Exercise

UNIT – 2

Solvable for p.

01. $4p^2 - 8p + 3 = 0$

$a = 4 \quad b = -8 \quad c = 3$

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$\boxed{\therefore p = \frac{dy}{dx}}$

$$p = \frac{8 \pm \sqrt{64 - 4(12)}}{2(4)}$$

$$p = \frac{8 \pm \sqrt{64 - 48}}{8} = \frac{8 \pm \sqrt{16}}{8}$$

$$p = \frac{8 \pm 4}{8} \Rightarrow \cancel{\frac{12}{8}}\frac{3}{2}, \cancel{\frac{4}{8}}\frac{1}{2}$$

$$p = \frac{3}{2} \quad \text{or} \quad p = \frac{1}{2}$$

$$\frac{dy}{dx} = \frac{3}{2} \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{2}$$

Integrating we get,

$$\int \frac{dy}{\cancel{dx}} = \int \frac{3}{2}\, dx \quad \text{or} \quad \int dy = \int \frac{1}{2}\, dx$$

$$y = \frac{3}{2}x + c \qquad y = \frac{1}{2}x + c$$

$$2y = 3x + c \qquad 2y = x + c$$

$$2y - 3x - c = 0 \qquad 2y - x - c = 0.$$

Hence the solution is

$$2y - 3x - c = 0 \ \& . \ 2y - x - c = 0.$$

02. $x^2p^2 + xyP - 6y^2 = 0$

$a = x^2 \quad b = xy \quad c = -6y^2$

$$P = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$P = \frac{-xy \pm \sqrt{x^2y^2 - 4x^2(-6y^2)}}{2x^2}$$

$$= \frac{-xy \pm \sqrt{x^2y^2 + 24x^2y^2}}{2x^2}$$

$$= \frac{-xy \pm \sqrt{25x^2y^2}}{2x^2} = \frac{-xy \pm 5xy}{2x^2}$$

$$P = \frac{\cancel{4}^{2}xy}{\cancel{2}x^{\cancel{2}}} \quad \text{or} \quad P = \frac{-\cancel{6}^{3}xy}{\cancel{2}x^{\cancel{2}}}$$

$$P = \frac{2y}{x} \quad \text{or} \quad P = \frac{-3y}{x}.$$

$$\frac{dy}{dx} = \frac{2y}{x} \quad \text{or} \quad \frac{dy}{dx} = \frac{-3y}{x}$$

$$\int dy = \int \frac{2y}{x}dx \quad \text{or} \quad \int dy = \int -\frac{3y}{x}dx$$

$$\int \frac{dy}{y} = 2\int \frac{dx}{x} \quad \text{or} \quad \int \frac{dy}{y} = -3\int \frac{dx}{x}.$$

$\log y = 2\log x + \log c$ (or) $\log y = -3\log x + \log c.$

$\log y = \log x^2 + \log c$ (or) $\log y = -\log x^3 + \log c$

$\cancel{\log} y = \cancel{\log}(cx^2)$ (or) $\log y + \log x^3 = \log c$

$y = cx^2$ (or) $\cancel{\log}(yx^3) = \cancel{\log} c$

$(y - cx^2)$ $\qquad$ $(yx^3 - c)$.

Hence the solution is

$$(y - cx^2)(yx^3 - c) = 0$$

03. $xp^2 + (y-x)P - y = 0$

$a = x \quad b = y - x \quad c = -y$

$$P = \frac{-(y-x) \pm \sqrt{(y-x)^2 - 4(x)(-y)}}{2x}$$

$$= \frac{-(y-x) \pm \sqrt{y^2 + x^2 - 2xy + 4xy}}{2x}$$

$$= \frac{-(y-x) \pm \sqrt{x^2 + y^2 + 2xy}}{2x}$$

$$= \frac{-y + x \pm \sqrt{(x+y)^2}}{2x} = \frac{-y + x \pm (x+y)}{2x}$$

$$P = \frac{\cancel{-y} + x + x \cancel{+y}}{\cancel{2x}} \quad \text{(or)} \quad P = \frac{-y \cancel{+x} \cancel{-x} - y}{2x}$$

$P = 1$ (or) $P = -y/x.$

$$\frac{dy}{dx} = 1 \qquad \frac{dy}{dx} = -y/x$$

$$\int dy = \int dx \qquad \int \frac{dy}{y} = \int -\frac{1}{x}dx$$

$y = x + c$

$(y - x - c) = 0$

$\log y = -\log x + \log c$

$\log y + \log x = \log c$

$\cancel{\log}(yx) = \cancel{\log} c$

$xy - c = 0$

Hence the solution is

$(y - x - c)(xy - c) = 0.$

Example-01.

