UNIT - III

Comparision test:

Statement:

If $u_1+u_2+\cdots$ $+v_1+v_2\ldots$ are two series of positive terms and the second series is convergent and if $u_n \le kv_n$ (where k is a constant) for all values of n, then the first series is also convergent and its sum is less than or equal to k times that of the second.

Proof:

(i) Given Σv_n is convergent.

$$v_1+v_2\cdots = t \text{ (finite)}$$

To prove.

Σu_n is convergent.

Given $u_n \le kv_n$.

$$u_1+u_2+\cdots \le k(v_1+v_2+v_3+\cdots)$$

$$\le kt \text{ (finite } [kt])$$

$\therefore u_1+u_2+\cdots$ is convergent and the sum of the first series is less than or equal to sum of the second series.

ii) If v_n is divergent.

Proof:

i) Σv_n is divergent.

$u_n \ge kv_n$.

To prove Σu_n is divergent.

$$u_1+u_2+\cdots \ge k(v_1+v_2+\cdots)$$

$\therefore \Sigma u_n$ is divergent.

Theorem 4:

If $\sum v_n$ is convergent and u_n / v_n tends to a limit other than zero as $n \to \infty$ then $\sum u_n$ is convergent.

Proof:

let $\lim\limits_{n \to \infty} \frac{u_n}{v_n}$ be k [k being positive and $\neq 0$].

$\therefore$ on and after a certain value of n, say m, the values of the series of numbers.

$$\frac{u_n}{v_n}, \frac{u_{n+1}}{v_{n+1}}, \frac{u_{n+2}}{v_{n+2}}, \ldots$$

lie in the ~~in the~~ interval $k-\epsilon$ to $k+\epsilon$ where ϵ is a very small finite +ve quantity.

$$\therefore \frac{u_n}{v_n} < k+\epsilon, \quad \forall n \geq m.$$

$$u_n < (k+\epsilon) v_n, \quad \forall n \geq m.$$

$\therefore \sum v_n$ is convergent.

$\therefore \sum u_n$ is convergent.

Theorem: 5.

If $\sum v_n$ is divergent and $\frac{u_n}{v_n}$ tends to a limit other than zero as $n \to \infty$ then $\sum u_n$ is divergent.

Proof:

let $\lim\limits_{n \to \infty} \frac{u_n}{v_n}$ be k (k being positive and $\neq 0$)

$$k-\epsilon < \frac{u_n}{v_n} < k+\epsilon \quad \forall n \geq m$$

where ϵ is a small finite quantity.

$$\therefore \frac{u_n}{v_n} > k-\epsilon$$

$$\therefore u_n > (k-\epsilon) v_n \quad \forall \ n \geq m.$$

Σv_n is divergent.

Σu_n is divergent.

Problems:

1. Test the convergence of the series $\Sigma \frac{1}{\sqrt{n^2+1}}$

Solution:

$$\text{Let } u_n = \frac{1}{\sqrt{n^2+1}}$$

$$= \frac{1}{\sqrt{n^2[1+1/n^2]}}.$$

$$\text{let } v_n = \frac{1}{n}.$$

By comparison test,

$$\frac{u_n}{v_n} = \frac{1}{\sqrt{n^2+1}} \times \frac{n}{1} \quad \text{— ① as } n \to \infty$$

Hence Σu_n and Σv_n converge (or) diverge together.

But $\Sigma 1/n$ is divergent.

$\Sigma \left\{ \frac{1}{\sqrt{n^2+1}} \right.$ is divergent.

[$\because$ we know that $1 + 1/2 + 1/3 + 1/5 + \cdots$ is divergent].

2. Test the convergence of the series.

Q.P

$$\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \frac{7}{4.5.6} + \cdots$$

Solution:

Let $U_n = \dfrac{2n-1}{n(n+1)(n+2)}$

$= \dfrac{n(2-1/n)}{n^3(1+1/n)(1+\frac{2}{n})}$

$= \dfrac{(2-1/n)}{n^2(1+1/n)(1+2/n)}$

$= \dfrac{1}{n^2}\left[\dfrac{(2-1/n)}{(1+\frac{1}{n})(1+\frac{2}{n})}\right]$

Let $V_n = \dfrac{1}{n^2}$

By comparision test,

$\dfrac{U_n}{V_n} = \dfrac{(2n-1)/n(n+1)(n+2)}{1/n^2}$

$= \dfrac{(2n-1)n^2}{n(n+1)(n+2)}$

$= \dfrac{n^3(2-1/n)}{n^3(1+1/n)(1+2/n)}$

Take limit as $n \to \infty$

$\lim_{n\to\infty} \dfrac{U_n}{V_n} = \lim_{n\to\infty} \dfrac{(2-1/n)}{(1+1/n)(1+2/n)}$

