UNIT - V

# 9 Planar Graphs

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## 9.1 PLANE AND PLANAR GRAPHS

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A graph is said to be *embeddable in the plane*, or *planar*, if it can be drawn in the plane so that its edges intersect only at their ends. Such a drawing of a planar graph $G$ is called a *planar embedding of G*. A planar embedding $\tilde{G}$ of $G$ can itself be regarded as a graph isomorphic to $G$; the vertex set of $\tilde{G}$ is the set of points representing vertices of $G$, the edge set of $\tilde{G}$ is the set of lines representing edges of $G$, and a vertex of $\tilde{G}$ is incident with all the edges of $\tilde{G}$ that contain it. We therefore sometimes refer to a planar embedding of a planar graph as a *plane graph*. Figure 9.1*b* shows a planar embedding of the planar graph in figure 9.1*a*.

It is clear from the above definition that the study of planar graphs necessarily involves the topology of the plane. However, we shall not attempt here to be strictly rigorous in topological matters, and will be content to adopt a naive point of view toward them. This is done so as not to obscure the combinatorial aspect of the theory, which is our main interest.

The results of topology that are especially relevant in the study of planar graphs are those which deal with Jordan curves. (A 2m *Jordan curve* is a continuous non-self-intersecting curve whose origin and terminus coincide.) The union of the edges in a cycle of a plane graph constitutes a Jordan curve; this is the reason why properties of Jordan curves come into play in planar graph theory. We shall recall a well-known theorem about Jordan curves and use it to demonstrate the nonplanarity of $K_5$.

Let $J$ be a Jordan curve in the plane. Then the rest of the plane is partitioned into two disjoint open sets called the *interior* and *exterior* of $J$. We shall denote the interior and exterior of $J$, respectively, by int $J$ and ext $J$, and their closures by Int $J$ and Ext $J$. Clearly Int $J \cap$ Ext $J = J$. The *Jordan curve theorem* states that any line joining a point in int $J$ to a point in ext $J$ must meet $J$ in some point (see figure 9.2). Although this theorem is intuitively obvious, a formal proof of it is quite difficult.

*Theorem* 9.1 $K_5$ is nonplanar. university

*Proof* By contradiction. If possible let $G$ be a plane graph corresponding to $K_5$. Denote the vertices of $G$ by $v_1$, $v_2$, $v_3$, $v_4$ and $v_5$. Since $G$ is complete, any two of its vertices are joined by an edge. Now the cycle $C = v_1v_2v_3v_1$ is a Jordan curve in the plane, and the point $v_4$ must lie either in int $C$ or ext $C$.

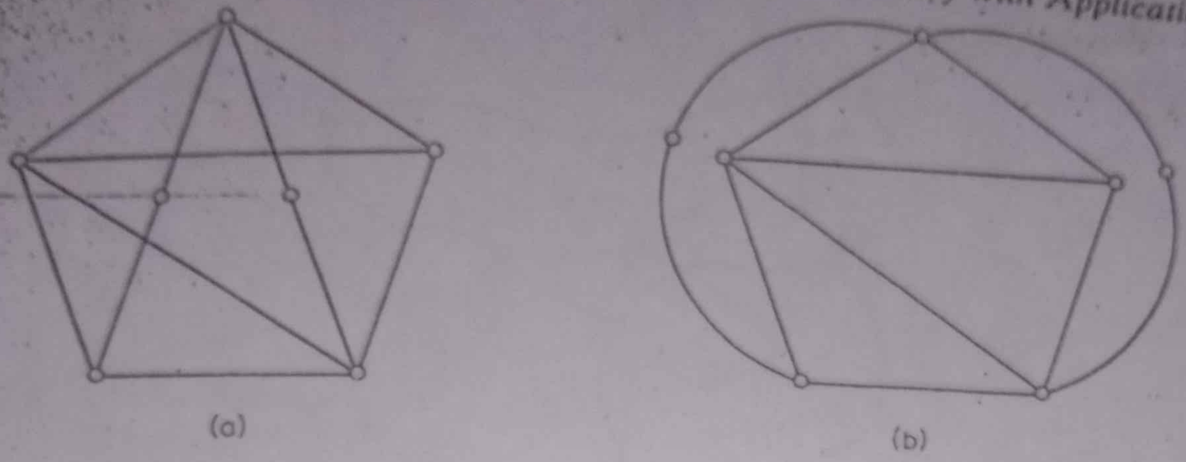

Figure 9.1. (*a*) A planar graph $G$; (*b*) a planar embedding of $G$

We shall suppose that $v_4 \in \text{int } C$. (The case where $v_4 \in \text{ext } C$ can be dealt with in a similar manner.) Then the edges $v_4v_1$, $v_4v_2$ and $v_4v_3$ divide int $C$ into the three regions int $C_1$, int $C_2$ and int $C_3$, where $C_1 = v_1v_4v_2v_1$, $C_2 = v_2v_4v_3v_2$ and $C_3 = v_3v_4v_1v_3$ (see figure 9.3).

Now $v_5$ must lie in one of the four regions ext $C$, int $C_1$, int $C_2$ and int $C_3$. If $v_5 \in \text{ext } C$ then, since $v_4 \in \text{int } C$, it follows from the Jordan curve theorem that the edge $v_4v_5$ must meet $C$ in some point. But this contradicts the assumption that $G$ is a plane graph. The cases $v_5 \in \text{int } C_i$, $i = 1, 2, 3$, can be disposed of in like manner □

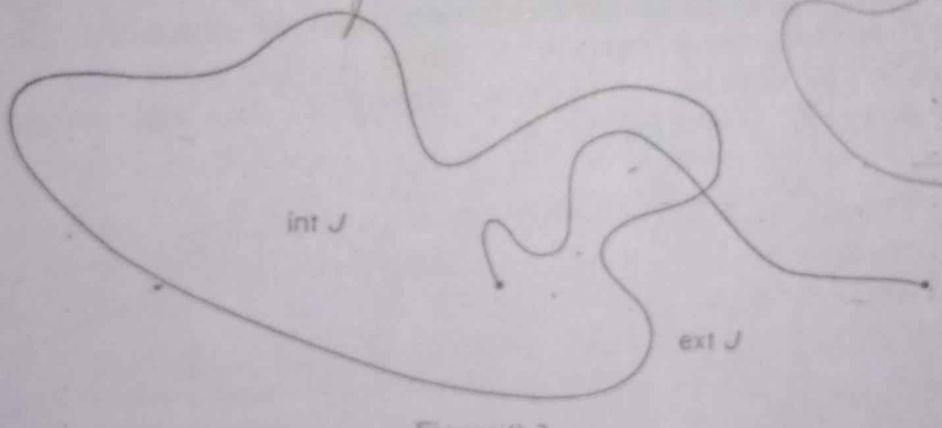


Figure 9.2

A similar argument can be used to establish that $K_{3,3}$, too, is nonplanar (exercise 9.1.1). We shall see in section 9.5 that, on the other hand, every nonplanar graph contains a subdivision of either $K_5$ or $K_{3,3}$.

