UNIT - I

Introduction

Outer Measure:

For each sets A of real numbers.

Consider the countable collection $\{I_n\}$ and open interval which cover A

That is collection for which $\underline{A \subset \cup I_n}$ and for each such collection. Consider the sum of the length of the interval in the collection.

Since length is '+ve' number, the number is uniquely defined independently of order of the terms.

[We define the outer measure of a set by $m^*(A) = \inf \sum l(I_n)$ where the infimum is taken over all finite or countable collection of intervals $\{I_n\}$ such that $A \subseteq \cup I_n$.]

I_1: $m^*(A) = \inf\{\sum l(I_n)\}$

I_2: $l[a,b] = l[a,b] = l[a,b) = l(a,b]$

$= l(a,b) = b-a$

I_3: $m^*(I_n)$ = length of I_n, where I_n is an interval.

I_4: For any sequence of subsets $\{E_i\}$

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i)$$

I_5: Equality occurs, where the above $\{E_i\}$ is disjoint.

I_6: A set whose outer measure is >0 is an $[0,1]$ uncountable set.

$m^*(A)=0$.

I_7: Every countable set has zero measure.

I_8: A set E is measurable $\Leftrightarrow m^*(A) = m^*(A \cap E) + m^*(A \cap CE)$

For every set A, where complement E.

I_9: If $m^*(E) = 0$, then E is measurable.

I_{10}: If E is measurable set, we write $m(E)$ in the place of $m^*(E)$

I_{11}: Lebesgue:

Let f be a extended real valued function defined on a measurable set E. Then f is a lebesgue measurable function (or) more briefly measurable function.

If for each $\alpha \in R$, this set collection of $\{x : f(x) > \alpha\}$ is measurable.

I_{12}: The following sets are equivalent.

i) f is measurable

ii) $\forall \alpha, \{x : f(x) < \alpha\}$ is measurable.

iii) $\forall \alpha, \{x : f(x) \geq \alpha\}$ is measurable.

iv) $\forall \alpha, \{x : f(x) \leq \alpha\}$ is measurable.

I_{13}: Characteristic function:

Let A be a subset of a measurable set E. The characteristic function φ_A of the set A is defined by,

$$\psi_A(x) = \begin{cases} 1 & ; \text{ if } x \in A \\ 0 & ; \text{ if } x \notin A \end{cases}$$

I_{14}: Almost everywhere:

If property holds expect on a set of measure zero. We say that it holds almost everywhere.

Theorem:

P.T i) $m^*(A) \geq 0$, (ii) $m^*(\phi) = 0$, (iii) $m^*(A) \leq m^*(B)$ if $A \subseteq B$ iv) $m^*([x]) = 0$ for any $x \in R$

Proof:

i) i) Let I_n be any interval, $A \subseteq \cup I_n$. $l(I_n) \geq 0$

(∴ the length of any interval is only +ve)

$\Rightarrow \Sigma\, l(I_n) \geq 0$

$= \inf \Sigma\, l(I_n) \geq 0 \Rightarrow m^*(A) \geq 0$.

ii) $\phi = I = [a, b)$

$m^*(\phi) = m^*(I) = l(b.a)$

$m^*(\phi) = l(a-a)$, where $b = a$.

$m^*(\phi) = 0$.

iii) $A \subset B$, ∃ a sequence $\{I_n\}$ & $\{I_n'\}$ ∋

$A \subseteq \cup I_n \subset B \in \cup I_n'$

$m^*(A) = \inf \Sigma\, l(I_n)$

$\leq \inf \Sigma\, l(I_n') = m^*(B)$

$m^*(A) \leq m^*(B)$.

iv) $\therefore x \in I_n = [x, x+1/n]$ for each x.

$m^*(x) = m^*[x, x+1/n]$ $(b-a)$

$= l[x, x+1/n]$ $l(I_n) = x + (1/n) - x = (1/n)$

$m^*[(x)] = \inf [\sum l(I_n)] = \inf_{n\to\infty} 1/n = 1/\infty = 0$

$m^*[(x)] = 0$, for any $x \in R$.

Ex:

S.T for any set $m^*(A) = m^*(A+x)$ where,

$A+x = \{y+x ; y \in A\}$ (or)

P.T the measure is translation invariant.

Soln:

For each $\epsilon > 0$ there exist a collection $\{I_n\}$ such that $A \subseteq \cup I_n$ &

$$m^*(A) \geq \sum l(I_n) - \epsilon \quad \text{—} (1)$$

Clearly,

$$A+x \subseteq \cup(I_n + x)$$

$$m^*(A+x) \leq \sum l(I_n + x)$$

$$\leq \sum l(I_n)$$

$$\leq m^*(A) + \epsilon \quad \text{by } (1)$$

Since ϵ is arbitrary.

$$m^*(A+x) \leq m^*(A) \quad \text{—} (2)$$

But $A = A + x - x$.

$$m^*(A) \leq m^*(A+x) - m^*(x)$$

$$\leq m^*(A+x) - 0$$

$$\leq m^*(A+x) \quad \text{———} (3)$$

Combining (2) & (3)

$$m^*(A+x) = m^*(A)$$

m^* is translation invariant.