$\lim_{n\to\infty} \dfrac{U_n}{V_n} = \dfrac{2}{1}$

$\lim_{n\to\infty} \dfrac{U_n}{V_n} = 2$

$\therefore \sum U_n$ & $\sum V_n$ converge (or) diverge together.

But $\sum 1/n^2$ is convergent.

$\sum U_n$ is convergent.

Hence proved.

3. Test for convergent or divergent of the series $\sum_{1}^{\infty} \sqrt{n^4+1} - \sqrt{n^4-1}$

Let $u_n = \sqrt{n^4+1} - \sqrt{n^4-1}$

$$u_n = \frac{(\sqrt{n^4+1} - \sqrt{n^4-1})(\sqrt{n^4+1} + \sqrt{n^4-1})}{(\sqrt{n^4+1} + \sqrt{n^4-1})}$$

$$= \frac{(\sqrt{n^4+1})^2 - (\sqrt{n^4-1})^2}{(\sqrt{n^4+1} + \sqrt{n^4-1})}$$

$$= \frac{n^4+1-n^4+1}{(\sqrt{n^4+1} + \sqrt{n^4-1})}$$

$$= \frac{2}{\sqrt{n^4+1} + \sqrt{n^4-1}} = \frac{2}{n^2\left(\sqrt{1+\frac{1}{n^4}} + \sqrt{1-\frac{2}{n^4}}\right)}$$

Let $v_n = \frac{1}{n^2}$ By comparision test.

$$\frac{u_n}{v_n} = \frac{\sqrt{n^4+1} - \sqrt{n^4-1}}{1/n^2}$$

$$\lim_{n\to\infty} \frac{u_n}{n} = \lim_{n\to\infty} \frac{2n^2}{n^2\left(\sqrt{1+\frac{1}{n^4}} + \sqrt{1-\frac{1}{n^4}}\right)} = 2$$

But $\sum \frac{1}{n^2}$ is convergent.

$\therefore u_n$ is convergent, so the given series is also convergent.

4. Prove that the series $\frac{1}{1\cdot3} + \frac{2}{3\cdot5} + \frac{3}{5\cdot7} + \frac{4}{7\cdot9} + \dots$ is divergent.

QP

Proof:

Let $u_n = \frac{n}{(2n-1)(2n+1)}$

$$u_n = \frac{n}{(2-\frac{1}{n})(2+\frac{1}{n})\cdot n^2}$$

$$u_n = \frac{1}{n(2-1/n)(2+1/n)}$$

let $v_n = \frac{1}{n}$.

But $u_n \le \frac{1}{n}$ is divergent.

$\therefore$ v_n is divergent. Hence u_n is divergent.

$\therefore$ The given series is divergent.

5. Discuss the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{(a+n)^p (b+n)^q} \quad a, b, p, q \text{ being all positive.}$$

Let $u_n = \frac{1}{(a+n)^p (b+n)^q}$

$$= \frac{1}{n^p \left(1+\frac{a}{n}\right)^p n^q \left(1+\frac{b}{n}\right)^q}$$

$$u_n = \frac{1}{n^{(p+q)} \left(1+\frac{a}{n}\right)^p \left(1+\frac{b}{n}\right)^q}.$$

Let $v_n = \frac{1}{n^{(p+q)}}$ By comparison test.

$$\frac{u_n}{v_n} = \frac{1}{n^{(p+q)} \left(1+\frac{a}{n}\right)^p \left(1+\frac{b}{n}\right)^q} \times \frac{n^{p+q}}{1}$$

$$= \frac{n^{p+q}}{n^{p+q} \left(1+\frac{a}{n}\right)^p \left(1+\frac{b}{n}\right)^q}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = 1.$$

Hence Σu_n and Σv_n converge or diverge together.

$\Sigma \frac{1}{n^{p+q}}$ is converges, if $p+q > 1$

$\sum \frac{1}{n^{p+q}}$ is diverges, if $p+q \leq 1$.

