we have $m^*(O - E) \le \epsilon$.

$\therefore \quad m^*(E - F) \le \epsilon$.

(ii)* $\rightarrow$ (iii)*:

For each n, let F_n be a closed set $F_n \subseteq E$ and $m^*(E - F_n) \le 1/n$.

[F_σ - one which is a countable union of closed sets]

Then if $F = \bigcup_{n=1}^{\infty} F_n$

F is an F_σ set.

Since $F \subseteq E$.

$m^*(E - F) \le m^*(E - F_n)$

$m^*(E - F) < 1/n$ for each n.

$m^*(E - F) = 0$.

Where F is a closed set such that $F \subseteq E$.

(iii)* $\rightarrow$ (i):

Since $m^*(E - F) = 0$.

$E - F$ is measurable

Since F is an F_σ - set, F is measurable.

$E = F \cup (E - F)$.

Since F & $(E - F)$ are measurable.

$\therefore$ E is measurable.

Completion of Measure.

Th: 11. If $m^*(E) < \infty$ then E is measurable if and only if, $\forall \epsilon > 0$, $\exists$ disjoint finite intervals $I_1, \dots, I_n$ such that $m^*(E \,\Delta\, \bigcup_{i=1}^{n} I_i) < \epsilon$. We may stipulate that the intervals I_i be open, closed or half-open.

Proof:-

Suppose that E is measurable.

Given $\epsilon > 0$, there exist an open set 'O' containing E

$O \supseteq E$.

$$m(O) \le m(E) + \epsilon$$

$$m(O) - m(E) \le \epsilon$$

$$m(O - E) = m(O) - m(E)$$

$$m(O-E) \le \epsilon \quad \text{--- (1)}$$

Since $m(E)$ is finite and $m(O)$ is infinite. 'O' is the union of disjoint open interval $\{I_i\}$ $(i = 1, 2, \ldots, \infty)$

$$O = \bigcup_{i=1}^{\infty} I_i$$

$m(I_i)$ is finite for each i,

$$m(O) = m\left(\bigcup_{i=1}^{\infty} I_i\right) = \sum_{i=1}^{\infty} l(I_i)$$

$m(O)$ is infinite $\exists$ 'O' an positive integer 'n' such that.

$$\sum_{i=n+1}^{\infty} l(I_i) < \epsilon$$

$$U = \bigcup_{i=1}^{n} I_i$$

$$E \Delta U = (E - U) \cup (U - E)$$

$$E \Delta U \subseteq (O - U) \cup (O - E) \quad \text{--- (2)}$$

$$O - U = \bigcup_{i=1}^{\infty} I_i - \bigcup_{i=1}^{n} I_i$$

$$= \bigcup_{i=n+1}^{\infty} I_i$$

Taking measure on both sides,

$$m(O - U) = m\left(\bigcup_{i=n+1}^{\infty} I_i\right)$$

$$m(O - U) = \sum_{i=n+1}^{\infty} m(I_i) < \epsilon \quad \text{--- (3)}$$

(2) $\Rightarrow$ $m(E \Delta U) \leq m(O-U) + m(O-E)$

$$\leq \epsilon + \epsilon$$

$$m(E \Delta U) \leq 2\epsilon$$

$$\therefore m(E \Delta U) < 2\epsilon$$

$$m(E \Delta \bigcup_{i=1}^{n} I_i) < \epsilon \quad \text{--- (4)}$$

We first obtain open interval $I_1, I_2, \ldots, I_n$ then for each i, choose a half-open interval $J_i \subset I_i$.