Theorem:

P.T the outer measure of an interval equal it's length

Proof:

Case (i): suppose that

S.T I is $[a,b]$

Then for each $\epsilon > 0$ we have $m^*([a,b]) \leq$

cont-open $[\;)$ $m^*[a, b+\epsilon]$

$[\because A \subseteq B \Rightarrow m^*(A) \leq m^*(B)]$ closed: $+\epsilon$ open: $-\epsilon$

$$m^*([a,b]) \leq b+\epsilon-a$$

$$\leq b-a+\epsilon$$

[Since ϵ is arbitrary]

$$\leq b-a$$

$$m^*[I] \leq l(I) \quad \text{--- (1)}$$

for each n, let $I_n' = (a_n - \frac{\epsilon}{2^n}, b_n)$

Then $\bigcup_{n=1}^{\infty} I_n' \supseteq I$

By Heine Borel Theorem:

" A finite sub collection of the I_n', say $J_1, J_2, \ldots, J_n$ where $J_k = (c_k, d_k)$ cover I "

Then as we say suppose that no J_k is contained in any other we have supposing that,

$c_1 < c_2 < \cdots < c_N$.

$$d_N - c_1 = \sum_{k=1}^{N} (d_k - c_k) - \sum_{k=1}^{N-1} (d_k - c_{k+1})$$

$$d_N - c_1 \leq \sum_{k=1}^{N} l[J_k]$$

Consider, $m^*(I) = \inf \sum_{n=1}^{\infty} l(I_n)$

$$m^*(I) \geq \sum_{n=1}^{\infty} l(I_n) - \epsilon \quad \text{--- (2)}$$

Consider. $\sum_{n=1}^{\infty} l(I_n) = \sum_{n=1}^{\infty} (b_n - a_n)$

Where $I_n = [a_n, b_n]$

$$\sum_{n=1}^{\infty} l(I_n') = \sum_{n=1}^{\infty} b_n - a_n + \epsilon \sum_{n=1}^{\infty} \frac{1}{2^n}$$

$$= \sum_{n=1}^{\infty} b_n - a_n + \epsilon(1).$$

$\sum (b_n - a_n) + \epsilon\left[\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots\right]$

$\Rightarrow \epsilon\left(\frac{1}{2}\right)\left[1 + \frac{1}{2} + \frac{1}{2^2} + \dots\right]$

$= \epsilon\left(\frac{1}{2}\right)\left[1 - \frac{1}{2}\right]^{-1}$

$= \epsilon\left(\frac{1}{2}\right)(2)$

$= \epsilon(1)$

$$\sum_{n=1}^{\infty} l(I_n') - \epsilon = \sum_{n=1}^{\infty} l(I_n)$$

$$\sum_{n=1}^{\infty} l(I_n) = \sum_{n=1}^{\infty} l(I_n' - \epsilon) \quad \text{—— (A)}$$

Sub (A) in (2)

$$m^*(I) \geq \sum l(I_n')$$
$$\geq l(J_k) - 2\epsilon$$
$$\geq d_N - c_1 - 2\epsilon$$
$$\geq d(I) - 2\epsilon$$

$l(I)$

Since ϵ is arbitrary.

$$m^*(I) \geq l(I) \quad \text{—— (3)}$$

From (1) & (3)

$$m^*(I) = l(I).$$

Case (ii):

$(a, b]$

Suppose $I = [a, b]$ & $a > -\infty$.

If $a = b$ then $I = \phi$

$$m^*(I) = \inf \sum_{n=1}^{\infty} l(I_n)$$

$$= \inf \sum_{n=1}^{\infty} (b_n - a_n)$$

$$= 0$$

$$m^*(I) = l(I)$$

$m^*(a,b) = m^*(a,b]$

$= l(b-a)$

(a,b)

$= l(b-b)$

$m^* = 0$

If $a<b$, Suppose that $0 < \epsilon < (b-a)$ ①

$I' = [a+\epsilon, b]$.

$$m^*(I) \geq m^*(I') \quad [\because I' \subseteq I]$$
$$\geq m^*[a+\epsilon, b]$$
$$\geq l[a+\epsilon, b]$$

$$m^*(I) \geq l(b-a-\epsilon) = l(b-a)-\epsilon$$

$$m^*(I) \geq l(I) - \epsilon \quad — ④$$

But $I \subseteq I'' = [a, b+\epsilon]$

$$m^*(I) \leq m^*(I'')$$
$$\leq l(b+\epsilon-a)$$
$$\leq b-a+\epsilon$$
$$\leq l(I)+\epsilon \quad — ⑤$$

Combining ④ & ⑤

$$l(I) - \epsilon \leq m^*(I) \leq l(I)+\epsilon$$

Since ϵ is arbitrary

$$m^*(I) = l(I).$$

Similarly we prove that the case $I = (a,b)$ and $I = [a,b)$

Case (iii):

Suppose I is an infinite interval, four types of interval occurs $[-\infty, a]$, $[a, \infty]$, $(-\infty, \infty)$, $(-\infty, a]$.