$\sum_{n=1}^{\infty} \frac{1}{(a+n)^p (b+n)^q}$ is converges, if $p+q > 1$.

$\sum_{n=1}^{\infty} \frac{1}{(a+n)^p (b+n)^q}$ is diverges, if $p+q \leq 1$.

6. Find whether the series in which $u_n = (n^3+1)^{1/3} - n$ is convergent or divergent

Let $u_n = (n^3+1)^{1/3} - n$

$= \left[n^3 \left(1+\frac{1}{n^3}\right)\right]^{1/3} - n$

$= (n^3)^{1/3} \left[1+\frac{1}{n^3}\right]^{1/3} - n$

$= n \left[1+\frac{1}{n^3}\right]^{1/3} - n$

$= n \left[\left(1+\frac{1}{n^3}\right)^{1/3} - 1\right]$

$= n \left[1+\frac{1}{3}\left(\frac{1}{n^3}\right) + \frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{n^3}\right)^2 + \cdots\right]$

$= n \left[1+\frac{1}{3n^3} - \frac{1}{9n^6} + \cdots\right]$

$= \frac{1}{3n^2} - \frac{1}{9n^5}$ [$\because$ neglecting higher power of n].

If we take $v_n = \frac{1}{n^2}$, we get.

$\frac{u_n}{v_n} = \frac{1}{3n^2} \times \frac{n^2}{1}$

$\lim_{n \to \infty} \frac{u_n}{v_n} = \frac{1}{3}$

But $\sum \frac{1}{n^2}$ is convergent.

$\therefore$ The given series is convergent.

7. S.T $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots$ is convergent.

Soln:

Given $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots$

let $u_{n+1} < \frac{1}{1.2.3\cdots n}$

$< \frac{1}{2^{n-1}}$

$1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots$ is convergent.

Since every term of the series less than or equal to the corresponding term of the convergent series.

$1+\frac{1}{1!}+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\cdots$

Theorem: 6

Show that the series $\frac{1}{1^k}+\frac{1}{2^k}+\frac{1}{3^k}+\cdots$ is convergent when k is greater than unity $(k>1)$ and divergent when k is equal to or less than unity.

QP

Solution:

Given series $\frac{1}{1^k}+\frac{1}{2^k}+\frac{1}{3^k}+\cdots$

case (i) : Let $k>1$.

[since each term of the series is less than the proceding term].

We have the following relation.

let $1/1^k = 1$

$$\frac{1}{2^K} + \frac{1}{3^K} < \frac{2}{2^K} < \frac{1}{2^{K-1}}$$

$$\frac{1}{4^K} + \frac{1}{5^K} + \frac{1}{6^K} + \frac{1}{7^K} < \frac{1}{4^{K-1}}$$

$$\frac{1}{8^K} + \frac{1}{9^K} + \dots + \frac{1}{15^K} < \frac{8}{8^K} < \frac{1}{8^{K-1}}$$

Adding we get.

$$\frac{1}{1^K} + \frac{1}{2^K} + \frac{1}{3^K} + \dots < 1 + \frac{1}{2^{K-1}} + \frac{1}{(2^2)^{K-1}} + \frac{1}{(2^3)^{K-1}}$$

$$< 1 + \frac{1}{2^{K-1}} + \left[\frac{1}{2^{K-1}}\right]^2 + \left[\frac{1}{2^{K-1}}\right]^3 + \dots$$

But the series is Geometric series whose common ratio is $\frac{1}{2^{K-1}}$

which is $\frac{1}{2^{K-1}} < 1$

Already we know that.

$1 + x + x^2 + x^3 + \dots$ for $x < 1$ is convergent

$\therefore$ The given series is convergent.

Case (ii) : $K = 1$

$K = 1$ we can group the series as follows.

$\frac{1}{1} = 1 \quad 1 \quad \frac{1}{2} = \frac{1}{2}$

$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$

- - - - - -

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots$

But $1 + \frac{1}{2} + \frac{1}{2} + \dots$ is divergent.

$\therefore$ The given series is $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$

is divergent.

Case (iii): $K<1$

let $\frac{1}{n^K} > \frac{1}{n}$

$\frac{1}{1^K} > \frac{1}{1}$

$\frac{1}{2^K} > \frac{1}{2}$

$\frac{1}{3^K} > \frac{1}{3}$

Adding we get,

$1/1^K + \frac{1}{2^K} + 1/3^K + \cdots > 1 + 1/2 + 1/3 + 1/4 + \cdots$

But $1 + 1/2 + 1/3 + 1/4 + \cdots$ is divergent.