$m(J_i) \leq m(J_i) + \epsilon/n$

$\ni:$ $m(I_i - J_i) < \epsilon/n$. $m(I_i) - m(J_i) \leq \epsilon/n$

Then the intervals J_i are disjoint,

$$m(E \Delta \bigcup_{i=1}^{n} J_i) \leq m(E \Delta \bigcup_{i=1}^{n} I_i) + m(\bigcup_{i=1}^{n} I_i \Delta \bigcup_{i=1}^{n} J_i) \quad \text{--- (5)}$$

We know that,

$$m(E \Delta \bigcup_{i=1}^{n} I_i) < \epsilon.$$

$$\bigcup_{i=1}^{n} I_i \Delta \bigcup_{i=1}^{n} J_i = \left(\bigcup_{i=1}^{n} J_i - \bigcup_{i=1}^{n} J_i\right) \cup \left(\bigcup_{i=1}^{n} J_i - \bigcup_{i=1}^{n} I_i\right) \quad \text{--- (A)}$$

Since $J_i - I_i = \phi$ for each i,

$$\bigcup_{i=1}^{n} (J_i - I_i) = \phi.$$

$$m\left[\bigcup_{i=1}^{n} (J_i - I_i)\right] = m(\phi).$$

$$m\left[\bigcup_{i=1}^{n} (J_i - I_i)\right] = 0 \quad \text{--- (B)}$$

(A) $\Rightarrow$

$$m\left(\bigcup_{i=1}^{n} I_i \Delta \bigcup_{i=1}^{n} J_i\right) \leq m\left(\bigcup_{i=1}^{n} I_i - \bigcup_{i=1}^{n} J_i\right) \cup m\left(\bigcup_{i=1}^{n} J_i - \bigcup_{i=1}^{n} I_i\right)$$

$$\leq \sum_{i=1}^{n} m(I_i - J_i) + 0.$$

$$\leq \sum_{i=1}^{n} \epsilon$$

$$< \epsilon.$$

Equation (5) becomes,

$m(E \Delta \bigcup_{i=1}^{n} J_i) \le \epsilon + \epsilon < 2\epsilon$

We can P.T the simpler result in the case of closed interval J_i where $J_i = (a_i + r, b_i - r)$, $r \in \epsilon/2$.

Conversely;

Any set E for every $\epsilon > 0$ then there exists an open set 'O' containing E such that,

$$m(O) \le m^*(E) + \epsilon \quad \text{--- (B)}$$

$$m(O) \le 1$$

Let, $J = \bigcup_{i=1}^{n} I_i$ and $V = O \cap J$ then,

$$m^*(O \Delta E) \le m^*(O \Delta U) + m^*(U \Delta E) \quad \text{--- (7)}$$

Case (a):

Since $U \subseteq J$ we have $U - E \subseteq J - E$. and

Since $E \subseteq O$ we have. $E - U \subseteq E - J$.

$$\text{So,} \quad U \Delta E = (U-E) \cup (E-U)$$

$$= (J-E) \cup (E-J)$$

$$\le J \Delta E$$

$$\therefore U \Delta E \subseteq J \Delta E$$

$$E \subseteq U \Rightarrow m^*(U \Delta E) \le m^*(J \Delta E)$$

$$\le m^*(E \Delta \bigcup_{i=1}^{n} J_i)$$

$$m^*(U \Delta E) < \epsilon \quad \text{--- (8)}$$

Since E is a subset of U.

$$E \subset U \cup (U \Delta E)$$

$$m^*(E) \le m^*(U) + m^*(U \Delta E)$$

$$m^*(E) \le m^*(U) + \epsilon \quad \text{--- (9)}$$

Case (b):

Consider, $O \Delta U = (O-U) \Delta (U-O)$

$= O-U \quad (\because U \subseteq O,\ U-O=\phi)$

$m^*(O \Delta U) = m^*(O-U) = m^*(O) - m^*(U)$

$m^*(O \Delta U) \leq m^*(E) + \epsilon - m^*(U)$ By (B)

$\leq m^*(U) + \epsilon + \epsilon - m^*(U)$ By (9)

$m^*(O \Delta U) \leq 2\epsilon$.

(7) $\Rightarrow$ $m^*(O \Delta E) \leq m^*(O \Delta U) + m^*(U \Delta E)$

$m^*(O \Delta E) \leq 2\epsilon + \epsilon$

$\leq 3\epsilon$.