Suppose $I = (-\infty, a]$

The other cases being similar.

For any $m>0$ there exist k such that the finite interval I_m, where $I_m = [k, k+m]$ is contained in I.

So $m^*(I) \geq m^*(I_m)$

$\geq l(I_m)$

$\geq k + m - k = m$

$m^*(I) = \infty = l(I)$.

Hence the theorem.

Theorem: Property of countable subadditivity

For any sequence of sets $\{E_i\}$: $m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i)$

Proof:

For each i and for any $\epsilon > 0$, there exist a sequence of intervals $\{I_{i,j}, j = 1, 2, \ldots\}$ such that $E_i \subseteq \bigcup_{j=1}^{\infty} I_{i,j}$ and

$$m^*(E_i) \geq \sum_{j=1}^{\infty} l(I_{i,j}) - \frac{\epsilon}{2^i}, \quad i = 1, 2, \ldots$$

$$\bigcup_{i=1}^{\infty} E_i \subseteq \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} I_{i,j}$$

(i.e) the set $[I_{i,j}]$ form a countable class covering $\bigcup_{i=1}^{\infty} E_i$.

$$\text{So, } m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} l(I_{i,j})$$

$$\leq \sum_{i=1}^{\infty} \left(m^*(E_i) + \frac{\epsilon}{2^i}\right)$$

$$\leq \sum_{i=1}^{\infty} m^*(E_i) + \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} \quad - (1)$$

$(\therefore \epsilon \cdot \frac{1}{2^i} = 1)$

$$m^*\left(\sum_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i) + \epsilon.$$

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i)$$

$(\because \epsilon$ is arbitrary$)$

Hence the theorem.

Example:

S.T for any set A & any $\epsilon > 0$ there is an open set 'O' containing A & such that $m^*(O) \leq m^*(A) + \epsilon$.

Soln:

Choose a sequence of intervals $\{I_n\}$ such that

$A \subseteq \bigcup_{n=1}^{\infty} I_n$ and $\sum_{n=1}^{\infty} l(I_n) - \epsilon/2 \leq m^*(A)$

$m^*(A) \geq \sum_{n=1}^{\infty} l(I_n) - \epsilon/2$

If $I_n = [a_n, b_n)$. semi-open

let $I_n' = (a_n - \frac{\epsilon}{2^{n+1}}, b_n)$

So that $A \subseteq \bigcup_{n=1}^{\infty} I_n'$.

Hence if put $O = \bigcup_{n=1}^{\infty} I_n'$. Countable union of open sets is open. 'O' is an open set and

$$m^*(O) = m^*\left(\bigcup_{n=1}^{\infty} I_n'\right)$$

$$\leq \sum_{n=1}^{\infty} m^*(I_n')$$

$$\leq \sum_{n=1}^{\infty} (l(I_n'))$$

$$m^*(O) \leq \sum_{n=1}^{\infty} \left(b_n - a_n + \frac{\epsilon}{2^{n+1}}\right)$$

$$\leq \sum_{n=1}^{\infty} (b_n - a_n) + \epsilon \sum_{n=1}^{\infty} \left(\frac{1}{2^{n+1}}\right)$$

$$\leq \sum_{n=1}^{\infty} (b_n - a_n) + \epsilon/2$$

$$\leq \sum_{n=1}^{\infty} l(I_n) + \epsilon/2$$

$$m^*(O) \leq m^*(A) + \epsilon/2 + \epsilon/2$$

$$\leq m^*(A) + \epsilon$$

$$\therefore m^*(O) \leq m^*(A) + \epsilon.$$

Problem:

1. S.T if $m^*(A) = 0$ then $m^*(A \cup B) = m^*(B)$ for any set B.

Soln:

Given $m^*(A) = 0$.

Consider $m^*(A \cup B) \leq m^*(A) + m^*(B)$

$m^*(A \cup B) \leq m^*(B)$ —— ① $\therefore (m^*(A) = 0)$

$\because B \subseteq A \cup B$.

$m^*(B) \leq m^*(A \cup B)$ —— ②

Combining ① & ②

$m^*(A \cup B) = m^*(B)$

2. S.T every countable set has measure zero.

Proof:

Let A be any countable set.

Let $A = \{a_1, a_2, \ldots\}$ say,

$A_i = \{a_i\} \ \forall i$

$A = \bigcup_{i=1}^{\infty} A_i$

$m^*(A) = m^*(\bigcup_{i=1}^{\infty} A_i)$

$\leq \sum_{i=1}^{\infty} m^*(A_i)$

$\leq \sum_{i=1}^{\infty} m^*(a_i)$ [$\because m^*(a_i) = 0$ for each i]

$m^*(A) \leq 0$.

$\therefore m^*(A) = 0$.

Note:

Since Q is a countable set $m(Q) = 0$.

Similarly $m^*(N) = 0$, $m^*(Z) = 0$.

$\therefore [0,1]$ is uncountable set.

$\therefore m^*[0,1] = 1$.