$1/n^K$ is divergent.

Hence the given series $1/n^K$ is convergent if $K>1$ and $1/n^K$ is divergent if $K<1$.

Theorem: 8

D. Alembert Ratio Test.

Statement:

A series is convergent if after any particular term, the ratio of each term to the proceeding is always less than some fixed quantity which is less than unity.

Proof:

let the ratio of each term after the r th to proceeding term be less than x.

where $K < 1$.

Then $\frac{U_{r+1}}{U_r} < K$, $\frac{U_{r+2}}{U_{r+1}} < K$, $\frac{U_{r+3}}{U_{r+2}} < K$,

Since $\frac{U_{r+1}}{U_r} < K$.

$U_{r+1} < K . U_r$

$U_{r+2} < K . U_{r+1} < K . K U_r < K^2 . U_r$.

$U_{r+3} < K . U_{r+2} < K^3 . U_r$.

$U_{r+4} < K . U_{r+3} < K^4 U_r$.

. . . . . .

Adding we get,

$U_r + U_{r+1} + U_{r+2} + \cdots < U_r + K U_r + K^2 U_r + K^3 U_r + \cdots$

[∵ Adding U_r on both sides]

$< U_r (1 + K + K^2 + K^3 + \cdots)$

$< U_r (1-K)^{-1}$

$< \frac{U_r}{(1-K)}$, $K < 1$.

Hence the sum of the terms beginning at the r^{th} term is finite.

Since the nature of the series is unaffected by omitting a finite number of the terms in the beginning.

The series Σu_n is convergent.

Corrollary:

1) If $\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = K$ where $K < 1$ when

Σu_n is convergent.

Proof:

Given $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = k$ (where $k<1$)

To prove that $\sum u_n$ is convergent

Now $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = k$.

We can find a natural number m. so large that $\frac{u_{n+1}}{u_n}$ differs from k by less than so long as $n \geq m$, $\frac{u_{n+1}}{u_n} \leq k+\epsilon$ for $n \geq m$.

$k<1$ and hence we can use ϵ positive and sufficiently small

$\ni : k+\epsilon < 1$.

$$\frac{u_{n+1}}{u_n} < k+\epsilon$$

$$\frac{u_{n+1}}{u_n} < 1 \quad \text{(by theorem 8)}$$

The series is convergent.

Theorem 9:

A series is divergent if after any particular term the ratio of each term to the proceeding term is either equal to unity or greater than unity.

Proof:

Let all the terms after the r^{th} term to be equal u_r, then

$U_{r+1} + U_{r+2} + \cdots + U_{n+r} = nU_r.$

Taking limit on both sides as $n \to \infty$

$$\lim_{n\to\infty} U_{r+1} + U_{r+2} + \cdots + U_{n+r} = \infty$$

Σu_n is divergent.

Let the ratio of each term after the r^{th} term to be proceeding term be greater than unity.

$$\frac{U_{r+1}}{U_r} > 1.$$

$$U_{r+1} > U_r$$

Then $U_{r+1} > U_r$, $U_{r+2} > U_r$, $U_{r+3} > U_r \cdots\cdots$

Adding we get.

$$U_{r+1} + U_{r+2} + \cdots + U_{n+r} > n.U_r$$

$$\lim_{n\to\infty} U_{r+1} + U_{r+2} + \cdots + U_{n+r} > \lim_{n\to\infty} n.U_r = \infty$$

The series is divergent.

Corallary:

If $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = K$ where $K > 1$ then the series Σu_n is divergent.

Proof:

Given $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = K$ (where $K > 1$)

To prove that Σu_n is divergent.

Now $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = K.$

We can find a natural number m.

so large that $\frac{U_{n+1}}{U_n}$ differs from k by greater than ϵ so long as $n \geq m$

$$\frac{U_{n+1}}{U_n} > K+\epsilon \quad \text{for } n \geq m.$$

$K > 1$ and have we can be use ϵ positive and sufficiently small $\ni$: $K+\epsilon > 1$.

$$\frac{U_{n+1}}{U_n} > K+\epsilon$$

$\frac{U_{n+1}}{U_n} > 1$ $\therefore$ the series is divergent.