$m^*(O-E) = m^*(O \Delta E) < 3\epsilon$

$\therefore$ $O-E$ is measurable

$\therefore$ E is measurable

Th: 12. The folly statements are equivalent:

i) f is a measurable function. ii) $\forall \alpha$, $[x : f(x) \geq \alpha]$ is mea.

iii) $\forall \alpha, [x : f(x) < \alpha]$ is mea iv) $\forall \alpha, [x : f(x) \leq \alpha]$ is mea.

Proof:

(i) $\Rightarrow$ (ii):

Assume that 'f' is measurable.

By definition, $\Rightarrow \forall \alpha, \{x : f(x) > \alpha\}$ is measurable

$\Rightarrow \forall \alpha, \{x : f(x) > \alpha - \frac{1}{n}\}$ is measurable

$\Rightarrow \forall \alpha, \bigcap_{n=1}^{\infty} \{x : f(x) > \alpha - \frac{1}{n}\}$ is measurable.

The intersection of a measurable set is measurable.

$\forall \alpha, [x : f(x) \geq \alpha]$ is measurable.

(ii) $\Rightarrow$ (iii):

Let $[x: f(x) \geq \alpha]$ be measurable.

$\Rightarrow \forall \alpha$, $C[x: f(x) \geq \alpha]$ is measurable.

$\Rightarrow \forall \alpha$, $[x: f(x) < \alpha]$ is measurable.

(iii) $\Rightarrow$ (iv):

If $\forall \alpha$, $[x: f(x) < \alpha]$ is measurable

$\Rightarrow \forall \alpha$, $[x: f(x) < \alpha + 1/n]$ is measurable.

$\Rightarrow \forall \alpha$, $\bigcap_{n=1}^{\infty} [x: f(x) < \alpha + 1/n]$

$\Rightarrow \forall \alpha$, $[x: f(x) \leq \alpha]$ is measurable.

(iv) $\Rightarrow$ (i):

If $[x: f(x) \leq \alpha]$ is measurable.

$\Rightarrow \forall \alpha$, $C[x: f(x) \leq \alpha]$ is measurable.

$\Rightarrow \forall \alpha$, $[x: f(x) > \alpha]$ is measurable.

$\therefore$ f is a measurable function.

Ex: 9 S.T if f is measurable, then $[x: f(x) = \alpha]$ is measurable for each extended real number, α.

Proof:-

Since f is measurable.

$\Rightarrow [x: f(x) > \alpha]$ is measurable.

$\Rightarrow [x: f(x) > \alpha - 1/n]$

$\Rightarrow \bigcap_{n=1}^{\infty} [x: f(x) > \alpha - 1/n]$

$\Rightarrow [x: f(x) \geq \alpha]$ is measurable.

Similarly,

$\Rightarrow [x: f(x) \leq \alpha]$ is measurable

Case (i):

If α is finite.

$[x : f(x) = \alpha] = [x : f(x) \geq \alpha] \cap [x : f(x) \leq \alpha]$ and is measurable.

Case (ii):

For $\alpha = \infty$.

Since f is measurable.

$\Rightarrow [x : f(x) > n]$ is measurable.

$\Rightarrow \bigcap\limits_{n=1}^{\infty} [x : f(x) > n]$ is measurable.

$\Rightarrow \bigcap\limits_{n=1}^{\infty} [x : f(x) > \alpha]$

$= [x : f(x) = \infty]$ is measurable

Case (iii):

For $\alpha = -\infty$.

$\Rightarrow [x : f(x) <\ ]$ is measurable.

$\cap [x : f(x) < n]$ is measurable.

$[x : f(x) = -\infty]$ is measurable.

Ex: 10. P.T the constant functions are measurable.

Proof:

Let f be a constant function defined on measurable set E such that,

$$f(x) = c, \quad \forall x \in E.$$

$$[x : f(x) > \alpha] = \begin{cases} R, & \text{if } c > \alpha \\ \phi, & \text{if } c \leq \alpha \end{cases}$$

Since E & f are measurable function

$\Rightarrow f$ is measurable.

$\Rightarrow$ c is measurable

Hence the constant functions are measurable.