The notion of a planar embedding extends to other surfaces.† A graph $G$ is said to be *embeddable* on a surface $S$ if it can be drawn in $S$ so that its

† A *surface* is a 2-dimensional manifold. Closed surfaces are divided into two classes, orientable and non-orientable. The sphere and the torus are examples of orientable surfaces; the projective plane and the Möbius band are non-orientable. For a detailed account of embeddings of graphs on surfaces the reader is referred to Fréchet and Fan (1967).

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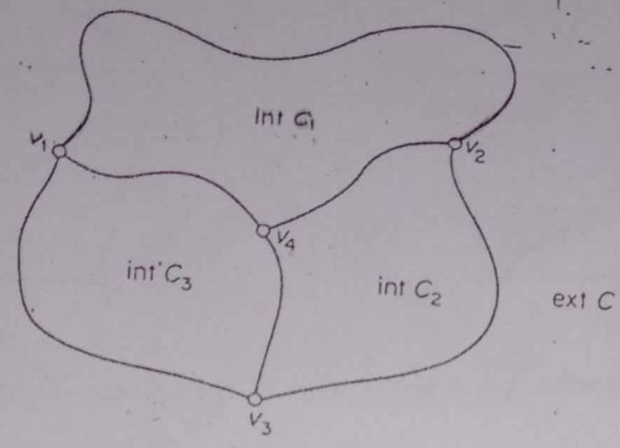


Figure 9.3

edges intersect only at their ends; such a drawing (if one exists) is called an *embedding* of $G$ on $S$. Figure 9.4$a$ shows an embedding of $K_5$ on the torus and figure 9.4$b$ an embedding of $K_{3,3}$ on the Möbius band. The torus is represented as a rectangle in which opposite sides are identified, and the Möbius band as a rectangle whose two ends are identified after one half-twist.

We have seen that not all graphs can be embedded in the plane; this is also true of other surfaces. It can be shown (see, for example, Fréchet and Fan, 1967) that, for every surface $S$, there exist graphs which are not embeddable on $S$. Every graph can, however, be 'embedded' in 3-dimensional space $\mathscr{R}^3$ (exercise 9.1.3).

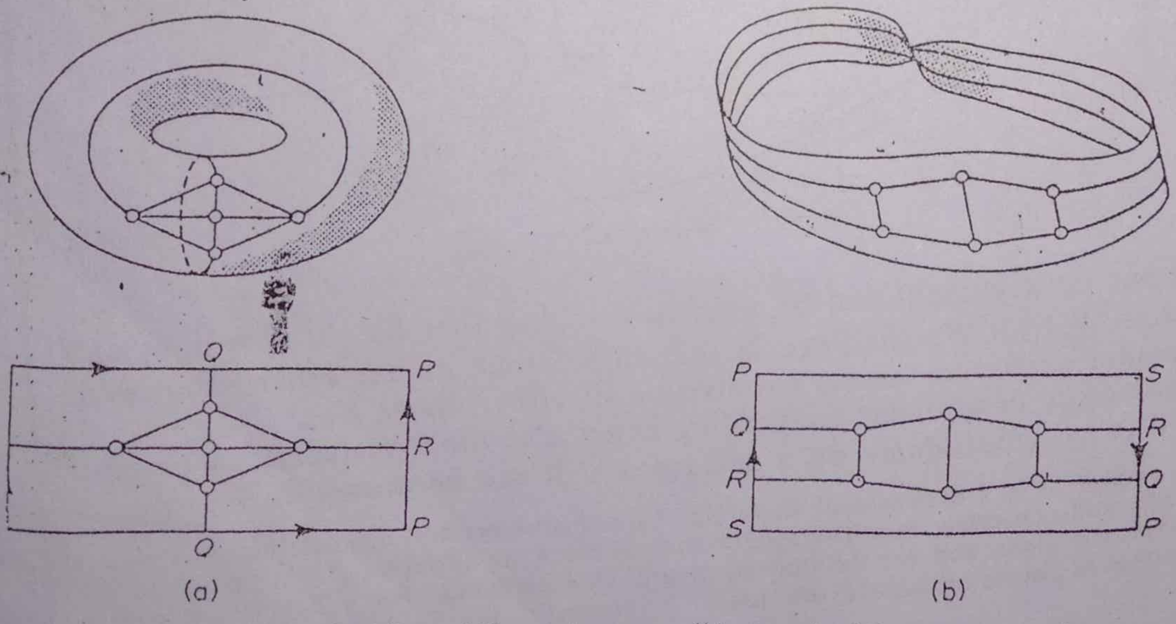


Figure 9.4. ($a$) An embedding of $K_5$ on the torus; ($b$) an embedding of $K_{3,3}$ on the Möbius band

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Planar graphs and graphs embeddable on the sphere are one and the same. To show this we make use of a mapping known as stereographic projection. Consider a sphere $S$ resting on a plane $P$, and denote by $z$ the point of $S$ that is diagonally opposite the point of contact of $S$ and $P$. The mapping $\pi : S\backslash\{z\} \to P$, defined by $\pi(s) = p$ if and only if the points $z$, $s$ and $p$ are collinear, is called *stereographic projection* from $z$; it is illustrated in figure 9.5.

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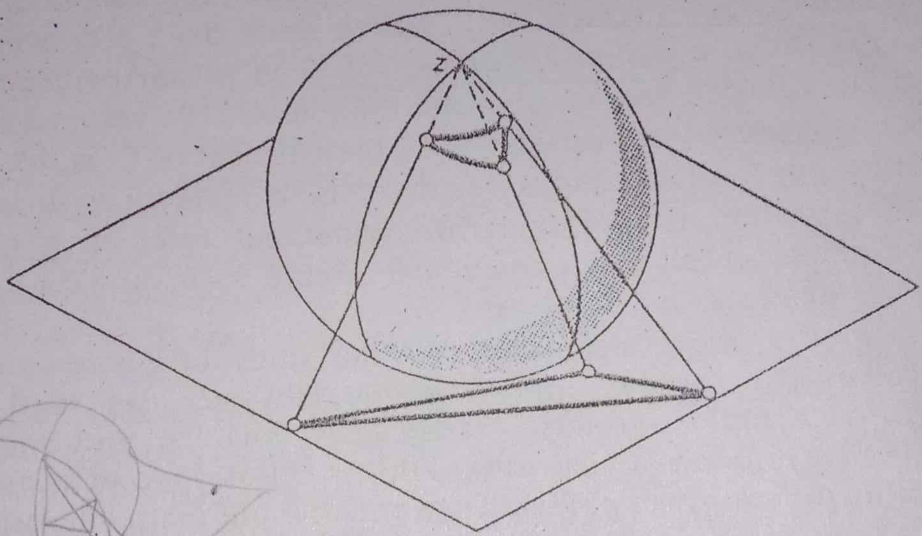



Figure 9.5. Stereographic projection

*Theorem 9.2* A graph $G$ is embeddable in the plane if and only if it is embeddable on the sphere. Note -84

*Proof* Suppose $G$ has an embedding $\tilde{G}$ on the sphere. Choose a point $z$ of the sphere not in $\tilde{G}$. Then the image of $\tilde{G}$ under stereographic projection from $z$ is an embedding of $G$ in the plane. The converse is proved converselly similarly □ If G' is a plane of a graph then keeping a sphere on the plane and selecting the pts z of its diagonally opposite to the point of contact of S.

