Theorem : 12. [Logrithemic test]

statement:

The series whose general term is u_n is convergent or divergent according as $\lim_{n \to \infty} [n \log(\frac{u_n}{u_{n+1}})]$ is greater than one or less than one.

Proof:

Let us compare the given series with the series whose general term is $1/n^p$ when $p>1$, $\sum 1/n^p$ is convergent.

$\sum u_n$ is convergent

If $\frac{u_{n+1}}{u_n} < \frac{n^p}{(n+1)^p}$

$$\frac{u_n}{u_{n+1}} > \frac{(n+1)^p}{n^p}$$

$$> \frac{n^p (1+1/n)^p}{n^p}$$

$$> (1+1/n)^p$$

(ie) if $\log\left(\frac{u_n}{u_{n+1}}\right) > \log(1+1/n)^p$

$> p \log(1+1/n)$ $[\because \log m^n = n \log m]$

(ie) $\log\left(\frac{u_n}{u_{n+1}}\right) > p\left(1/n - 1/2n^2 + \dots\right)$

$$> p/n - p/2n^2 + \dots$$

$$n \log\left(\frac{u_n}{u_{n+1}}\right) > p - \frac{p}{2n} + \dots$$

$$\lim_{n\to\infty} \left(\frac{u_n}{u_{n+1}}\right) > p \quad \text{Here } p > 1$$

$\therefore \sum u_n$ is convergent.

If $\lim_{n\to\infty} \left[n \log\left(\frac{u_n}{u_{n+1}}\right)\right] > 1$.

Similarly we can show that $\sum u_n$ is divergent

$$\lim_{n\to\infty} \left[n \log\left(\frac{u_n}{u_{n+1}}\right)\right] < 1.$$

1. Test for convergency and divergency series

$$1 + \frac{2x}{2!} + \frac{3^2x^2}{3!} + \frac{4^3x^3}{4!} + \dots$$

$$u_n = \frac{n^{n-1} x^{n-1}}{n!}$$

$$u_{n+1} = \frac{(n+1)^n \cdot x^n}{(n+1)!}$$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^n x^n}{(n+1)!} \times \frac{n!}{n^{n-1} x^{n-1}}$$

$$= \frac{n^n (1+1/n)^n x^n}{(n+1)\, n!} \times \frac{n!}{n^n n^{-1} x^n x^{-1}}$$

$$= \frac{(1+1/n)^n nx}{(n+1)} = \frac{(1+1/n)^n \cdot nx}{\cancel{n}(1+1/n)}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(1+1/n)^n x}{(1+1/n)} \quad \lim_{n\to\infty} (1+1/n)^n = e$$

$$= xe \quad \begin{array}{l} < 1 \text{ convergent} \\ > 1 \text{ divergent} \\ = 1 \text{ test fails.} \end{array}$$

$$\frac{u_{n+1}}{u_n} = (1+1/n)^{n-1} x.$$

$$\frac{u_n}{u_{n+1}} = \frac{1}{x(1+1/n)^{n-1}} = \frac{e}{\left(\frac{n+1}{n}\right)^{n-1}} \quad [\because \log e = 1]$$

$$\log \frac{u_n}{u_{n+1}} = \log \left[\frac{e}{\left(\frac{n+1}{n}\right)^{n-1}} \right]$$

$$= \log e - \log\left(\frac{n+1}{n}\right)^{n-1} \quad [\because \log(1+x)]$$

$$= 1 - (n-1)\log(1+1/n)$$

$$= 1 - (n-1)\left[1/n - 1/2n^2 + 1/3n^2 + \cdots\right]$$

$$= 1 - \frac{n-1}{n} + \frac{n-1}{2n^2} - \frac{(n-1)}{3n^3} + \cdots$$

$$= 1 - 1 + \frac{1}{n} + \frac{(n-1)}{2n^2} \cdots$$

$$= \frac{1}{n} + \frac{n}{2n^2} - \frac{1}{2n^2} - \frac{n-1}{3n^3} + \cdots$$

$$n \log \frac{u_n}{u_{n+1}} = \frac{n}{n} + \frac{n}{2n} - \frac{n}{2n^2} - \frac{n(n-1)}{2n^3} + \cdots$$

$$= 1 + \frac{1}{2} - \frac{1}{2n} - \frac{(n-1)}{2n^2} + \cdots$$

$$\lim_{n\to\infty} n\log\left[\frac{u_n}{u_{n+1}}\right] = \lim_{n\to\infty} 1+\frac{1}{2}-\frac{1}{2n}-\cdots$$

$$= 1+\frac{1}{2}$$

$$= \frac{3}{2} > 1 \Rightarrow \text{convergent.}$$

The given series is convergent for the case $x = 1/e$.

Finally we conclude that $x < 1/e$ the given series is converges.