Note: 1

* $K < 1$ The series is convergent.
* $K > 1$ the series is divergent.
* $K = 1$ the De'Alembert's ratio test fails.

Note: 2

If $\lim_{n \to \infty} \frac{U_{n+1}}{U_n} = 1$. The series may be convergent or it may be divergent and we say that the test fails.

Consider the series.

$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ (dgt) and $1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$ (cgt).

The first series is divergent and the second series is convergent and in bothe we have $\lim_{n \to \infty} \frac{U_{n+1}}{U_n} = 1$.

When $\lim_{n \to \infty} \frac{U_{n+1}}{U_n} = 1$ the natural of the series is generally consider from elementary principle or form comparison test.

Problems:

1. Test for convergency of the series $\sum_{n=0}^{\infty} \frac{n^3+1}{2^n+1}$

Solution:

$$\text{Let } U_n = \frac{n^3+1}{2^n+1}$$

$$U_{n+1} = \frac{(n+1)^3+1}{2^{n+1}+1}$$

By ratio test.

$$\frac{U_{n+1}}{U_n} = \frac{(n+1)^3+1}{2^{n+1}+1} \times \frac{2^n+1}{n^3+1}$$

$$= \frac{(n+1)^3+1}{2^n \cdot 2^1+1} \times \frac{2^n+1}{n^3+1}$$

$$= \frac{n^3\left[(1+1/n)^3 + 1/n^3\right]}{n^3(1+1/n^3)} \times \frac{2^n(1+1/2^n)}{2^n(2+1/2^n)}$$

$$= \frac{(1+1/n)^3 + 1/n^3}{1+1/n^3} \times \frac{1+1/2^n}{2+1/2^n}$$

Taking limit as $n \to \infty$

$$\lim_{n \to \infty} \frac{U_{n+1}}{U_n} = \frac{1}{1} \times \frac{1}{2} = 1/2 < 1$$

The series is convergent.

2. Examine the convergently of the series

Q.P $\sum_{1}^{\infty} \left(\frac{n}{n+1}\right)^{1/2} x^n$

solution:

$$\text{let } u_n = \left[\frac{n}{n+1}\right]^{1/2} x^n$$

$$u_{n+1} = \left[\frac{n+1}{n+2}\right]^{1/2} . x^{n+1}.$$

By Ratio's test.

$$\frac{u_{n+1}}{u_n} = \left[\frac{n+1}{n+2}\right]^{1/2} . x^{n+1} \times \frac{(n+1)^{1/2}}{n^{1/2} . x^n}$$

$$= \frac{(n+1)^{1/2} . x^{n+1}}{(n+2)^{1/2}} \times \frac{(n+1)^{1/2}}{n^{1/2} . x^n}$$

$$= \frac{n^{1/2}(1+1/n)^{1/2} . x^n . x}{n^{1/2}(1+2/n)^{1/2}} \times \frac{(1+1/n)^{1/2} . n^{1/2}}{n^{1/2} . x^n}$$

$$= \frac{(1+1/n)^{1/2} . x}{(1+2/n)^{1/2}} \times \frac{(1+1/n)^{1/2}}{1}.$$

Taking limit as $n \to \infty$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1/1 \times x \times 1$$

$$= x.$$

$x < 1$, The series is convergent.

$x > 1$, The series is divergent.

$x = 1$, The test fails

If $x = 1$ the series becomes.

$$\sum_1^\infty \left(\frac{n}{n+1}\right)^{1/2} x^n = \sum_1^\infty \left(\frac{n}{n+1}\right)^{1/2} (1)^n$$

$$= \sum_1^\infty \left(\frac{n}{n+1}\right)^{1/2}$$

$$\text{let } u_n = \left(\frac{n}{n+1}\right)^{1/2}$$

$$= \frac{n^{1/2}}{n^{1/2}(1+1/n)^{1/2}}$$

$$= \frac{1}{(1+1/n)^{1/2}}$$

$\lim_{n \to \infty} U_n = 1$.

But we Know in a convergent series the n^{th} term tends to zero as $n \to \infty$.

$\therefore$ The series is divergence.

Hence $x < 1$, the series is convergent

$x \geq 1$ The series is divergent.

QP

Raabe's Test:

The series whose general term is U_n is convergent (or) divergent according as

$\lim_{n \to \infty} \left\{ n \left(\frac{U_n}{U_{n+1}} - 1 \right) \right\} > 1$ or < 1.