On many occasions it is advantageous to consider embeddings of planar graphs on the sphere; one instance is provided by the proof of theorem 9.3 in the next section.

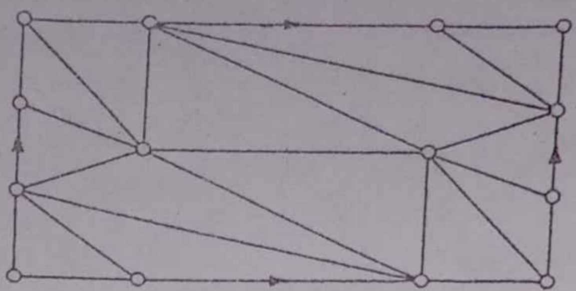
*Exercises*

**9.1.1** Show that $K_{3,3}$ is nonplanar.

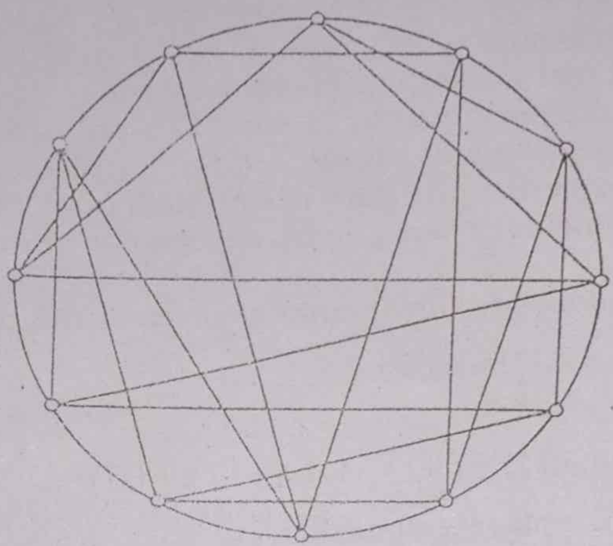
**9.1.2** (*a*) Show that $K_5 - e$ is planar for any edge $e$ of $K_5$.
(*b*) Show that $K_{3,3} - e$ is planar for any edge $e$ of $K_{3,3}$.

9.1.3 Show that all graphs are 'embeddable' in $\mathfrak{R}^3$.

9.1.4 Verify that the following is an embedding of $K_7$ on the torus:



9.1.5 Find a planar embedding of the following graph in which each edge is a straight line.
(Fáry, 1948 has proved that every simple planar graph has such an embedding.)



## 9.2 DUAL GRAPHS

A plane graph $G$ partitions the rest of the plane into a number of connected egions; the closures of these regions are called the *faces* of $G$. Figure 9.6 hows a plane graph with six faces, $f_1$, $f_2$, $f_3$, $f_4$, $f_5$ and $f_6$. The notion of a face pplies also to embeddings of graphs on other surfaces. We shall denote by $(G)$ and $\phi(G)$, respectively, the set of faces and the number of faces of a lane graph $G$.

Each plane graph has exactly one unbounded face, called the *exterior face:* the plane graph of figure 9.6, $f_1$ is the exterior face.

*heorem* 9.3 Let $v$ be a vertex of a planar graph $G$. Then $G$ can be mbedded in the plane in such a way that $v$ is on the exterior face of the mbedding.

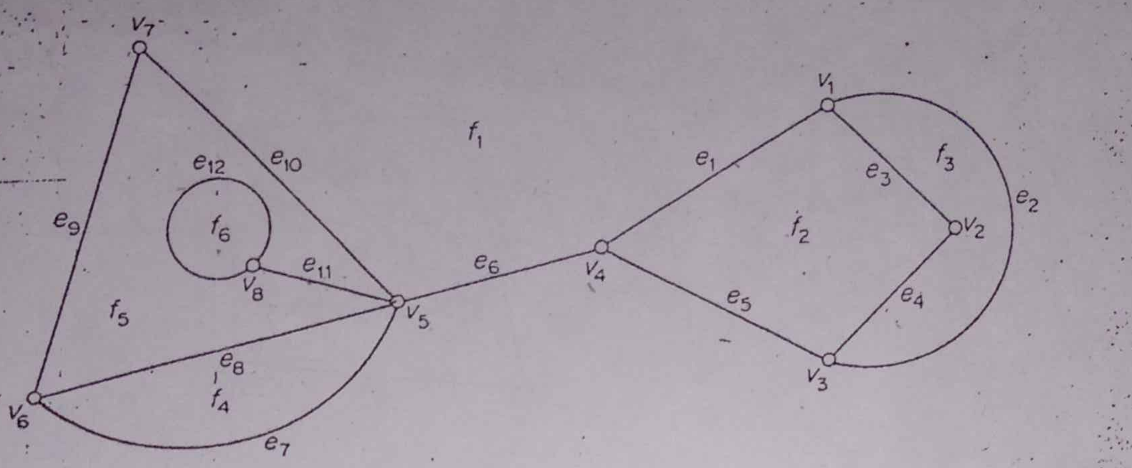



Figure 9.6. A plane graph with six faces

*Proof* Consider an embedding $\tilde{G}$ of $G$ on the sphere; such an embedding exists by virtue of theorem 9.2. Let $z$ be a point in the interior of some face containing $v$, and let $\pi(\tilde{G})$ be the image of $\tilde{G}$ under stereographic projection from $z$. Clearly $\pi(\tilde{G})$ is a planar embedding of $G$ of the desired type □

b(f)

We denote the boundary of a face $f$ of a plane graph $G$ by $b(f)$. If $G$ is connected, then $b(f)$ can be regarded as a closed walk in which each cut edge of $G$ in $b(f)$ is traversed twice; when $b(f)$ contains no cut edges, it is a cycle of $G$. For example, in the plane graph of figure 9.6,

$$b(f_2) = v_1e_3v_2e_4v_3e_5v_4e_1v_1$$

and

$$b(f_5) = v_7e_{10}v_5e_{11}v_8e_{12}v_8e_{11}v_5e_8v_6e_9v_7$$

A face $f$ is said to be *incident* with the vertices and edges in its boundary. If $e$ is a cut edge in a plane graph, just one face is incident with $e$; otherwise, there are two faces incident with $e$. We say that an edge *separates* the faces incident with it. The *degree*, $d_G(f)$, of a face $f$ is the number of edges with which it is incident (that is, the number of edges in $b(f)$), cut edges being counted twice. In figure 9.6, $f_1$ is incident with the vertices $v_1$, $v_3$, $v_4$, $v_5$, $v_6$, $v_7$ and the edges $e_1$, $e_2$, $e_5$, $e_6$, $e_7$, $e_9$, $e_{10}$; $e_1$ separates $f_1$ from $f_2$ and $e_{11}$ separates $f_5$ from $f_5$; $d(f_2) = 4$ and $d(f_5) = 6$.