Characteristic Function:-

Let A be a subset of measurable set E. The characteristic function χ_A is of the set A is defined by,

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

Ex: 11 P.T the characteristic function χ_A of the set A is measurable iff A is measurable.

Proof:-

Case (i):

Let A be measurable

The characteristic function χ_A of the set A is defined by,

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

$$[x : \chi_A(x) > \alpha] = \begin{cases} \phi & \text{if } \alpha \geq 1 \\ A & \text{if } 0 \leq \alpha < 1 \\ R & \text{if } \alpha < 0 \end{cases}$$

Since ϕ, A, R are measurable function.

$[x : \chi_A(x) > \alpha]$ is a measurable set

(i.e) χ_A is measurable function

Case (ii):

Conversely,

χ_A is measurable function

Since χ_A is measurable. Then by definition we have,

$[x : \chi_A(x) > \alpha]$ is measurable.

$[x : \chi_A(x) > 0] = A$ is measurable.

$\therefore$ A is measurable.

Ex: 12. P.T continuous functions are measurable.

Proof:-

If f is continuous the set of all $[x: f(x) > \alpha]$ is an open set.

$\therefore$ Every open set is measurable.

$[x: f(x) > \alpha]$ is measurable.

$\therefore f$ is measurable.

Th: 13. Let c be any real no & let f & g be real-valued mea function defined on the same mea set E. Then $f+c$, cf, $f+g$ & $f-g$ & fg are also measurable.

Proof:-

Given f & g are measurable function.

For each α, $[x: f(x) > \alpha]$ and $[x: g(x) > \alpha]$ are measurable

Part (i):

For each α, $[x: f(x)+c > \alpha] = [x: f(x) > \alpha - c]$ is a measurable set.

$\therefore f+c$ is measurable.

Part (ii):

If $c=0$, then cf is measurable.

[$\because$ We K.T constant functions are measurable]

If $c > 0$, then $[x: cf(x) > \alpha] = [x: f(x) > c^{-1}(\alpha)]$ is measurable set. $\therefore cf$ is measurable.

If $c < 0$, then $[x: -cf(x) > -\alpha] = [x: cf(x) < \alpha]$

$= [x: f(x) < c^{-1}(\alpha)]$

$\therefore$ So cf is always measurable.

Part (iii):

Next we have to prove that $f+g$ is measurable.

Observe that,

$$x \in A = [x : f(x) + g(x) > \alpha] \text{ only}$$
$$= [x : f(x) > \alpha - g(x)]$$

$x \in A$ only if $f(x) > \alpha - g(x)$

(i.e) only if there exists a rational r_i such that

$$f(x) > r_i > \alpha - g(x).$$

where $\{r_i, i = 1, 2, \ldots\}$ is an enumeration of Q.

But then, $g(x) > \alpha - r_i$ and so.

$$x \in [x : f(x) > r_i] \cap [x : g(x) > \alpha - r_i]$$

Hence,

$$A \subseteq B = \bigcup_{i=1}^{\infty} \left([x : f(x) > r_i] \cap [x : g(x) > \alpha - r_i]\right).$$

is a measurable set. Since A clearly contains B we have $A = B$.

$\therefore$ $f+g$ is measurable.

Then, $f - g = f + (-g)$

$\therefore$ $f - g$ is measurable.

Part (iv):

$fg = \frac{1}{4}[(f+g)^2 - (f-g)^2]$ so it is sufficient to show that f^2 is measurable.

If $\alpha < 0$, then $[x : f^2 > \alpha] = R$ is measurable.

If $\alpha \geq 0$, then $[x : f^2(x) > \alpha] = [x : f(x) > \sqrt{\alpha}] \cap [x : f(x) < -\sqrt{\alpha}]$

Since f is measurable, $[x : f(x) > \sqrt{\alpha}]$ & $[x : f(x) < -\sqrt{\alpha}]$ is are measurable.

$[x : f(x) > \sqrt{\alpha}] \cap [x : f(x) < -\sqrt{\alpha}]$ is measurable.

(i.e) $[x : f^2(x) > \alpha]$ is measurable.

$\therefore$ fg is measurable.