Proof:

Case (1) First we prove that

If ΣU_n and ΣV_n are two series of positive terms and if.

$\frac{U_{n+1}}{U_n} < \frac{V_{n+1}}{V_n}$, $\forall n$ after a certain stage.

Show that ΣU_n will convergent if ΣV_n is converges.

Since the omission of a finite number of terms from a series does not effect convergence

We can assume that the inequality holds for all positive integral value of 'n'

$\frac{U_2}{U_1} < \frac{V_2}{V_1}$; $\frac{U_3}{U_2} < \frac{V_3}{V_2}$; $\frac{U_4}{U_3} < \frac{V_4}{V_3}$, ...

Now,

$U_1 + U_2 + U_3 + \cdots = U_1 \left(1 + \frac{U_2}{U_1} + \frac{U_3}{U_1} + \frac{U_4}{U_1} + \cdots \right)$

$$= u_1\left[1+\frac{u_2}{u_1}+\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}+\frac{u_4}{u_3}\cdot\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}\cdots\right]$$

$$< u_1\left[1+\frac{v_2}{v_1}+\frac{v_3}{v_2}\cdot\frac{v_2}{v_1}+\frac{v_4}{v_3}\cdot\frac{v_3}{v_2}\cdot\frac{v_2}{v_1}+\cdots\right]$$

$$< u_1\left(\frac{v_1+v_2+v_3+\cdots}{v_1}\right)$$

$$< \frac{u_1}{v_1}(v_1+v_2+v_3+\cdots)$$

Since $\frac{u_1}{v_1}$ is convergent.

Σv_n is convergent. It follows that Σu_n is convergent.

Case (ii)

Next we prove that Σu_n is divergent and $\frac{u_{n+1}}{u_n} > \frac{v_{n+1}}{v_n}$ then Σu_n is divergent if Σv_n is devergent and $\frac{u_{n+1}}{u_n} > \frac{v_{n+1}}{v_n}$.

Since the omission of a finite number of terms from a series does not effect divergent.

We can assume that the inequality holds for all positive integral values of 'n'.

$$\frac{u_2}{u_1} > \frac{v_2}{v_1},\quad \frac{u_3}{u_2} > \frac{v_3}{v_2},\quad \frac{u_4}{u_3} > \frac{v_4}{v_3}\cdots$$

Now,

$$u_1+u_2+u_3+\cdots = u_1\left[1+\frac{u_2}{u_1}+\frac{u_3}{u_1}+\frac{u_4}{u_1}+\cdots\right]$$

$$= u_1\left[1+\frac{u_2}{u_1}+\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}+\frac{u_4}{u_3}\cdot\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}+\cdots\right]$$

$$> u_1\left[1+\frac{v_2}{v_1}+\frac{v_3}{v_2}\cdot\frac{v_2}{v_1}+\frac{v_4}{v_3}\cdot\frac{v_3}{v_2}\cdot\frac{v_2}{v_1}+\cdots\right]$$

$> u_1 \left[\frac{v_1 + v_2 + v_3 + \cdots}{v_1} \right]$

$> \frac{u_1}{v_1} [v_1 + v_2 + v_3 + \cdots]$

Since $\frac{u_1}{v_1}$ is divergent Σv_n is divergent.

It follows that Σu_n is divergent.

Step 8:

Let us compare the series Σu_n with the series $\Sigma \frac{1}{n^p}$

To prove that $\Sigma \frac{1}{n^p}$ is convergent when $p > 1$.

$\Sigma \frac{1}{n^p}$ is divergent when $p \leq 1$

Proof:

Step 1:

Σu_n is convergent

$\frac{u_{n+1}}{u_n} = \frac{1/(n+1)^p}{1/n^p}$

$\frac{u_{n+1}}{u_n} = \frac{n^p}{n^p(1+1/n)^p}$

$u_n > (1+1/n)^p$

$\frac{u_n}{u_{n+1}} > p/n + \frac{p(p-1)}{2} (1/n)^2 + \cdots$

[$\because$ Ascending Binomial theorem for rational index].

$\frac{u_n}{u_{n+1}} - 1 > 1/n \left[p + \frac{p(p-1)}{2n} + \cdots \right]$

$n \left[\frac{u_n}{u_{n+1}} - 1 \right] > p + \frac{p(p-1)}{2n} + \cdots$

Taking limit as $n \to \infty$.