Given a plane graph $G$, one can define another graph $G^*$ as follows: corresponding to each face $f$ of $G$ there is a vertex $f^*$ of $G^*$, and corresponding to each edge $e$ of $G$ there is an edge $e^*$ of $G^*$; two vertices $f^*$ and $g^*$ are joined by the edge $e^*$ in $G^*$ if and only if their corresponding faces $f$ and $g$ are separated by the edge $e$ in $G$. The graph $G^*$ is called the *dual* of $G$. A plane graph and its dual are shown in figures 9.7$a$ and 9.7$b$. It is easy to see that the dual $G^*$ of a plane graph $G$ is planar: in fact,

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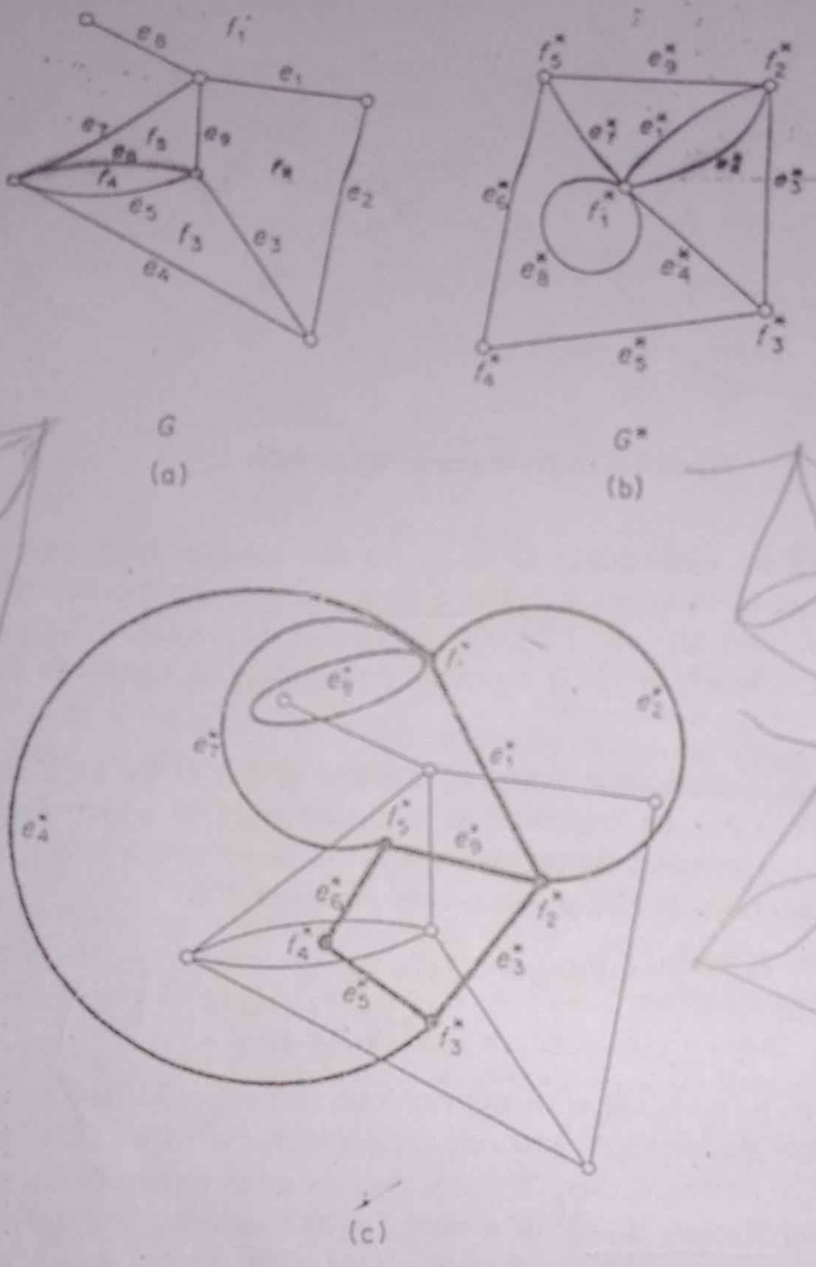


Figure 9.7. A plane graph and its dual

there is a natural way to embed $G^*$ in the plane. We place each vertex $f^*$ in the corresponding face $f$ of $G$, and then draw each edge $e^*$ in such a way that it crosses the corresponding edge $e$ of $G$ exactly once (and crosses no other edge of $G$). This procedure is illustrated in figure 9.7c, where the dual is indicated by heavy points and lines. It is intuitively clear that we can always draw the dual as a plane graph in this way, but we shall not prove this fact. Note that if $e$ is a loop of $G$, then $e^*$ is a cut edge of $G^*$, and vice versa.

Although defined abstractly, it is sometimes convenient to regard the dual

(a) (b)

Figure 9.8. Isomorphic plane graphs with nonisomorphic duals

$G^*$ of a plane graph $G$ as a plane graph (embedded as described above). One can then consider the dual $G^{**}$ of $G^*$, and it is not difficult to prove that, when $G$ is connected, $G^{**} \cong G$ (exercise 9.2.4); a glance at figure 9.7$c$ will indicate why this is so.

It should be noted that isomorphic plane graphs may have nonisomorphic duals. For example, the plane graphs in figure 9.8 are isomorphic, but their duals are not—the plane graph of figure 9.8$a$ has a face of degree five, whereas the plane graph of figure 9.8$b$ has no such face. Thus the notion of a dual is meaningful only for plane graphs, and cannot be extended to planar graphs in general.

The following relations are direct consequences of the definition of $G^*$:

$$\begin{aligned} \nu(G^*) &= \phi(G) \\ \varepsilon(G^*) &= \varepsilon(G) \qquad\qquad\qquad (9.1) \\ d_{G^*}(f^*) &= d_G(f) \quad \text{for all} \quad f \in F(G) \end{aligned}$$

4 5m

*Theorem 9.4* If $G$ is a plane graph, then

$$\sum_{f \in F} d(f) = 2\varepsilon$$

*Proof*.– Let $G^*$ be the dual of $G$. Then

$\Sigma v(u) = 2E$

$$\begin{aligned} \sum_{f \in F(G)} d(f) &= \sum_{f^* \in V(G^*)} d(f^*) && \text{by (9.1)} \\ &= 2\varepsilon(G^*) && \text{by theorem 1.1} \\ &= 2\varepsilon(G) && \text{by (9.1)} \quad \square \end{aligned}$$

## Exercises

**9.2.1** ($a$) Show that a graph is planar if and only if each of its blocks is planar.

($b$) Deduce that a minimal nonplanar graph is a simple block.

**9.2.2** A plane graph is *self-dual* if it is isomorphic to its dual.

($a$) Show that if $G$ is self-dual, then $\varepsilon = 2\nu - 2$.

($b$) For each $n \geq 4$, find a self-dual plane graph on $n$ vertices.

9.2.3 (*a*) Show that $B$ is a bond of a plane graph $G$ if and only if $\{e^* \in E(G^*) \mid e \in B\}$ is a cycle of $G^*$.

(*b*) Deduce that the dual of an eulerian plane graph is bipartite.

9.2.4 Let $G$ be a plane graph. Show that

(*a*) $G^{**} \cong G$ if and only if $G$ is connected;

(*b*) $\chi(G^{**}) = \chi(G)$.

9.2.5 Let $T$ be a spanning tree of a connected plane graph $G$, and let $E^* = \{e^* \in E(G^*) \mid e \notin E(T)\}$. Show that $T^* = G^*[E^*]$ is a spanning tree of $G^*$.