Measurable Set:

The set E is lebesgue measurable (or) measurable if for each set A we have

$m^*(A) = m^*(A \cap E) + m^*(A \cap CE)$.

where CE is the complement of the set E.

Example:

S.T if $m^*(E) = 0$ then E is measurable.

Proof:

Given $m^*(E) = 0$

By known Theorem: $A \subseteq B$, $m^*(A) \leq m^*(B)$

$\therefore A \cap E \subseteq E$

$m^*(A \cap E) \leq m^*(E)$

$m^*(A \cap E) = 0$ — (1)

$A \cap CE \subseteq A \Rightarrow m^*(A \cap CE) \leq m^*(A)$ — (2).

(1) + (2)

$m^*(A \cap E) + m^*(A \cap CE) \leq m^*(A)$ — (3)

Consider,

$A = (A \cap E) \cup (A \cap CE)$

$m^*(A) = m^*((A \cap E) \cup (A \cap CE))$

$m^*(A) \leq m^*(A \cap E) + m^*(A \cap CE)$ — (4).

Combining (3) & (4) we get,

$m^*(A) = m^*(A \cap E) + m^*(A \cap CE)$.

Hence by definition E is measurable.

Definition: [σ- algebra]

A class of subsets of an arbitrary space X is said to be σ- algebra (or) a σ-field, if X belongs to the class and the class is closed under the function unions and of complements.

M is a class of lebesgue measurable set.

Theorem:

The class of M is a σ - algebra.

Proof:

Since R is a countable and measurable

By definition (i.e) $R \in M$.

If $E \in M$ then $m^*(A) = m^*(A \cap E) + m^*(A \cap CE)$

$m^*(A) = m^*(A \cap CE) + m^*(A \cap CCE)$ (I)

$\therefore$ CE is measurable.

$\therefore$ $CE \in M$.

Hence $E \in M \Rightarrow CE \in M$.

It is enough to prove that $\{E_1, E_2 \ldots\} \in M$.

$\Rightarrow E = \bigcup_{i=1}^{\infty} E_i \in M$.

Let A be any arbitrary constant then we have,

$m^*(A) = m^*(A \cap E_1) + m^*(A \cap \overset{c}{E_1})$ — (1) (CE replace E_1)

and

$m^*(A \cap CE_1) = m^*(A \cap CE_1 \cap E_2) + m^*(A \cap CE_1 \cap CE_2)$ — (2)

(E_1 replacely E_2 & A replace by $A \cap CE_2$)

Sub this previous step, we get.

$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_2 \cap CE_1) + m^*(A \cap CE_1 \cap CE_2)$

Containing in this way, we obtain $n \geq 2$

$m^*(A) = m^*(A \cap E_1) + \sum_{i=2}^{n} m^*(A \cap E_i \cap \bigcap_{j<i} CE_j) +$

$m^*(A \cap \bigcap_{j=1}^{n} CE_j)$

$= m^*(A \cap E_1) + \sum_{i=2}^{n} m^*(A \cap E_i \cap C \bigcup_{j<i} E_j) + m^*(A \cap C \bigcup_{j=1}^{n} E_j)$

$\geq m^*(A \cap E_1) + \sum_{i=2}^{n} m^*(A \cap E_i \cap C \bigcup_{j<i} E_j) + m^*(A \cap C \bigcup_{j=1}^{\infty} E_j)$ — (1)

$\therefore \bigcup_{i=1}^{n} (E_i) = \bigcup_{i=1}^{n} (E_i \cap C \bigcup_{j<i} E_j)$

$$A \cap \bigcup_{i=1}^{\infty} E_i = A \cap \bigcup_{i=1}^{\infty} (E_i \cap C \bigcup_{j<1} E_j)$$

$$m^*(A \cap \bigcup_{i=1}^{\infty} E_i) = m^*(A \cap \bigcup_{i=1}^{\infty} (E_i \cap C \bigcup_{j<1} E_j)$$

$$\leq m^*(A \cap E_1) + m^*(A \cap \bigcup_{i=2}^{\infty} E_i \cap C \bigcup_{j<i} E_j)$$

$$\leq m^*(A \cap E_1) + \sum_{i=2}^{\infty} m^*(A \cap E_i \cap C \bigcup_{j<i} E_j) \quad — (2)$$

Sub (2) in (1) we get,

$$m^*(A) \geq m^*(A \cap \bigcup_{i=1}^{\infty} E_i) + m^*(A \cap C \bigcup_{i=1}^{\infty} E_i) \quad — (3)$$

$$m^*(A) = (A \cap \bigcup_{i=1}^{\infty} E_i) \cup (A \cap C \bigcup_{i=1}^{\infty} E_i)$$

$$m^*(A) \leq m^*(A \cap E_i) + m^*(A \cap C \bigcup_{i=1}^{\infty} E_i) \quad — (4)$$

Combining (3) & (4) we get,

$$m^*(A) = m^*(A \cap \bigcup_{i=1}^{\infty} E_i) + m^*(A \cap C \bigcup_{i=1}^{\infty} E_i)$$

$\Rightarrow \bigcup_{i=1}^{\infty} E_i$ is measurable.