Solve $\Rightarrow p^2 - 3p + 2 = 0$

we have $a = 1,\ b = -3,\ c = 2$

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{3 \pm \sqrt{9 - 8}}{2}$$

$$= \frac{3 \pm 1}{2}$$

$\boxed{p = 2}$ or $\boxed{p = 1}$

$\frac{dy}{dx} = 2$ or $dy/dx = 1$

$\int dy = \int 2\, dx$ $\qquad \int dy = \int dx$

$y = 2x + c_1$ $\qquad y = x + c_2$

$y - 2x - c_1 = 0$ $\qquad y - x - c_2 = 0$

Hence the solution

$(y - 2x - c)(y - x - c) = 0$

Example - 2.

$p^3 - 7p - 6 = 0 \Leftarrow$ solve

The given equation canbe factorised as $(p+1)(p+2)(p-3) = 0.$

$p + 1 = 0$ (or) $p + 2 = 0$ (or) $p - 3 = 0$

$p = -1$ $\qquad p = -2$ $\qquad p = 3$

$\frac{dy}{dx} = -1$ $\qquad \frac{dy}{dx} = -2$ $\qquad \frac{dy}{dx} = 3$

$\int dy = \int -dx$ $\qquad \int dy = \int -2\, dx$ $\qquad \int dy = \int 3\, dx$

$y = -x + c$ $\qquad y = -2x + c$ $\qquad y = 3x + c$

Hence the solution is.

$(y + x - c)(y + 2x - c)(y - 3x - \cancel{x}) = 0$

Example - 3.

Solve: $y(1 - p^2) = 2px.$

Given $y(1 - p^2) = 2px$

$y - p^2 y = 2px$

$p^2 y + 2px - y = 0 \longrightarrow (1)$

$\boxed{a = y}$ $\boxed{b = 2x}$ $\boxed{c = -y}$

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow p = \frac{-2x \pm \sqrt{4x^2 + 4y^2}}{2y}$$

$$\Rightarrow p = \frac{-2x \pm 2\sqrt{x^2 + y^2}}{2y} \Rightarrow \frac{dy}{dx} = \frac{-x \pm \sqrt{x^2 + y^2}}{y}$$

$$y\,dy = \left(-x \pm \sqrt{x^2+y^2}\right) dx$$
$$y\,dy = -x\,dx \pm \sqrt{x^2+y^2}\,dx$$
$$x\,dx + y\,dy = \pm\sqrt{x^2+y^2}\,dx \longrightarrow (2)$$
Put : $x^2+y^2 = t. \longrightarrow (3)$
$$2x + 2y\frac{dy}{dx} = \frac{dt}{dx}$$
$$\frac{2x\,dx + 2y\,dy}{dx} = \frac{dt}{dx}$$
$$2(x\,dx + y\,dy) = dt$$
$$x\,dx + y\,dy = \frac{dt}{2} \longrightarrow (4)$$
Substitute (4) & (3) in (2) we get
$$\frac{dt}{2} = \pm\sqrt{t}\,dx$$
$$\frac{dt}{\sqrt{t}} = \pm 2\,dx$$
$$\int \frac{dt}{\sqrt{t}} = \pm\int 2\,dx$$
$$2\sqrt{t} = \pm 2x + 2c$$
$$2\sqrt{x^2+y^2} = \pm 2(x+c)$$
$$\sqrt{x^2+y^2} = \pm(x+c)$$
$$x^2+y^2 = x^2+c^2+2cx$$
$$y^2 = \cancel{-x^2}+\cancel{x^2}+c^2+2cx$$
$$y^2 = 2cx + c^2$$
Hence the solution.

Example - 4.

Solve : $2p^2 - (x+2y^2)p + xy^2 = 0$

$a = 2 \quad b = -(x+2y^2) \quad c = xy^2$

$$p = \frac{(x+2y^2) \pm \sqrt{(x+2y^2)^2 - 8xy^2}}{4}$$
$$= \frac{(x+2y^2) \pm \sqrt{x^2+4y^4+4xy^2 - 8xy^2}}{4}$$
$$= \frac{(x+2y^2) \pm \sqrt{x^2+4y^4-4xy^2}}{4}$$
$$= \frac{(x+2y^2) \pm \sqrt{(x-2y^2)^2}}{4}$$
$$= \frac{(x+2y^2) \pm (x-2y^2)}{4}$$
$$= \frac{x+\cancel{2y^2}+x-\cancel{2y^2}}{4} \quad \text{or} \quad \frac{\cancel{x}+2y^2-\cancel{x}+2y^2}{4}$$
$$= \frac{2x}{4} \quad \text{or} \quad \frac{4y^2}{4}$$
$$p = x/2 \quad \text{or} \quad p = y^2$$
$$\frac{dy}{dx} = \frac{x}{2} \quad \text{(or)} \quad \frac{dy}{dx} = y^2$$
$$\int dy = \int \frac{x}{2}\,dx \quad \text{(or)} \quad \int \frac{dy}{y^2} = \int dx$$
$$y = \frac{x^2}{4} + c \qquad -1/y = x + c$$
$$(4y - x^2 - 4c) = 0 \qquad -1 = xy + cy.$$

$p = 0 \Rightarrow \frac{dy}{dx} = 0 \qquad dy = 0$

$y = c.$

$(y-c)(xy+cy+1)(y+x^2-c) = 0$

Hence the solution.