9.2.6 A *plane triangulation* is a plane graph in which each face has degree three. Show that every simple plane graph is a spanning subgraph of some simple plane triangulation ($\nu \geq 3$).

9.2.7 Let $G$ be a simple plane triangulation with $\nu \geq 4$. Show that $G^*$ is a simple 2-edge-connected 3-regular planar graph.

9.2.8* Show that any plane triangulation $G$ contains a bipartite subgraph with $2\varepsilon(G)/3$ edges.

(F. Harary, D. Matula)

## 9.3 EULER'S FORMULA

There is a simple formula relating the numbers of vertices, edges and faces n a connected plane graph. It is known as *Euler's formula* because Euler stablished it for those plane graphs defined by the vertices and edges of lyhedra.

**Theorem 9.5** If $G$ is a connected plane graph, then

$$\nu - \varepsilon + \phi = 2$$

*Proof* By induction on $\phi$, the number of faces of $G$. If $\phi = 1$, then each edge of $G$ is a cut edge and so $G$, being connected, is a tree. In this case $\varepsilon = \nu - 1$, by theorem 2.2, and the theorem clearly holds. Suppose that it is ue for all connected plane graphs with fewer than $n$ faces, and let $G$ be a onnected plane graph with $n \geq 2$ faces. Choose an edge $e$ of $G$ that is not a ut edge. Then $G - e$ is a connected plane graph and has $n - 1$ faces, since ie two faces of $G$ separated by $e$ combine to form one face of $G - e$. By the iduction hypothesis

If G is a tree, then $\varepsilon = \nu - 1$

$$\nu(G - e) - \varepsilon(G - e) + \phi(G - e) = 2$$

id, using the relations

$$\nu(G - e) = \nu(G) \qquad \varepsilon(G - e) = \varepsilon(G) - 1 \qquad \phi(G - e) = \phi(G) - 1$$

e obtain

$$\nu(G) - \varepsilon(G) + \phi(G) = 2$$

ie theorem follows by the principle of induction □

*Corollary 9.5.1* All planar embeddings of a given connected planar graph have the same number of faces.

*Proof* Let $G$ and $H$ be two planar embeddings of a given connected planar graph. Since $G \cong H$, $\nu(G) = \nu(H)$ and $\varepsilon(G) = \varepsilon(H)$. Applying theorem 9.5, we have $\nu(G) - \varepsilon(G) + \phi(G) = 2 \Rightarrow \phi(G) = 2 - \nu(G) + \varepsilon(G)$

$$\phi(G) = \varepsilon(G) - \nu(G) + 2 = \varepsilon(H) - \nu(H) + 2 = \phi(H) \quad \square$$

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*Corollary 9.5.2* If $G$ is a simple planar graph with $\nu \geq 3$, then $\varepsilon \leq 3\nu - 6$.

*Proof* It clearly suffices to prove this for connected graphs. Let $G$ be a simple connected graph with $\nu \geq 3$. Then $d(f) \geq 3$ for all $f \in F$, and

$$\sum_{f \in F} d(f) \geq 3\phi$$

By theorem 9.4

$$2\varepsilon \geq 3\phi$$

Thus, from theorem 9.5

$$\nu - \varepsilon + 2\varepsilon/3 \geq 2 \Rightarrow \nu - \varepsilon/3 \geq 2 \Rightarrow -\varepsilon/3 \geq 2 - \nu$$

or

$$\varepsilon \leq 3\nu - 6 \quad \square$$

*Corollary 9.5.3* If $G$ is a simple planar graph, then $\delta \leq 5$.

*Proof* This is trivial for $\nu = 1, 2$. If $\nu \geq 3$, then, by theorem 1.1 and corollary 9.5.2,

$$\delta\nu \leq \sum_{v \in V} d(v) = 2\varepsilon \leq 6\nu - 12 \leq 6 - \frac{12}{\nu} \leq 5$$

It follows that $\delta \leq 5$ $\square$

We have already seen that $K_5$ and $K_{3,3}$ are nonplanar (theorem 9.1 and exercise 9.1.1). Here, we shall derive these two results as corollaries of theorem 9.5.

USM

*Corollary 9.5.4* $K_5$ is nonplanar.

$\sum d(v) = 2(10) = 20 \Rightarrow 6 \leq 12$

*Proof* If $K_5$ were planar then, by corollary 9.5.2, we would have

$$10 = \varepsilon(K_5) \leq 3\nu(K_5) - 6 = 9$$

Thus $K_5$ must be nonplanar $\square$

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*Corollary 9.5.5* $K_{3,3}$ is nonplanar.

*Proof* Suppose that $K_{3,3}$ is planar and let $G$ be a planar embedding of $K_{3,3}$. Since $K_{3,3}$ has no cycles of length less than four, every face of $G$ must have degree at least four. Therefore, by theorem 9.4, we have

$$4\phi \leq \sum_{f\in F} d(f) = 2\varepsilon = 18$$

That is

$$\phi \leq 4$$

Theorem 9.5 now implies that

$$2 = \nu - \varepsilon + \phi \leq 6 - 9 + 4 = 1$$

which is absurd □

*Exercises*

9.3.1 (*a*) Show that if $G$ is a connected planar graph with girth $k \geq 3$, then $\varepsilon \leq k(\nu-2)/(k-2)$.

(*b*) Using (*a*), show that the Petersen graph is nonplanar.

**9.3.2** Show that every planar graph is 6-vertex-colourable.

9.3.3 (*a*) Show that if $G$ is a simple planar graph with $\nu \geq 11$, then $G^c$ is nonplanar.

(*b*) Find a simple planar graph $G$ with $\nu = 8$ such that $G^c$ is also planar.

9.3.4 The *thickness* $\theta(G)$ of $G$ is the minimum number of planar graphs whose union is $G$. (Thus $\theta(G) = 1$ if and only if $G$ is planar.)

(*a*) Show that $\theta(G) \geq \{\varepsilon/(3\nu-6)\}$.

(*b*) Deduce that $\theta(K_\nu) \geq \{\nu(\nu-1)/6(\nu-2)\}$ and show, using exercise 9.3.3*b*, that equality holds for all $\nu \leq 8$.

9.3.5 Use the result of exercise 9.2.5 to deduce Euler's formula.

9.3.6 Show that if $G$ is a plane triangulation, then $\varepsilon = 3\nu - 6$.

9.3.7 Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of $n \geq 3$ points in the plane such that the distance between any two points is at least one. Show that there are at most $3n-6$ pairs of points at distance exactly one.

## 9.4 BRIDGES

In the study of planar graphs, certain subgraphs, called bridges, play an important rôle. We shall discuss properties of these subgraphs in this section.

Let $H$ be a given subgraph of a graph $G$. We define a relation $\sim$ on $E(G)\backslash E(H)$ by the condition that $e_1 \sim e_2$ if there exists a walk $W$ such that

(i) the first and last edges of $W$ are $e_1$ and $e_2$, respectively, and

(ii) $W$ is internally-disjoint from $H$ (that is, no internal vertex of $W$ is a vertex of $H$).