$\sum_{i=1}^{\infty} E_i$ is belongs to M.

(i.e) $E_1, E_2, \ldots \in M$.

Hence the class M is σ-algebra.

Every interval is measurable.

Proof:

Suppose the interval to be of the form $[a, \infty)$.

For any set A we have to show that.

$$m^*(A) \geq m^*(A \cap (-\infty, a)) + m^*(A \cap [a, \infty)) \quad — (A)$$

Let,

$A_1 = A \cap (-\infty, a)$ and $A_2 = A \cap [a, \infty)$

We have to p.t,

$$m^*(A) \geq m^*(A_1) + m^*(A_2)$$

For any $\epsilon > 0$, there exist intervals I_n such that

$$A \subseteq \bigcup_{n=1}^{\infty} I_n \text{ and } m^*(A) \geq \sum_{n=1}^{\infty} l(I_n) - \epsilon \quad \text{--- (1)}$$

Write,

$$I_n' = I_n \cap (-\infty, a) \text{ and}$$

$$I_n'' = I_n \cap [a, \infty)$$

So that $\quad l(I_n) = l(I_n') + l(I_n'')$.

$$\therefore \quad A_1 \subseteq \bigcup_{n=1}^{\infty} I_n' \text{ and } A_2 \subseteq \bigcup_{n=1}^{\infty} I_n''.$$

$$m^*(A_1) \leq \sum_{n=1}^{\infty} l(I_n')$$

$$m^*(A_2) \leq \sum_{n=1}^{\infty} l(I_n'')$$

So, $\quad m^*(A_1) + m^*(A_2) = \sum_{n=1}^{\infty} l(I_n') + \sum_{n=1}^{\infty} l(I_n'')$

$$\leq \sum_{n=1}^{\infty} l(I_n)$$

$$m^*(A_1) + m^*(A_2) \leq m^*(A) + \epsilon \quad \text{(by (1))}$$

$$m^*(A_1) + m^*(A_2) \leq m^*(A) \quad [\text{since } \epsilon \text{ is arbitrary}]$$

$\therefore [a, \infty)$ is measurable.

$\Rightarrow \mathbb{R} \in \mathcal{M} \Rightarrow \mathbb{R} - [a, \infty) = (-\infty, a] \in \mathcal{M}$.

Using union, intersection and complement as defined in σ- algebra we prove that all kinds of intervals are measurable.

Hence every interval is measurable.

Ex 5 Show that if $F \in \mathcal{M}$ and $m^*(F \Delta G) = 0$ then G is measurable

Proof:-

We know that, $m^*(E) = 0$ then E is measurable.

Given, $m^*(F \Delta G) = 0$.

$\therefore$ $F \Delta G$ is measurable.

(i.e) $(F-G) \cup (G-F)$ is measurable.

$\because$ Subset of a measurable set is measurable.

$$G = (F \cap G) \cup (G-F)$$

$$= F-(F-G) \cup (G-F)$$

Complement of a measurable set is measurable.

$F-(F-G)$ is measurable.

Union of measurable set is measurable.

$F-(F-G) \cup (G-F)$ is measurable.

$\therefore$ Hence G is measurable.

Th: 7. Let $\mathcal{A}$ be a class of subsets of a space X. Then $\exists$ a smallest σ-algebra $\mathcal{S}$ containing $\mathcal{A}$. We say that $\mathcal{S}$ is the σ-algebra generated by $\mathcal{A}$.

Proof:

Let the sequence $[\mathcal{S}_\alpha]$ be any collection of σ-algebras of subsets of X containing $\mathcal{A}$.

$$\mathcal{S} = \bigcap_\alpha \mathcal{S}_\alpha.$$

(i) $X \in \mathcal{S}_\alpha$ for each $\alpha \Rightarrow X \in \bigcap_\alpha \mathcal{S}_\alpha$

$\Rightarrow X \in \mathcal{S}$.

(ii) $A_1, A_2, \dots, \in \mathcal{S} \Rightarrow A_1, A_2, \dots \in \mathcal{S}_\alpha \; \forall \alpha$.

$\Rightarrow \bigcap_{i=1}^{\infty} A_i \in \mathcal{S}_\alpha$ for each α.

$\Rightarrow \bigcap_{i=1}^{\infty} A_i \in \mathcal{S}$

(i.e) $A_1, A_2, \ldots \in S$

$$\bigcap_{i=1}^{\infty} A_i \in S.$$

$A \in S \Rightarrow A \in \bigcap_{\alpha} S_\alpha$

$\Rightarrow A \in S_\alpha$ for each α.

$\Rightarrow CA \in S_\alpha$ for each α.

$\Rightarrow CA \in \bigcap_{\alpha} S_\alpha$

$A \in S \Rightarrow CA \in S$.

S is the smallest σ-algebra containing $\mathcal{A}$, namely the class of all subsets of X.

So taking the intersection of the σ-algebra containing $\mathcal{A}$ we get a σ-algebra, necessarily the smallest, containing $\mathcal{A}$.