$x > 1/e$ the given series is divergent.

Theorem: 3

Statement:

The series whose general term is u_n is convergent or divergent according as

$$\lim_{n\to\infty} \left[\left(n\left(\frac{u_n}{u_{n+1}}\right)-1\right)\log n\right] > 1 \text{ or } < 1$$

Proof:

Compare the series $\sum \frac{1}{n(\log n)^p}$.

This convergent if $p>1$ and divergent if $p \leq 1$.

This series $\sum u_n$ is convergent.

$$\text{if } \left(\frac{u_{n+1}}{u_n} < \frac{(n)(\log n)^p}{(n+1)(\log(n+1))^p}\right)$$

$$\Rightarrow \frac{u_n}{u_{n+1}} > \frac{n+1}{n}\left[\frac{\log(n+1)}{\log n}\right]^p \quad \text{--- (1)}$$

Now $\log(n+1) - \log n = \log\left[\frac{n+1}{n}\right]$

$$= \log\left[\frac{n}{n}+\frac{1}{n}\right]$$

$$= \log\left[1+\frac{1}{n}\right]$$

$$= 1+\frac{1}{2n^2}+\cdots$$

$= \frac{1}{n}$ [nearly when n is large]

$\therefore \log(n+1) - \log n = \frac{1}{n}$

$\log(n+1) = \log n + \frac{1}{n}$ — ①

[nearly for large value of n]

$\therefore$ this series is convergent.

① implies

If $\frac{u_n}{u_{n+1}} > \frac{n+1}{n}\left\{\frac{\log(n+1)}{\log n}\right\}^p$ [for large value of n]

$> \left(\frac{n+1}{n}\right)\left\{\frac{\log n + 1/n}{\log n}\right\}^p$ [$\because$ By ①].

$> (1+1/n)\left[\frac{\log n}{\log n} + \frac{1/n}{\log n}\right]^p$

$> (1+1/n)\left[1 + \frac{1}{n\log n}\right]^p$

$> \left(1+\frac{1}{n}\right)\left[1 + \frac{p}{n} \times \frac{1}{\log n} + \cdots\right]$

$> \left(1 + 1/n + \frac{p}{n\log n} + \cdots\right)$

(ie) If $\frac{u_n}{u_{n+1}} > 1 + \frac{1}{n} + \frac{p}{n\log n} + \cdots$

if $\frac{u_n}{u_{n+1}} - 1 > \frac{1}{n} + \frac{p}{n\log n} + \cdots$

if $n\left(\frac{u_n}{u_{n+1}} - 1\right) > 1 + \frac{p}{\log n} + \cdots$

if $\left(n\left[\frac{u_n}{u_{n+1}} - 1\right] - 1\right) > \frac{p}{\log n} + \cdots$

$\log n \left[n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right] > p$

$\therefore \Sigma u_n$ is convergent.

If $\lim_{n\to\infty} \log n \left[n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right] > 1$.

||ly we can prove that $\sum u_n$ is divergent

If $\lim_{n\to\infty} \log n \left[n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right] < 1$.

Logrithmic test problems.

Examine the convergently of $\frac{1^2}{2^2}+\frac{1^2.3^2}{2^2.4^2}+\cdots$

Let us apply the result (3)

$$u_n = \frac{1^2.3^2\ldots(2n-1)^2}{2^2.4^2\ldots(2n)^2}$$

$$u_{n+1} = \frac{1^2.3^2\ldots(2n-1)^2(2n+1)^2}{2^2.4^2\ldots(2n)^2(2n+2)^2}$$

$$\frac{u_n}{u_{n+1}} = \frac{1^2.3^2\ldots(2n-1)^2}{2^2.4^2\ldots(2n)^2} \times \frac{2^2.4^2\ldots(2n)^2(2n+2)^2}{1^2.3^2\ldots(2n-1)^2(2n+1)^2}$$

$$= \frac{(2n+2)^2}{(2n+1)^2} \Rightarrow \frac{u_n}{u_{n+1}}-1$$

$$= \frac{(2n+2)^2}{(2n+1)^2}-1 \Rightarrow \frac{(2n+2)^2-(2n+1)^2}{(2n+1)^2}$$

$$= \frac{4n^2+8n+4-4n^2-4n-1}{(2n+1)^2} = \frac{4n+3}{(2n+1)^2}.$$

$$n\left(\frac{u_n}{u_{n+1}}-1\right)-1 = n\left(\frac{4n+3}{(2n+1)^2}\right)-1$$

$$= \frac{4n^2+3n-4n^2-4n-1}{(2n+1)^2} = \frac{-n-1}{(2n+1)^2}$$

$$\lim_{n\to\infty}\left[n\left(\frac{u_n}{u_{n+1}}-1\right)\right]\log n = \lim_{n\to\infty} \frac{-(n+1)}{(2n+1)^2}\log n.$$

$$= \lim_{n\to\infty} \frac{-(n+1)}{n^2(2+1/n)^2}\log n.$$

$$=0$$

$< 1 \Rightarrow$ divergent.