Th:14 Let $\{f_n\}$ be a sequence of measurable functions. Then $\sup f_i$, $\inf f_i$, $\sup f_n$, $\inf f_n$, $\lim\sup f_n$, $\lim\inf f_n$ is measurable.

Proof:

(i) Since each f_i is measurable

$\Rightarrow [x : f_i > \alpha]$ is measurable for each i.

$$[x : f_i > \alpha] = \bigcup_{i=1}^{n} [x : f_i > \alpha] \text{ is measurable.}$$

$$[x : \sup_{1 \le i \le n} f_i(x) > \alpha] = \bigcup_{i=1}^{n} [x : f_i(x) > \alpha]$$

we have,

$\sup\limits_{1 \le i \le n} f_i$ is measurable.

(ii) Since each f_i is measurable

$\Rightarrow [x : f_i(x) < \alpha]$ is measurable for each i.

$$[x : f_i(x) < \alpha] = \bigcap_{i=1}^{n} [x : f_i(x) < \alpha] \text{ is measurable.}$$

$[x : \inf\limits_{1 \le i \le n} f_i(x) < \alpha]$ is measurable.

$$[x : \inf_{1 \le i \le n} f_i(x) < \alpha] = \bigcap_{i=1}^{n} [x : f_i(x) < \alpha]$$

we have

$\inf\limits_{1 \le i \le n} f_i$ is measurable.

(iii) Since each f_n is measurable.

$[x : f_n > \alpha]$ is measurable.

$$[x : f_n > \alpha] = \bigcup_{n=1}^{\infty} [x : f_n > \alpha]$$

$= [x : \sup f_n > \alpha]$

$\therefore \bigcup_{n=1}^{\infty} [x : f_n > \alpha] = [x : \sup f_n > \alpha]$ is measurable.

(iv) Since each f_n is measurable.

$\Rightarrow [x : f_n < \alpha]$ is measurable.

$[x : f_n < \alpha] = \bigcap_{n=1}^{\infty} [x : f_n < \alpha]$

$= [x : \inf f_n < \alpha]$

$\therefore \bigcap_{n=1}^{\infty} [x : f_n < \alpha] = [x : \inf f_n < \alpha]$ is measurable

(v) & (vi):

Inf f_n & sup f_n is measurable.

Inf (sup f_n) & sup (Inf f_n) is measurable.

Inf (sup f_n) & = lim (sup f_n) is measurable.

Sup (Inf f_n) = lim (Inf f_n) is measurable.

Th. 15. Let f be a measurable function & let $f = g$ (a.e). Then g is measurable.

Proof:-

Given f is a measurable function.

By known definition:-

" Let f be an extended real-valued function defined on a measurable set E. Then f is a measurable fn if for each $\alpha \in R$, the set $[x : f(x) > \alpha]$ is measurable."

Put $F = [x : f(x) > \alpha]$ & $G = [x : g(x) > \alpha]$

By known Theorem:

" If $F \in M$ & $m^*(F \Delta G) = 0$, then G is measurable."

$F \Delta G = [x : f(x) > \alpha] \Delta [x : g(x) > \alpha]$

$[x : f(x) > \alpha] \Delta [x : g(x) > \alpha] \subseteq [x : f(x) \neq g(x)]$

$F \Delta G = E$ where $m^*(E) = 0$.

$0 \leq m^*(F \Delta G) \leq m^*(E) \leq 0$.

$m^*(F \Delta G) = m^*(E)$.

$m^*(F \Delta G) = 0$.

$\therefore$ F is measurable

Since f is measurable function & $m^*(F \Delta G) = 0$.

$\therefore$ G is measurable.

Ex: 13 Let $\{f_i\}$ be a sequence of measurable functions converging a.e to f, then f is measurable, since $f = \lim \sup f_i$ (a.e)

Proof:

Given, let $\{f_i\}$ be a sequence of measurable functions convergence (a.e) to f.