Ex: 6. For any set A $\exists$ a measurable set E containing A and such that $m^*(A) = m(E)$.

Soln:-

For any set A and for any $\epsilon > 0$ then there exists an open set 'O' containing A such that,

$$O \subseteq A.$$

$$m^*(O) \leq m^*(A) + \epsilon.$$

We take $\epsilon = 1/n$, and ~~take~~ the open set 'O_n' for the corresponding open set. Then the set $E = \bigcap_{n=1}^{\infty} O_n$ has the required properties.

$$m^*(A) \geq \sum l(I_n) - \epsilon.$$

For every n, $E \subseteq O_n$

$$m^*(E) = m(E) \leq m(O_n) \leq m^*(A)+\epsilon$$

$$\leq m^*(A) + 1/n \qquad [\because \epsilon = 1/n]$$

$$m(E) \leq m^*(A) \quad -① $$

$$A \subseteq \bigcap_{n=1}^{\infty} O_n$$

(i.e) $A \subseteq E$.

$$m^*(A) \leq m^*(E)$$

$$m^*(A) \leq m(E) \quad — ② \qquad [\because m^*(E) = m(E)]$$

From ① & ② we get,

$$m^*(A) = m(E)$$

Hence proved.

Th:9 Let $\{E_i\}$ be a sequence of measurable sets then.

(i). If $E_1 \subseteq E_2 \subseteq \ldots$ we have $m(\lim E_i) = \lim m(E_i)$

ii) If $E_1 \supseteq E_2 \supseteq \ldots$ and $m(E_i) < \infty$ then $m(\lim E_i) = \lim m(E_i)$

Proof:

(i) Write, $F_1 = E_1$, $F_2 = E_2 - E_1$, $F_3 = E_3 - E_2$.

$$F_i = E_i - E_{i-1} \text{ for } i > 1$$

Then we have,

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} F_i$$

$\therefore$ E_i is a measurable sets and F_i is also measurable and disjoint.

$$m(\lim E_n) = m\left(\bigcup_{i=1}^{\infty} E_i\right) = m\left(\bigcup_{i=1}^{\infty} F_i\right)$$

$$= \sum_{i=1}^{\infty} m(F_i) = \lim \sum_{i=1}^{n} m(F_i)$$

$$= \lim m\left(\bigcup_{i=1}^{n} F_i\right)$$

$= \lim m(\bigcup_{i=1}^{n} E_i)$

$m(\lim E_n) = \lim m(E_n)$

(i.e) $m(\lim E_i) = \lim m(E_i)$ — ①

(ii) We have,

$E_1 - E_1 \subseteq E_1 - E_2 \subseteq E_1 - E_3 \subseteq \cdots$ by (i)

$m(\lim(E_1 - E_i)) = \lim m(E_1 - E_i)$

$= m \lim(E_1) - m \lim(E_i)$

$= m(E_1) - \lim m(E_i)$ — ②

But, $\lim(E_1 - E_i) = \bigcup_{i=1}^{\infty}(E_1 - E_i)$

$= E_1 - \bigcap_{i=1}^{\infty} E_i$

$= E_1 - \lim E_i$

So taking measure of both sides we get,

$m(\lim(E_1 - E_i)) = m(E_1) - m(\lim E_i)$ — ③

Since $m(E_i) < \infty$,

comparing ② & ③ we get,

$m(E_1) - \lim m(E_i) = m(E_1) - m(\lim E_i)$

$m(\lim E_i) = \lim m(E_i)$

Ex: 7
i) S.T every non-empty open set has positive measure

Proof:-

We know that the outer measure of an interval greater than '0' and intervals are measurable sets.

$\therefore$ measurable of intervals is > 0

Every non-empty open set in $\mathbb{R}$ is the union of disjoint intervals.

$\therefore$ The open sets are measurable and measure of an open sets is > 0.

(ii) The rationals $\mathbb{Q}$ are enumerated as $q_1, q_2, \ldots$ and the set G is defined by $G = \bigcup_{n=1}^{\infty} (q_n - \frac{1}{n^2}, q_n + \frac{1}{n^2})$.

P.T, for any closed set F, $m(G \Delta F) > 0$.

Proof:

$$G = \bigcup_{n=1}^{\infty} \left(q_n - \frac{1}{n^2}, q_n + \frac{1}{n^2}\right) \quad \text{——} \; (1)$$

$$G \Delta F = (G - F) \cup (F - G)$$

$$\{ G, F \in \mathcal{M} \Rightarrow (G - F) \,\&\, (F - G) \in \mathcal{M}.$$

$$\Rightarrow (G - F) \cup (F - G) \in \mathcal{M}.$$

$$\Rightarrow G \Delta F \in \mathcal{M}. \}.$$

$$m(G \Delta F) = m\{(G - F) \cup (F - G)\}.$$

$$= m(G - F) + m(F - G)$$

It is enough to prove that, $m(G - F) > 0$

If not, $m(G - F) = 0$

F is closed $\Rightarrow$ $G - F$ is an open set.