$\lim_{n \to \infty} \left\{ n \left(\frac{u_n}{u_{n+1}} - 1 \right) \right\} > p.$

But the auxillary series is convergent if $p > 1$

$\therefore \sum u_n$ is convergent if $p > 1$

Step 2:

$\sum u_n$ is divergent

$$\frac{u_{n+1}}{u_n} > \frac{1/(n+1)^p}{1/n^p}$$

$$\frac{u_{n+1}}{u_n} > \frac{n^p}{n^p(1+1/n)^p}$$

$$\frac{u_{n+1}}{u_n} > \frac{1}{(1+1/n)^p}$$

$$\frac{u_n}{u_{n+1}} < (1+1/n)^p$$

$$\frac{u_n}{u_{n+1}} < 1+p/n + \frac{p(p-1)}{2}(1/n)^2 + \cdots$$

$$\frac{u_n}{u_{n+1}} - 1 < 1/n\left[p + \frac{p(p-1)}{2n} + \cdots\right]$$

taking limit as $n \to \infty$

$$\lim_{n\to\infty}\left\{n\left[\frac{u_n}{u_{n+1}} - 1\right]\right\} < p.$$

But the auxillary series is divergent if $p < 1$.

$\therefore \sum u_n$ is divergent if $p < 1$.

Problem 8:

1. Discuss the convergency of the series

Q.P

$$1 + \frac{(1!)^2}{2!}x + \frac{(2!)^2}{4!}x^2 + \frac{(3!)^2}{6!}x^3 + \cdots$$

Soln:

$$\text{let } u_n = \frac{(n!)^2 x^n}{(2n!)}$$

$$u_{n+1} = \frac{[(n+1)!]^2 x^{n+1}}{2(n+1)!}$$

By ratio test

$$\frac{u_{n+1}}{u_n} = \frac{[(n+1)!]^2 x^{n+1}}{(2n+2)!} \times \frac{(2n)!}{(n!)^2 x^n}$$

$$= \frac{[(n+1)!]^2 . x^n . x . 2n!}{(2n+2)! (n!)^2 x^n}$$

$$= \frac{(n+1)^2 (n!)^2 . x^n . x . 2n!}{(2n+2)(2n+1)(2n)! x^n (n!)^2}$$

$$= \frac{x(n+1)^2}{(2n+2)(2n+1)}$$

$$= \frac{x(n+1)(n+1)}{2(n+1)(2n+1)}$$

$$\frac{u_{n+1}}{u_n} = \frac{x(n+1)}{2(2n+1)}$$

taking limit as $n \to \infty$.

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)x}{(2n+1)(2)}$$

$$= \lim_{n\to\infty} \frac{n(1+1/n)x}{2n(2+1/n)}$$

$$= \lim_{n\to\infty} 1/2 \left[\frac{(1+1/n)x}{2+1/n}\right]$$

$$= 1/2 \, x . 1/2$$

$$= x/4$$

$x/4 < 1$

$x < 4$

* If $x < 4$ is convergent
* If $x > 4$ is divergent
* If $x = 4$ is test fails.

we can apply Raabee's test.

when $x = 4$

$$\frac{u_n}{u_{n+1}} - 1 = \frac{2(2n+1)}{(n+1)4} - 1$$

$$= \frac{(2n+1)}{2(n+1)} - 1$$

$$\frac{u_n}{u_{n+1}} - 1 = \frac{2n+1-2n-2}{2(n+1)}$$

$$n\left[\frac{u_n}{u_{n+1}} - 1\right] = \frac{-1}{2(n+1)} \cdot n$$

$$n\left[\frac{u_n}{u_{n+1}} - 1\right] = \frac{-n}{2(n+1)}$$

$$= \frac{-n}{2n(1+1/n)} = \frac{-1}{2(1+1/n)}$$

$$\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} = \lim_{n\to\infty}\left\{\frac{-1}{2(1+1/n)}\right\}$$

$= -1/2 < 1 \Rightarrow$ divergent.

Finally.

$x \geq 4$ divergent.

$x < 4$ convergent.

Theorem : 12. [Logrithemic test]

statement:

The series whose general term is u_n is convergent or divergent according as $\lim_{n\to\infty}\left[n \log\left(\frac{u_n}{u_{n+1}}\right)\right]$ is greater than one or less than one.

Proof:

Let us compare the given series with the series whose general term is $1/n^p$ when $p>1$, $\Sigma 1/n^p$ is convergent.