r(k-1, l)-1$ vertices.

Consequently either case(i) or case(ii) above holds are therefore G contains either a clique of k-vertices or an independent set of $l-1$ vertices.

Thus, $r(k, l) \leq r(k, l-1) + r(k-1, l)-1$

$r(k, l) \leq r(k, l-1) + r(k-1, l)$

Hence the theorem.

Finding Ramsey number:

The determination of Ramsey numbers is in general a very difficult unsolved problem. But we can find the lower bound of Ramseys number by constructing suitable graph.

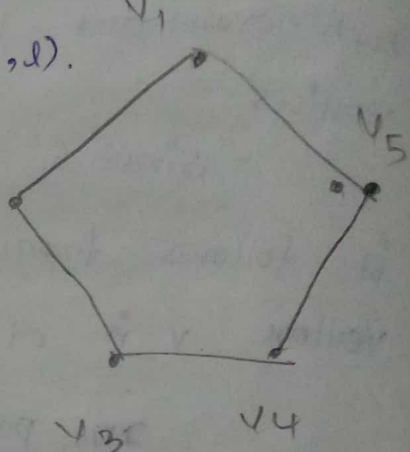
Ex: 1

Consider the 5-cycle.

It is clear that this contains no clique of ~~three~~ three vertices and no independent set of 3 vertices.

[i.e neither a clique of 3 vertices nor an independent set of 3 vertices] (i.e) $r(3,3) \geq 6 \longrightarrow ①$.

But by Ramsey's theorem,



$$r(k,l) \leq r(k,l-1) + r(k-1,l).$$

$$r(3,3) \leq r(3,2) + r(2,3)$$

$$\leq 3+3$$

$$r(3,3) \leq 6 \longrightarrow ②.$$

From ① & ② we get,

$$r(3,3) = 6.$$

Ex: 2.

From the graph it is evident that it has no 3 clique and a 5 independent set.

To get a graph with 3 clique or ~~3 clique and a 5 i~~ 3-independent set we should atleast add one vertex.

$$r(3,5) \geq 14 \longrightarrow ①.$$

But by, Ramsey's theorem,

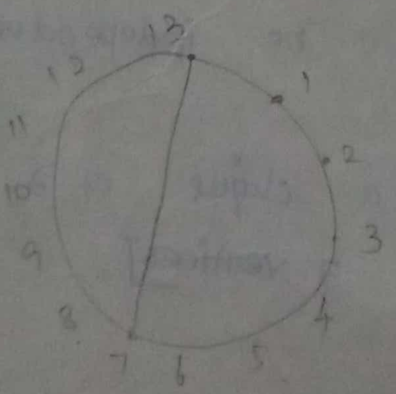
$$r(k,l) \leq r(k,l-1) + r(k-1,l)$$

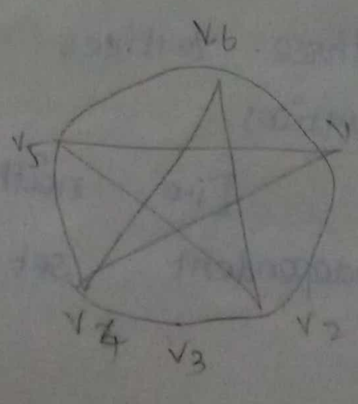
$$r(3,5) \leq r(3,4) + r(2,5)$$

$$\leq 9+5$$

$$r(3,5) \leq 14 \longrightarrow ②.$$

From ① & ② we get,





Ex: 3

$r(3,4) \geq 9 \rightarrow (1).$

But from Ramsey's theorem

$r(k,l) \leq r(k,l-1) + r(k-1,l)$

$r(3,4) \leq r(3,3) + r(2,2)$

$= 6+4$

$r(3,4) \leq 10 \rightarrow (2).$

From (1) & (2) We get,

$r(3,4) = 9.$

j/k	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7
3	1	3	6	9	14	18	23
4	1	4	9	18			