By (i), $G \subseteq F$

But G contains $\mathbb{Q}$

whose closure,

$$m(\mathbb{Q}) = m(\mathbb{R}) = \infty. \Rightarrow m(\mathbb{R}) = \infty.$$

In (1) apply measure on both sides we get,

$$m(G) = \sum_{n=1}^{\infty} m\left(q_n - \frac{1}{n^2}, q_n + \frac{1}{n^2}\right).$$

$$\leq \sum_{n=1}^{\infty} l(q_n - 1/n^2, q_n + 1/n^2)$$

$$\leq \sum_{n=1}^{\infty} \left(q_n + \frac{1}{n^2} - q_n + \frac{1}{n^2} \right)$$

$$= 2 \sum_{n=1}^{\infty} \left(\frac{1}{n^2} \right) \Rightarrow 2 \left(\frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \right)$$

$$= 2 \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \right)$$

$m(G) < \infty$

$m(F-G) = \infty \Rightarrow m(G-F) > 0.$

$m(G \Delta F) > 0.$

If $m(G-F) = \infty \Rightarrow m(F-G) \geq \infty$

$\therefore m(G \Delta F) > 0$

Ex: 8 S.T there exist uncountable sets of zero measure.

Soln:-

We show that the Cantor set 'P' from the interval $[0,1]$. First remove $\left(\frac{1}{3}, \frac{2}{3}\right)$ then $\left(\frac{1}{9}, \frac{2}{9}\right)$ and $(7/9, 8/9)$.

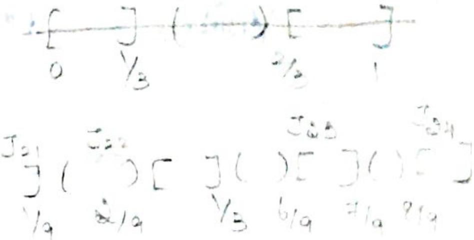
Removing at each stage the open intervals consisting the middle third stage of the closed interval, left at the previous stage. This gives a special case of the previous construction with the residual closed interval at the n^{th} stage $J_{n+1}, J_{n-2}, \ldots, J_n$ each of length $1/3^n$.

The open intervals $I_{n,r}$ also being of length $\frac{1}{3^n}$

Stage 1:-

Remove $I_{11} = \left(\frac{1}{3}, \frac{2}{3}\right)$

$l(I_{11}) = (\frac{2}{3}, -\frac{1}{3}) = \frac{1}{3}$



$J_{11} = [0, \frac{1}{3}]$

$l(J_{11}) = \frac{1}{3} - 0 = \frac{1}{3}$

$J_{12} = [\frac{2}{3}, 1]$

$l(J_{12}) = 1 - \frac{2}{3} = \frac{1}{3}$.

Stage 2:-

Remove $I_{21} = (\frac{1}{9}, \frac{2}{9})$ and $I_{22} = (\frac{7}{9}, \frac{8}{9})$

Residues,

$J_{21} = [0, \frac{1}{9}] \Rightarrow l(J_{21}) = \frac{1}{9} - 0 = \frac{1}{9}$

$J_{22} = [\frac{2}{9}, \frac{1}{3}] \Rightarrow l(J_{22}) = \frac{1}{3} - \frac{2}{9} = \frac{1}{9}$.

$J_{23} = [\frac{2}{3}, \frac{7}{9}] \Rightarrow l(J_{23}) = \frac{7}{9} - \frac{2}{9} = \frac{1}{9}$

$J_{24} = [\frac{8}{9}, 1] \Rightarrow l(J_{24}) = 1 - \frac{8}{9} = \frac{1}{9}$

$l(I_{21}) + l(I_{22}) = \frac{1}{9} + \frac{1}{9} = \frac{2}{9}$.

Stage 3:

We remove 2^2 open set, we get $J_{31}, J_{32}, \dots J_{38}$ closed intervals (residues) and the nth stage 2^{n-1} open intervals namely,

$I_{n,k}$; $k = 1, 2, \dots, 2^{n-1}$

We get residue $J_{n1}, J_{n2}, \dots$ and each of length is $\frac{1}{3^n}$.

Define $P_n = \bigcup_{r=1}^{2^{n-1}} I_{n,r}$

$\therefore$ P_n is measurable. $\Rightarrow m^*(P) = 0$.

Next we define, $P = \bigcap_{n=1}^{\infty} P_n$,

where P is a cantor set

(1)

Write, $P^* = [0,1] - P$

$\Rightarrow [0,1] - \bigcap_{n=1}^{\infty} P_n$

$$P^* = \bigcup_{n=1}^{\infty} \bigcup_{r=1}^{2^{n-1}} I_{n,r}$$

which is a union of disjoint sets.