Since $f = \lim \sup f_i$ (a.e)

$f' = \lim \sup f_i$

$A = [x : f(x) > \alpha] = [x : \lim \sup f_i (a.e) > \alpha]$

$A' = [x : f'(x) > \alpha] = [x : \lim \sup f_i > \alpha]$

If $\{f_i\}$ is a sequence of measurable function. Then, $\lim \sup f_i$ is also a measurable function.

Since $A' = \lim \sup f_i$ is a measurable function

A' is a measurable set.

Since, $A \subset A'$ and A' is a measurable set.

A is also a measurable set.

Hence $f = \lim \sup f_i$ (a.e) is a measurable function.

Ex:15. The set of points on which a sequence of measurable fns $\{f_n\}$ converges, is measurable.

Proof:-

Assume that $A = [x : f_n(x) \to f(x)]$

$\lim\sup f_n = \lim\inf f_n = f$ on the set A.

The set:

$[x : \lim\sup f_n(x) - \lim\inf f_n(x) = 0]$

By known theorem:

"[let $\{f_n\}$ be a sequence of measurable functions. Then, $\lim\sup f_n$ & $\lim\inf f_n$ are measurable]"

Then the set:

$[x : \lim\sup f_n(x) - \lim\inf f_n(x) = 0]$ is measurable.

$[x : f(x) = 0]$

By known theorem:

" If f is measurable then $[x : f(x) = \alpha]$ is measurable for each extended real number α."

$[x : f(x) = 0]$ is a measurable set.

f is measurable.

Ex: 16. S.T $f \leq \text{ess sup } f$. (a.e)

Proof:

(i) If ess sup $f = \infty$ then $f \leq \infty$.

(ii) If ess sup $f = -\infty$ then $f \leq n$, $\forall n \in \mathbb{Z}$. $(f \leq -\infty)$

Suppose that ess sup f is finite.

Write, $E_n = [x : f(x) > 1/n + \text{ess sup } f]$ and

$E = [x : f(x) > \text{ess sup } f]$.

$$\therefore \quad E = \bigcup_{n=1}^{\infty} E_n$$

By known definition:

$$m(E_n) = 0 \quad \Rightarrow \quad m(E) = 0.$$

$f >$ ess sup f on the set E where $m(E) = 0$

$$f \leq \text{ess sup } f \quad (a.e).$$

Ex: 17 S.T for any measurable function f & g,

$$\text{ess sup}(f+g) \leq \text{ess sup } f + \text{ess sup } g$$

and give an example of strict inequality.

Proof:-

Write previous example

From the previous example:

$$f \leq \text{ess sup } f \quad (a.e) \quad \text{—} \; (1)$$

$$g \leq \text{ess sup } g \quad (a.e) \quad \text{—} \; (2).$$

$$f+g \leq \text{ess sup } f + \text{ess sup } g, \; (a.e)$$

$$(i.e) \; f+g \leq \text{ess sup}(f+g).$$

$$\text{ess sup}(f+g) \leq \text{ess sup } f + \text{ess sup } g \quad (a.e)$$

Example:

For inequality take $f = \chi_{[-1,0)} - \chi_{[0,1]}$ and

$g = -f$.

Then L.H.S $\Rightarrow$ ess sup $(f+g) =$ ess sup $(f-f) = 0$.

R.H.S $\Rightarrow$ ess sup f + ess sup g.

$= 2$.

Then the left-hand side is zero and the right-hand side is '2'.

Ex: 18 $\quad Ess\ Sup\ f = -\ ess\ \inf(-f)$.

Proof:-

$$Ess\ sup\ f = \inf[\alpha : f \leq \alpha\ a.e]$$
$$= \inf[\alpha : -f \geq -\alpha\ a.e]$$
$$= -\sup[-\alpha : -f \geq -\alpha\ a.e]$$
$$Ess\ Sup\ f = -ess\ \inf(-f).$$

Hence proved.

Ex: 19. Let f be a measurable function & B a Borel set, then $f^{-1}(B)$ is a measurable set.

Proof:-

Given B is a Borel set.