It is easy to verify that $\sim$ is an equivalence relation on $E(G)\backslash E(H)$. A subgraph of $G - E(H)$ induced by an equivalence class under the relation $\sim$

is called a *bridge* of $H$ in $G$. It follows immediately from the definition that if $B$ is a bridge of $H$, then $B$ is a connected graph and, moreover, that any two vertices of $B$ are connected by a path that is internally-disjoint from $H$. It is also easy to see that two bridges of $H$ have no vertices in common except, possibly, for vertices of $H$. For a bridge $B$ of $H$, we write $V(B) \cap V(H) = V(B, H)$, and call the vertices in this set the *vertices of attachment* of $B$ to $H$. Figure 9.9 shows a variety of bridges of a cycle in a graph; edges of different bridges are represented by different kinds of lines.

In this section we are concerned with the study of bridges of a cycle $C$. Thus, to avoid repetition, we shall abbreviate 'bridge of $C$' to 'bridge' in the coming discussion; all bridges will be understood to be bridges of a given cycle $C$.

In a connected graph every bridge has at least one vertex of attachment, and in a block every bridge has at least two vertices of attachment. A bridge with $k$ vertices of attachment is called a $k$-*bridge*. Two $k$-bridges with the same vertices of attachment are *equivalent* $k$-bridges; for example, in figure 9.9, $B_1$ and $B_2$ are equivalent 3-bridges.

The vertices of attachment of a $k$-bridge $B$ with $k \geq 2$ effect a partition of $C$ into edge-disjoint paths, called the *segments* of $B$. Two bridges *avoid* one another if all the vertices of attachment of one bridge lie in a single segment of the other bridge; otherwise they *overlap*. In figure 9.9, $B_2$ and $B_3$ avoid one another, whereas $B_1$ and $B_2$ overlap. Two bridges $B$ and $B'$ are *skew* if there are four distinct vertices $u$, $v$, $u'$ and $v'$ of $C$ such that $u$ and $v$ are vertices of attachment of $B$, $u'$ and $v'$ are vertices of attachment of $B'$, and the four vertices appear in the cyclic order $u, u', v, v'$ on $C$. In figure 9.9, $B_3$ and $B_4$ are skew, but $B_1$ and $B_2$ are not.

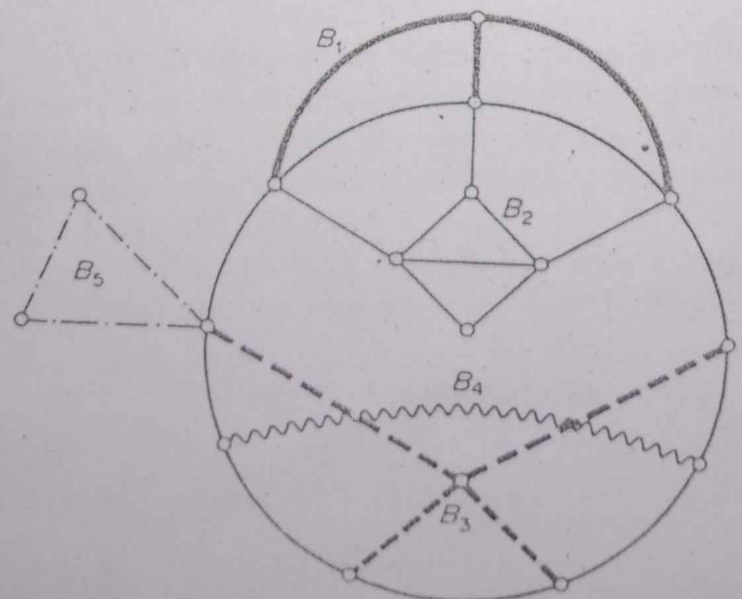


Figure 9.9. Bridges in a graph

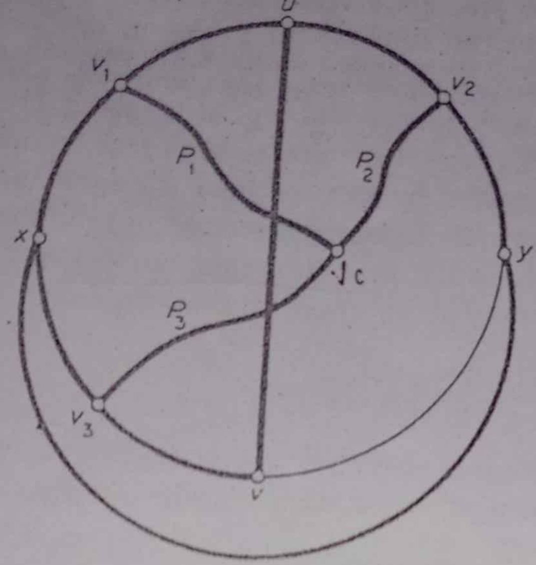


Figure 9.20

*Case 2*  $B$ has no vertex of attachment other than $u$, $v$, $x$ and $y$. Since $B$ is skew to both $uv$ and $xy$, it follows that $u$, $v$, $x$ and $y$ must all be vertices of attachment of $B$. Therefore (exercise 9.4.2) there exists a $(u, v)$-path $P$ and an $(x, y)$-path $Q$ in $B$ such that (i) $P$ and $Q$ are internally-disjoint from $C$, and (ii) $|V(P) \cap V(Q)| \geq 1$. We consider two subcases, depending on whether $P$ and $Q$ have one or more vertices in common.

*Case 2a*  $|V(P) \cap V(Q)| = 1$. In this case $(C \cup P \cup Q) + \{uv, xy\}$ is a subdivision of $K_5$ in $G$, again a contradiction (see figure 9.21).

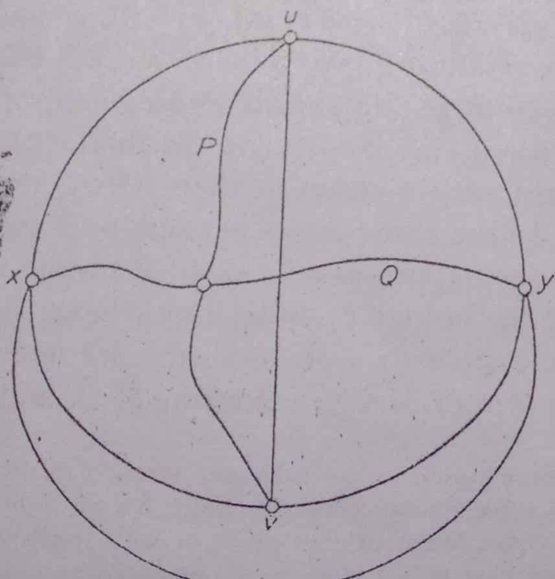


Figure 9.21

*Case 2b* $|V(P)\cap V(Q)|\geq 2$. Let $u'$ and $v'$ be the first and last vertices of $P$ on $Q$, and let $P_1$ and $P_2$ denote the $(u, u')$- and $(v', v)$-sections of $P$. Then $(C\cup P_1\cup P_2\cup Q)+\{uv, xy\}$ contains a subdivision of $K_{3,3}$ in $G$, once more a contradiction (see figure 9.22).