The given series is divergent.

H.W.

1. Examine the convergence of the series of $\frac{1}{1^k} + \frac{x}{3^k} + \frac{x^2}{5^k} + \cdots + \frac{x^{n-1}}{(2n-1)^k} + \cdots$

Soln:

$$U_n = \frac{x^{n-1}}{(2n-1)^k}$$

$$U_{n+1} = \frac{x^{(n+1)-1}}{[2(n+1)-1]^k} = \frac{x^n}{[2n+2-1]^k} = \frac{x^n}{[2n+1]^k}$$

$$\therefore \frac{U_{n+1}}{U_n} = \frac{x^n}{(2n+1)^k} \times \frac{(2n-1)^k}{x^{n-1}}$$

$$\frac{U_{n+1}}{U_n} = \frac{(2n-1)^k}{(2n+1)^k} \times \frac{\cancel{x^n}}{\cancel{x^n} \cdot x^{-1}} = \frac{(2n-1)^k}{(2n+1)^k} \times \frac{x}{1}$$

$$= \frac{n^k(2 - 1/n^k)}{n^k(2 + 1/n^k)} \cdot x$$

$$\frac{U_{n+1}}{U_n} = \frac{(2 - 1/n^k)}{(2 + 1/n^k)} \cdot x.$$

Taking limit on both sides as $n \to \infty$,

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{(2 - 1/n^k)}{(2 + 1/n^k)} \cdot x$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \frac{\cancel{2} \cdot x}{\cancel{2}} \Rightarrow \boxed{\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = x}$$

Hence if $x < 1$, the series is convergent.

if $x > 1$, the series is divergent.

if $x = 1$, the test fails.

In that case the series becomes

$$\frac{1}{1^k} + \frac{1}{3^k} + \frac{1}{5^k} + \cdots + \frac{1}{(2n-1)^k} +$$

compare the series with the series

$$\frac{1}{1^k} + \frac{1}{2^k} + \frac{1}{3^k} + \cdots + \frac{1}{n^k}$$

Σv_n is the second series

& $$\frac{u_n}{v_n} = \frac{1}{(2n-1)^k} \times \frac{n^k}{1}$$

$$= \frac{n^k}{n^k (2 - 1/n)^k}$$

$$\boxed{\frac{u_n}{v_n} = \frac{1}{(2 - 1/n)^k}}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{(2-\frac{1}{n})^k}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{2^k} = \text{a finite quantity}$$

$\therefore$ Σu_n & Σv_n behave alike.

$\therefore$ If $x < 1$, $\forall k$ the series is convergent.

If $x > 1$, $\forall k$ the series is divergent.

If $x = 1$, for $k > 1$, the series is convergent

for $k < 1$, The series is divergent.

2. Discuss the convergency of the series.

$$\frac{1}{1+x} + \frac{1}{1+2x^2} + \frac{1}{1+3x^3} + \frac{1}{1+4x^4} + \cdots$$ for +ve values of x.

Soln:

$$u_n = \frac{1}{1+nx^n}$$

$$u_{n+1} = \frac{1}{1+nx^{n+1}} = \frac{1}{1+(n+1)x^{n+1}}$$

$$\frac{u_{n+1}}{u_n} = \frac{1+nx^n}{1+(n+1)x^{n+1}}$$

$$= \frac{nx^n \left[1 + 1/nx^n\right]}{nx^n \left[1/nx^n + x + \frac{x}{n}\right]}$$

$$= \frac{nx^n\left[1+\frac{1}{nx^n}\right]}{nx^n\left\{\frac{1}{nx^n}+\left[1+\frac{1}{n}\right]x\right\}}$$

$$= \frac{\left[1+\frac{1}{nx^n}\right]}{\frac{1}{nx^n}+\left[1+\frac{1}{n}\right]\cdot x}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{\left[1+\frac{1}{nx^n}\right]}{\frac{1}{nx^n}+\left[1+\frac{1}{n}\right]x}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \frac{1}{x}.$$

If $x>1$, $\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n} < 1$

The series is convergent when $x>1$.

If $x=1$ the series becomes $\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$ which is divergent.

If $x<1$, $1+x<2$ $\therefore \frac{1}{1+x}>\frac{1}{2}$.

$1+2x^2<3$ $\therefore \frac{1}{1+2x^3}>\frac{1}{3}$

$1+3x^3<4$ $\therefore \frac{1}{1+3x^3}>\frac{1}{4}$.