Assume that $B = \bigcup_{i=1}^{\infty} A_i$ where each A_i is an open set.

$$f^{-1}(B) = f^{-1}\left[\bigcup_{i=1}^{\infty} A_i\right] = \bigcup_{i=1}^{\infty} f^{-1}(A_i)$$

If $B = CA$ then $f^{-1}(B) = f^{-1}(CA) = C f^{-1}(A)$.

So the class of sets whose inverse images under 'f' are measurable forms a σ-algebra. But this class contains the intervals. So it must contain all Borel sets.

$f^{-1}(B)$ is also a Borel set.

Since Borel sets are subset of measurable set.

Hence $f^{-1}(B)$ is a measurable set.

Borel and Lebesgue Measurability:-

Th: 16. Let E be a measurable set. Then for each y the set $E+y = [x+y;\ x \in E]$ is measurable and the measures are the same.

Proof:

By known Theorem:

Since E is a measurable set, $\forall \epsilon > 0$, $\exists$ an open set O, $O \supseteq E$ and $m(O-E) < \epsilon$. — ①

If 'O' is an open set, then the set $O+y$ is also an open set and $O+y \supseteq E+y$.

$(O+y) - (E+y) = (O-E) + y$.

Taking measures on both sides,

$m[(O+y) - (E+y)] = m[(O-E)+y]$

$m[(O-E)+y] \leq m(O-E) + m(y)$

$\leq m(O-E) + 0$. [By ①]

$m[(O-E)+y] \leq \epsilon$ — ②

$O+y$ is an open set, then $E+y$ is also an open set.

$E+y$ is measurable.

$m^*(E+y) = m(E+y) = m^*(E)$.

$m^*(E+y) = m^*(E)$ — ③

$m^*(E+y) = m(E)$

$\Rightarrow m(E+y) = m(E)$ — ④.

Hence the measures are the same.

Th: 17. There exists a non-measurable set.

Proof:-

If $x, y \in [0,1]$

let $x \sim y$ if $y - x \in Q_1 = Q \cap [-1,1]$.

Then '$\sim$' is said to be an equivalence relation on $[0,1]$

$[0,1] = \cup E_\alpha$.

where E_α disjoint sets such that x & y are in the same E_α, if and only if $x \sim y$.

Since Q_1 is countable, each E_α is a countable set.

Since $[0,1]$ is uncountable there are uncountable many sets E_α.

We consider a set V in $[0,1]$ containing just one element x_α from each E_α. Let $\{r_i\}$ be an enumeration of Q_1 and for each n write $V_n = V + r_n$.

Case (i):

$$V_n \cap V_m = \phi \quad \text{for } n \neq m.$$

Assume that $y \in V_n \cap V_m \Rightarrow y \in V_n$ & $y \in V_m$.

there exist $x_\alpha, x_\beta \in V$ such that $y = x_\alpha + r_n$ & $y = x_\beta + r_m$

$$x_\alpha + r_n = x_\beta + r_m$$

$$x_\beta - x_\alpha = r_n - r_m \in Q_1.$$

$$\therefore x_\beta - x_\alpha \in Q_1.$$

$$\text{(i.e)} \quad x_\beta = x_\alpha.$$

By definition of V and we have.

$$r_n = r_m \Rightarrow \boxed{n = m}$$

$$y \in V_n \cap V_m \quad \text{for } n = m.$$

So, $V_n \cap V_m = \phi$ for $n \neq m$

Case (ii):

$$[0,1] \subseteq \bigcup_{n=1}^{\infty} V_n.$$

Since $\forall x \in [0,1]$, $x \in E_\alpha$ for some α and then $x = x_\alpha + r_n$.

$$x \in V_n \Rightarrow x \in \bigcup_{n=1}^{\infty} V_n.$$

$$[0,1] \subseteq \bigcup_{n=1}^{\infty} V_n \subseteq [-1,2] \quad \text{--- (1)}$$

This is not a measurable set.

Case (iii):

To prove that: V is not a measurable set.

Suppose V is a measurable set.