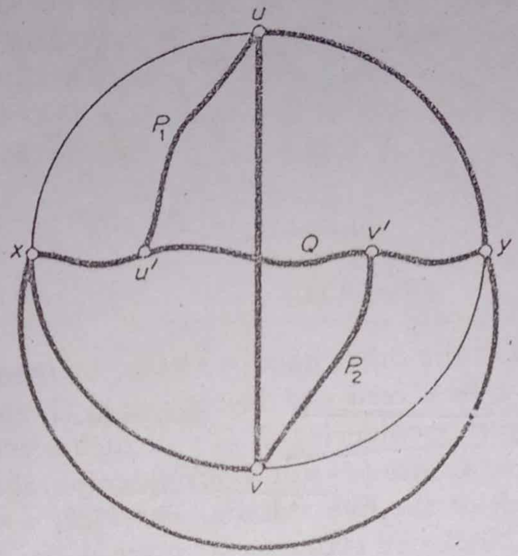


Figure 9.22

Thus all the possible cases lead to contradictions, and the proof is complete □

There are several other characterisations of planar graphs. For example, Wagner (1937) has shown that a graph is planar if and only if it contains no subgraph contractible to $K_5$ or $K_{3,3}$.

*Exercises*

9.5.1 Prove lemmas 9.10.1 and 9.10.2.

9.5.2 Show, using Kuratowski's theorem, that the Petersen graph is nonplanar.

## 9.6 THE FIVE-COLOUR THEOREM AND THE FOUR-COLOUR CONJECTURE

As has already been noted (exercise 9.3.2), every planar graph is 6-vertex-colourable. Heawood (1890) improved upon this result by showing that one can always properly colour the vertices of a planar graph with at most five colours. This is known as the *five-colour theorem*.

*Theorem 9.11* Every planar graph is 5-vertex-colourable.

*Proof* By contradiction. Suppose that the theorem is false. Then there exists a 6-critical plane graph $G$. Since a critical graph is simple, we see from

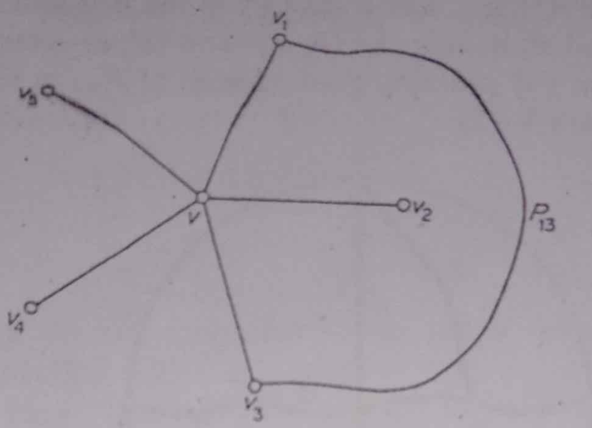


Figure 9.23

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corollary 9.5.3 that $\delta \leq 5$. On the other hand we have, by theorem 8.1, that $\delta \geq 5$. Therefore $\delta = 5$. Let $v$ be a vertex of degree five in $G$, and let $(V_1, V_2, V_3, V_4, V_5)$ be a proper 5-vertex colouring of $G - v$; such a colouring exists because $G$ is 6-critical. Since $G$ itself is not 5-vertex-colourable, $v$ must be adjacent to a vertex of each of the five colours. Therefore we can assume that the neighbours of $v$ in clockwise order about $v$ are $v_1$, $v_2$, $v_3$, $v_4$ and $v_5$, where $v_i \in V_i$ for $1 \leq i \leq 5$.

Denote by $G_{ij}$ the subgraph $G[V_i \cup V_j]$ induced by $V_i \cup V_j$. Now $v_i$ and $v_j$ must belong to the same component of $G_{ij}$. For, otherwise, consider the component of $G_{ij}$ that contains $v_i$. By interchanging the colours $i$ and $j$ in this component, we obtain a new proper 5-vertex colouring of $G - v$ in which only four colours (all but $i$) are assigned to the neighbours of $v$. We have already shown that this situation cannot arise. Therefore $v_i$ and $v_j$ must belong to the same component of $G_{ij}$. Let $P_{ij}$ be a $(v_i, v_j)$-path in $G_{ij}$, and let $C$ denote the cycle $vv_1P_{13}v_3v$ (see figure 9.23).

Since $C$ separates $v_2$ and $v_4$ (in figure 9.23, $v_2 \in \text{int } C$ and $v_4 \in \text{ext } C$), it follows from the Jordan curve theorem that the path $P_{24}$ must meet $C$ in some point. Because $G$ is a plane graph, this point must be a vertex. But this is impossible, since the vertices of $P_{24}$ have colours 2 and 4, whereas no vertex of $C$ has either of these colours. □

The question now arises as to whether the five-colour theorem is best possible. It has been conjectured that every planar graph is 4-vertex-colourable; this is known as the *four-colour conjecture*. The four-colour conjecture has remained unsettled for more than a century, despite many attempts by major mathematicians to solve it. If it were true, then it would, of course, be best possible because there do exist planar graphs which are not 3-vertex-colourable ($K_4$ is the simplest such graph). For a history of the four-colour conjecture, see Ore (1967).

The problem of deciding whether the four-colour conjecture is true or false is called the *four-colour problem*.† There are several problems in graph theory that are equivalent to the four-colour problem; one of these is the case $n = 5$ of Hadwiger's conjecture (see section 8.3). We now establish the equivalence of certain problems concerning edge and face colourings with the four-colour problem. A *k-face colouring* of a plane graph $G$ is an assignment of $k$ colours $1, 2, \ldots, k$ to the faces of $G$; the colouring is *proper* if no two faces that are separated by an edge have the same colour. $G$ is *k-face-colourable* if it has a proper $k$-face colouring, and the minimum $k$ for which $G$ is $k$-face-colourable is the *face chromatic number* of $G$, denoted by $\chi^*(G)$. It follows immediately from these definitions that, for any plane graph $G$ with dual $G^*$,

$$\chi^*(G) = \chi(G^*) \tag{9.2}$$

**Theorem 9.12** The following three statements are equivalent:

(i) every planar graph is 4-vertex-colourable;
(ii) every plane graph is 4-face-colourable;
(iii) every simple 2-edge-connected 3-regular planar graph is 3-edge-colourable.

*Proof* We shall show that (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(i).

let G* be a dual graph of G.

(a) (i)$\Rightarrow$(ii). This is a direct consequence of (9.2) and the fact that the dual of a plane graph is planar. No. of vertices of G = No. of faces of G*
G is 4-vertex colourable ⇒ G* is 4-face colourable

(b) (ii)$\Rightarrow$(iii). Suppose that (ii) holds, let $G$ be a simple 2-edge-connected 3-regular planar graph, and let $\tilde{G}$ be a planar embedding of $G$. By (ii), $\tilde{G}$ has a proper 4-face-colouring. It is, of course, immaterial which symbols are used as the 'colours', and in this case we shall denote the four colours by the vectors $c_0 = (0, 0)$, $c_1 = (1, 0)$, $c_2 = (0, 1)$ and $c_3 = (1, 1)$, over the field of integers modulo 2. We now obtain a 3-edge-colouring of $\tilde{G}$ by assigning to each edge the sum of the colours of the faces it separates (see figure 9.24). If $c_i$, $c_j$ and $c_k$ are the three colours assigned to the three faces incident with a vertex $v$, then $c_i + c_j$, $c_j + c_k$ and $c_k + c_i$ are the colours assigned to the three edges incident with $v$. Since $\tilde{G}$ is 2-edge-connected, each edge separates two distinct faces, and it follows that no edge is assigned the colour $c_0$ under this scheme. It is also clear that the three edges incident with a given vertex are assigned different colours. Thus we have a proper 3-edge-colouring of $\tilde{G}$, and hence of $G$.