$\therefore \frac{1}{1+x}+\frac{1}{1+2x^2}+\frac{1}{1+3x^3}+\cdots > \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$

$\therefore$ the series is divergent

3. settle the range of values of x for which the following series converge:

i) $\sum \frac{x^n}{1+x^n}$.

soln:

$$u_n = \frac{x^n}{1+x^n}$$

$$u_{n+1} = \frac{x^{n+1}}{1+x^{n+1}}.$$

$$\frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{1+x^{n+1}} \times \frac{1+x^n}{x^n}$$

$$= \frac{1+x^n}{1+x^{n+1}} \times x$$

If $x < 1$,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{1+x^n}{1+x^{n+1}} \times x$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = x. \text{ which is } < 1.$$

$\therefore \sum u_n$ is convergent

$x > 1$.

$$\lim_{n\to\infty} \cancel{\frac{u_{n+1}}{u_n}}\ u_n = \lim_{n\to\infty} \frac{x^n}{1+x^n}.$$

$$= \frac{x^n}{x^n\left[1+1/x^n\right]} = \frac{1}{1+1/x^n}.$$

$$\lim_{n\to\infty} u_n = 1$$

The nth term does not tend to zero as $n\to\infty$

$\therefore \sum u_n$ is divergent.

If $x=1$, the series becomes $\frac{1}{2}+\frac{1}{2}+\dots$

which is divergent

$\therefore \sum \frac{x^n}{1+x^n}$ is convergent only when $0 < x < 1$.

ii) $\sum \frac{x^n}{1+n^2x^{2n}}.$

Soln:

$$u_n = \frac{x^n}{1+n^2x^{2n}}.$$

$$u_{n+1} = \frac{x^{n+1}}{1+(n+1)^2 x^{2(n+1)}}$$

$$= \frac{x^{n+1}}{1+(n+1)^2 x^{2n+2}}.$$

$$\frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{1+(n+1)^2 x^{2n+2}} \times \frac{1+n^2x^{2n}}{x^n}$$

$$= \frac{1+n^2x^{2n}}{1+(n+1)^2 x^{2n+2}} \times x.$$

If $x<1$,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{1+n^2x^{2n} \times x}{1+(n+1)^2 x^{2n+2}}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = x < 1$$

$\therefore$ Σu_n is convergent

If $x>1$,

$$\frac{u_{n+1}}{u_n} = \frac{\frac{1}{n^2x^{2n}}\left[1+ \frac{1}{n^2x^{2n}}\right]}{\frac{1}{n^2x^{2n}}\left[\frac{1}{n^2x^{2n}} + x^2 + \frac{x^2}{n}\right]} x$$

$$= \frac{1+ \frac{1}{n^2x^{2n}}}{\frac{1}{n^2x^{2n}} + \left[1+\frac{1}{n}\right]x^2} x$$

$\frac{1}{x^{2n}} \to 0$ as $n\to\infty$ when $x>1$.

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{1+\frac{1}{n^2x^{2n}}}{\frac{1}{n^2x^{2n}} + \left[1+\frac{1}{n}\right]x^2} x.$$

$$= \frac{x}{x^2} = \frac{1}{x} < 1 \quad \boxed{\text{since } x>1}$$

$\therefore$ Σu_n is convergent.

If $x=1$. the series becomes $\Sigma \frac{1}{1+n^2}$ which is convergent.

$\therefore$ $\Sigma \frac{x^n}{1+n^2x^{2n}}$ is convergent, $\forall x$.

iii) $\Sigma \frac{x^n}{1+x^{2n}}$

Soln:

$$u_n = \frac{x^n}{1+x^{2n}}$$

$$u_{n+1} = \frac{x^{n+1}}{1+x^{2n+2}}$$

$$\frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{1+x^{2n+2}} \times \frac{1+x^{2n}}{x^n}.$$

$$= \frac{1+x^{2n}}{1+x^{2n+2}} \times x$$

If $x < 1$, $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{1+x^{2n}}{1+x^{2n+2}} \times x$

$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = x$. which is < 1

If $x > 1$, $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{1+x^{2n}}{1+x^{2n+2}} \times x$.

$$= \frac{x^{2n}\left[1+1/x^{2n}\right]}{x^{2n}\left[\frac{1+x^2}{x^{2n}}\right]} x$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n}. = \lim_{n\to\infty} \frac{\left[1+1/x^{2n}\right]}{\left[1/x^{2n}+x^2\right]} x.$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n}. = \frac{x}{x^2} = \frac{1}{x}. < 1$$

$\therefore$ If $x > 1$ or < 1, $\lim_{n\to\infty} \frac{u_{n+1}}{u_n}$ tends to a limit < 1

$\therefore \sum u_n$ is convergent.

If $x=1$, the series becomes $\frac{1}{2}+\frac{1}{2}+\cdots$

which is divergent.