$\Rightarrow V_n$ is also measurable

$$m(V) = m(V_n)$$

From (1) $\Rightarrow$ $$[0,1] \subseteq \bigcup_{n=1}^{\infty} V_n \subseteq [-1,2]$$

Taking measure on both sides,

$$m[0,1] \leq \sum_{n=1}^{\infty} m(V_n) \leq m[-1,2]$$

$$1 \leq \sum_{n=1}^{\infty} m(V_n) \leq 3.$$

$$1 \leq m(V_1) + m(V_2) + \cdots + m(V_n) + \cdots \leq 3.$$

$$\sum_{n=1}^{\infty} m(V_n) = \begin{cases} 0, & \text{if } m(V) = 0 \\ \infty, & \text{if } m(V) = 1 \end{cases}$$

which is a contradiction.

$\therefore V$ is not measurable.

Th: 18. Not every measurable set is a Borel set.

Proof:

Write each $x \in [0,1]$, in binary form

$$x = \sum_{n=1}^{\infty} \frac{\epsilon_n}{2^n}$$

with $\epsilon_n = 0$ or 1.

Choose a non-terminating expansion for each $x > 0$. Define the function 'f' by,

$$f(x) = \sum_{n=1}^{\infty} \frac{2\epsilon_n}{3^n}.$$

Then the values of 'f' are entirely in the cantor set P.

'f' is called a cantor's function.

Since ϵ_n is a measurable function of x, f is measurable.

Case (i):

f is a one-to-one mapping from $[0,1]$ onto its range. Since the value $f(x)$ defines the sequence $\{\epsilon_n\}$ in the expansion

$$\sum_{n=1}^{\infty} \frac{2\epsilon_n}{3^n} \text{ uniquely.}$$

So, x is determined uniquely.

Case (ii):

To prove that:

Not every measurable set is a Borel set or $[\mathcal{B} \subsetneq \mathcal{M}]$.

Suppose $\mathcal{B} = \mathcal{M}$. Then by known theorem:

" Let f be a measurable function & B is a Borel set then $f^{-1}(B)$ is a measurable set."

By this theorem:

$f^{-1}(B)$ would be measurable for any measurable set B and any measurable function f.

Let f be the cantor function and V a non-measurable set in $[0,1]$. Then $B = f(V)$ lies in P.

$$m^*(B) = m^*(f(V)) = 0.$$

$$m^*[f(V)] = 0. \Rightarrow m^*(B) = 0.$$

It has a measure zero. $f(V)$ is a measurable set.

(ie) B is measurable.

But since f is one-to-one, $f^{-1}(B) = V$ which is non-measurable.

$\therefore$ $\mathcal{B}$ is strictly contained in $\mathcal{M}$ $(\mathcal{B} \subseteq \mathcal{M})$

This is a contradiction.

Our assumption is wrong $\mathcal{B} \neq \mathcal{M}$

$\mathcal{B} \subsetneq \mathcal{M}$

$\therefore$ Every measurable set is not a Borel set.

UNIT - III

Inequalities & L^p - Spaces:.

Th: I Let $f, g \in L^p(\mu)$ and let a, b be const, Then $af + bg \in L^p(\mu)$.

Proof:-

Given $f, g \in L^p(\mu)$.

$\Rightarrow \int |f|^p d\mu < \infty$ and $\int |g|^p d\mu < \infty$.

Then, $af \in L^p(\mu)$.

Similarly $bg \in L^p(\mu)$

Let $af = F \in L^p(\mu)$ & $bg = G \in L^p(\mu)$

We have to prove that:

$F + G \in L^p(\mu)$

$|F+G|^p \leq 2^p \max\{|F|^p, |G|^p\}$

$\leq 2^p \{|F|^p + |G|^p\}$.

$\int |F+G|^p d\mu \leq 2^p \int (|F|^p + |G|^p) d\mu$.

$\leq 2^p \int |F|^p d\mu + 2^p \int |G|^p d\mu$.

$\int |F+G|^p d\mu < \infty$

$[\because G, F \in L^p(\mu)$

$\int |F|^p d\mu < \infty$

$\int |G|^p d\mu < \infty]$

$\therefore F + G \in L^p(\mu)$

$\Rightarrow af + bg \in L^p(\mu)$