† The four-colour problem is often posed in the following terms: can the countries of any map be coloured in four colours so that no two countries which have a common boundary are assigned the same colour? The equivalence of this problem with the four-colour problem follows from theorem 9.12 on observing that a map can be regarded as a plane graph with its countries as the faces.

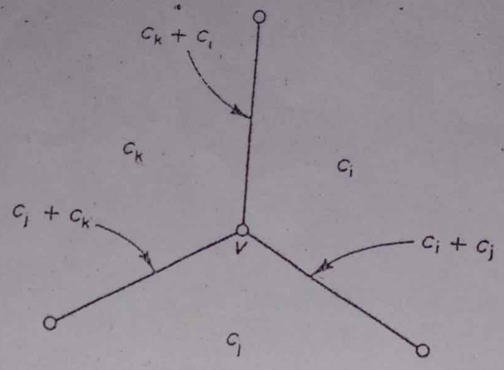



Figure 9.24

c) (iii)$\Rightarrow$(i). Suppose that (iii) holds, but that (i) does not. Then there is a 5-critical planar graph $G$. Let $\tilde{G}$ be a planar embedding of $G$. Then (exercise 9.2.6) $\tilde{G}$ is a spanning subgraph of a simple plane triangulation $H$. The dual $H^*$ of $H$ is a simple 2-edge-connected 3-regular planar graph (exercise 9.2.7). By (iii), $H^*$ has a proper 3-edge colouring $(E_1, E_2, E_3)$. For $i \neq j$, let $H^*_{ij}$ denote the subgraph of $H^*$ induced by $E_i \cup E_j$. Since each vertex of $H^*$ is incident with one edge of $E_i$ and one edge of $E_j$, $H^*_{ij}$ is a union of disjoint cycles and is therefore (exercise 9.6.1) 2-face-colourable. Now each face of $H^*$ is the intersection of a face of $H^*_{12}$ and a face of $H^*_{23}$. Given proper 2-face colourings of $H^*_{12}$ and $H^*_{23}$ we can obtain a 4-face colouring of $H^*$ by assigning to each face $f$ the pair of colours assigned to the faces whose intersection is $f$. Since $H^* = H^*_{12} \cup H^*_{23}$ it is easily verified that this 4-face colouring of $H^*$ is proper. Since $H$ is a supergraph of $G$ we have

$$5 = \chi(G) \leq \chi(H) = \chi^*(H^*) \leq 4$$

This contradiction shows that (i) does, in fact, hold $\square$

That statement (iii) of theorem 9.12 is equivalent to the four-colour roblem was first observed by Tait (1880). A proper 3-edge colouring of a -regular graph is often called a *Tait colouring*. In the next section we shall iscuss Tait's ill-fated approach to the four-colour conjecture. Grötzsch 958) has verified the four-colour conjecture for planar graphs without iangles. In fact, he has shown that every such graph is 3-vertex-colourable.

*xercises*

6.1 Show that a plane graph $G$ is 2-face-colourable if and only if $G$ is eulerian.

6.2 Show that a plane triangulation $G$ is 3-vertex colourable if and only if $G$ is eulerian.

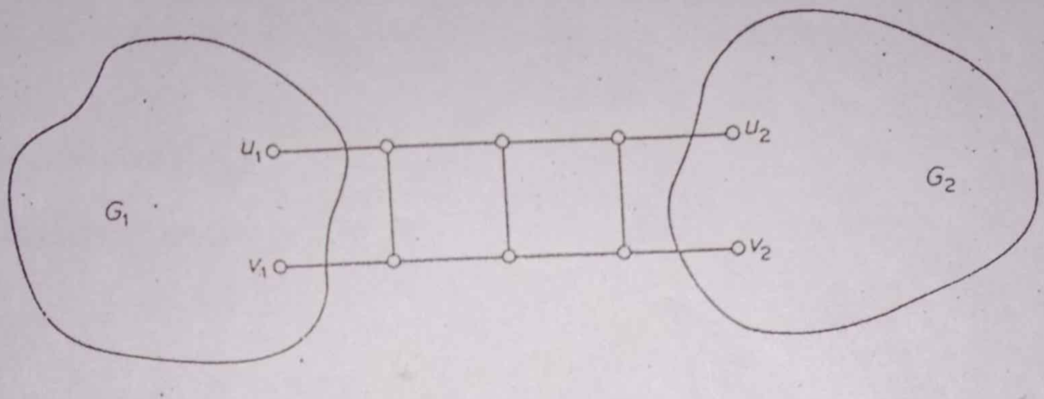
6.3 Show that every hamiltonian plane graph is 4-face-colourable.

6.4 Show that every hamiltonian 3-regular graph has a Tait colouring.

9.6.5 Prove theorem 9.12 by showing that (iii)$\Rightarrow$(ii)$\Rightarrow$(i)$\Rightarrow$(iii).

**9.6.6** Let $G$ be a 3-regular graph with $\kappa' = 2$.

(*a*) Show that there exist subgraphs $G_1$ and $G_2$ of $G$ and non-adjacent pairs of vertices $u_1, v_1 \in V(G_1)$ and $u_2, v_2 \in V(G_2)$ such that $G$ consists of the graphs $G_1$ and $G_2$ joined by a 'ladder' at the vertices $u_1$, $v_1$, $u_2$ and $v_2$:




(*b*) Show that if $G_1 + u_1v_1$ and $G_2 + u_2v_2$ both have Tait colourings, then so does $G$.

(*c*) Deduce, using theorem 9.12, that the four-colour conjecture is equivalent to *Tait's conjecture*: every simple 3-regular 3-connected planar graph has a Tait colouring.

9.6.7 Give an example of

(*a*) a 3-regular planar graph with no Tait colouring;

(*b*) a 3-regular 2-connected graph with no Tait colouring.

## 9.7 NONHAMILTONIAN PLANAR GRAPHS

In his attempt to prove the four-colour conjecture, Tait (1880) observed that it would be enough to show that every 3-regular 3-connected planar graph has a Tait colouring (exercise 9.6.6). By mistakenly assuming that every such graph is hamiltonian, he gave a 'proof' of the four-colour conjecture (see exercise 9.6.4). Over half a century later, Tutte (1946) showed Tait's proof to be invalid by constructing a nonhamiltonian 3-regular 3-connected planar graph; it is depicted in figure 9.25.

Tutte proved that his graph is nonhamiltonian by using ingenious *ad hoc* arguments (exercise 9.7.1), and for many years the Tutte graph was the only known example of a nonhamiltonian 3-regular 3-connected planar graph. However, Grinberg (1968) then discovered a necessary condition for a plane graph to be hamiltonian. His discovery has led to the construction of many nonhamiltonian planar graph