$$m(P^*) = m\left(\bigcup_{n=1}^{\infty} \bigcup_{r=1}^{2^{n-1}} I_{n,r}\right)$$

$$= \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} \Rightarrow \sum_{n=1}^{\infty} \frac{2^n \cdot 2^{-1}}{3^n}$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$$

$$\Rightarrow \frac{1}{2}\left[\frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots\right]$$

$$\Rightarrow \frac{1}{2} \times \frac{2}{3}\left[1 + \left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 + \cdots\right]$$

$$= \frac{1}{3}\left[1 - \frac{2}{3}\right]^{-1}$$

$$m(P^*) = 1/3 \times \frac{3}{1} \Rightarrow 1$$

$$m(P^*) = m[0,1] - m(P)$$

$$m(P) = m[0,1] - m(P^*) \Rightarrow l[0,1] - 1$$

$$= 1 - 1$$

$$m(P) = 0.$$

Regularity:-

Th: 10 The follg statements the set E are equivalent:

(i) E is measurable

(ii) $\forall \epsilon > 0, \exists$ 'O' an open set, $O \supseteq E \ni: m^*(O - E) \leq \epsilon$

(iii) $\exists G$; a G_δ-set, $G \supseteq E \ni: m^*(G - E) = 0$.

(ii)* $\forall \epsilon > 0, \exists F$, a closed set, $F \subseteq E \ni: m^*(E - F) \leq \epsilon$

(iii)* $\exists F$, an F_σ-set, $F \subseteq E \ni: m^*(E - F) = 0$

Proof:-

(i) $\to$ (ii)

Assume that E is measurable.

Case (i):-

Suppose that $m(E) < \infty$.

By known example:-

There is an open set $O \supseteq E$ such that $m(O) \leq m(E) + \epsilon$

(i.e) $m(O-E)$ $m(O) - m(E) \leq \epsilon$

$m(O-E) \leq \epsilon$

Case (ii):-

If $m(E) = \infty$.

Write $R = \bigcup_{n=1}^{\infty} I_n$.

a union of disjoint finite intervals. Then if,

$E_n = E \cap I_n$.

E_n are disjoint, $E_n \subset I_n$, $m(I_n) < \infty$ we have $m(E_n) < \infty$. So there is an open set $O_n \supseteq E_n$ such that,

$$m(O_n - E_n) \leq \epsilon/2^n$$

Write, $O = \bigcup_{n=1}^{\infty} O_n$ & $E = \bigcup_{n=1}^{\infty} E_n$.

which is an open set.

$$O - E = \bigcup_{n=1}^{\infty} O_n - \bigcup_{n=1}^{\infty} E_n.$$

$$\subseteq \bigcup_{n=1}^{\infty} (O_n - E_n)$$

$$m(O-E) = \sum_{n=1}^{\infty} m(O_n - E_n)$$

$$\leq \sum_{n=1}^{\infty} \epsilon/2^n$$

$$\le \epsilon \sum_{n=1}^{\infty} 1/2^n$$

$$\le \epsilon\ (1)$$

$$m(O-E) \le \epsilon.$$

(ii) $\to$ (iii)

For each n,

Let O_n be an open set, $O_n \supseteq E$

such that $m^*(O_n - E) < 1/n$.

Then if $G = \bigcap_{n=1}^{\infty} O_n$

[G_δ - one which is a countable intersection of open sets]

G is a G_δ - set, $E \subseteq G$.

And $m^*(G-E) \le m^*(O_n - E)$

$m^*(G-E) \le 1/n$ for each n.

$m^*(G-E) = 0$.

(iii) $\to$ (i):-

Since $m^*(G-E) = 0$.

$G-E$ is measurable.

Let $E = G - (G-E)$ is measurable.

G & $G-E$ are measurable.

$\therefore$ E is measurable.

(i) $\to$ (ii)*

Assume that E is measurable.

$\therefore$ CE is measurable.

Since CE is measurable, there exists an open set 'O'

such that $O \supseteq CE$ and $m(O - CE) \le \epsilon$.

But $O - CE = E - CO$ take $F = CO$

$m^*(O-CE) = m^*(E - CO)$

$= m^*(E-F)$

we have $m^*(O-E) \le \epsilon$.

$\therefore$ $m^*(E-F) \le \epsilon$.

(ii)* $\Rightarrow$ (iii)*:

For each n, let F_n be a closed set $F_n \subseteq E$ and $m^*(E-F_n) \le 1/n$.

[F_σ - one which is a countable union of closed sets]

Then if $F = \bigcup_{n=1}^{\infty} F_n$

F is an F_σ set.

Since $F \subseteq E$.

$m^*(E-F) \le m^*(E-F_n)$

$m^*(E-F) < 1/n$ for each n.

$m^*(E-F) = 0$.

Where F is a closed set such that $F \subseteq E$.

(iii)* $\to$ (i):

Since $m^*(E-F) = 0$.

$E-F$ is measurable

Since F is an F_σ - set, F is measurable.

$E = F \cup (E-F)$.

Since F & $(E-F)$ are measurable.

$\therefore$ E is measurable.

Completion of Measure.

Th: 11. If $m^*(E) < \infty$ then E is measurable if and only if, $\forall \epsilon > 0$, $\exists$ disjoint finite intervals $I_1, \dots, I_n$ such that $m^*(E \Delta \bigcup_{i=1}^{n} I_i) < \epsilon$. We may stipulate that the intervals I_i be open, closed or half-open.