Equation solvable for x.

Ex: 1. solve $x = 1 - \frac{p}{\sqrt{p^2+1}}$ $\qquad \frac{u}{v} = \frac{vu' - uv'}{v^2}$

Given $x = 1 - \frac{p}{\sqrt{p^2+1}} \longrightarrow (1)$

$$\frac{dx}{dy} = -\left[\frac{\sqrt{p^2+1}\,\frac{dp}{dy} - p\,\frac{1}{2\sqrt{p^2+1}}\,2p\,\frac{dp}{dy}}{\left(\sqrt{p^2+1}\right)^2}\right]$$

$$\frac{dx}{dy} = -\frac{dp}{dy}\left[\frac{\sqrt{p^2+1} - \frac{\cancel{2}p^2}{\cancel{2}\sqrt{p^2+1}}}{(p^2+1)}\right]$$

$$\frac{dx}{dy} = -\frac{dp}{dy}\left[\frac{\left(\sqrt{p^2+1}\right)^2 - p^2}{\sqrt{p^2+1}\,(p^2+1)}\right]$$

$$\frac{dx}{dy} = -\frac{dp}{dy}\left[\frac{(p^2+1) - p^2}{(p^2+1)^{3/2}}\right]$$

$$\frac{dx}{dy} = -\frac{dp}{dy}\cdot\frac{1}{(p^2+1)^{3/2}}$$

$$\frac{1}{p} = -\frac{dp}{dy}\cdot\frac{1}{(p^2+1)^{3/2}}$$

$$\int dy = -\int \frac{p}{(p^2+1)^{3/2}}\,dp$$

put $p^2+1 = t^2$

$2p\,dp = 2t\,dt$

$p = t\frac{dt}{dp}$

$$\int dy = -\int \frac{t\,\frac{dt}{dp}}{(t^2)^{3/2}}\,dp$$

$$\int dy = -\int \frac{t\,dt}{t^3} = -\int \frac{1}{t^2}\,dt$$

$$y = \frac{1}{t} + c$$

$$y = \frac{1}{\sqrt{p^2+1}} + c \longrightarrow (2)$$

$$(y-c)^2 = \frac{1}{p^2+1} \longrightarrow (3)$$

from (1) we get.

$$x - 1 = -\frac{p}{\sqrt{p^2+1}}$$

$$(x-1)^2 = \frac{p^2+1-1}{p^2+1}$$

$$(x-1)^2 = 1 - \frac{1}{p^2+1} \longrightarrow (4)$$

Substituting (3) in (4) we get

$$(x-1)^2 = 1 - (y-c)^2.$$

(or)

$$(x-1)^2 + (y-c)^2 = 1$$

Hence the solution.

Ex.02 Solve $x = y + a \log P$.

Given $x = y + a \log P \longrightarrow (1)$

Differentiating (1) w.r.t "y" we get.

$$\frac{dx}{dy} = 1 + \frac{a}{p}\frac{dP}{dy}$$

$$\frac{1}{P} - 1 = \frac{a}{P}\frac{dP}{dy}$$

$$(1-P)\,dy = \frac{a\cancel{P}}{\cancel{P}}\,dP$$

$$dy = -\frac{a}{(P-1)}\,dP$$

$$\int dy = -a\int \frac{1}{P-1}\,dP$$

$$y = -a\log(P-1) + C \longrightarrow (2)$$

Substituting (2) in (1) we get

$$x = -a\log(P-1) + a\log P + C$$

$$x = C - a\log(P-1) + a\log P$$

$$x = C + a\left[\log P - \log(P-1)\right]$$

$$x = C + a\log\frac{P}{P-1}$$

elimination of 'P' from (2) and (3) is Very difficult.

$$x = C + a\log\frac{P}{P-1}$$

$$y = C - a\log(P-1)$$ Hence the Solution

Ex-03. Solve $x = p^2 + y$

Given $x = p^2 + y \longrightarrow (1)$

$$\frac{dx}{dy} = 2P\frac{dP}{dy} + 1$$

$$\frac{1}{P} = 2P\frac{dP}{dy} + 1$$

$$\frac{1}{P} - 1 = 2P\frac{dP}{dy}$$

$$dy(1-P) = 2P^2\,dP$$

$$dy = \frac{2P^2}{1-P}\,dP$$

$$\int dy = 2\int \frac{P^2+1-1}{1-P}\,dP$$

$$\int dy = 2\int \frac{P^2+1}{1-P} - \frac{1}{1-P}\,dP$$

$$\int dy = 2\int \frac{(P+1)(P-1)}{1-P} - \frac{1}{1-P}\,dP$$

$$\int dy = 2\int -\frac{(P+1)}{} - \frac{1}{1-P}\,dP$$

$$\int dy = -2\int\left(P + 1 + \frac{1}{1-P}\right)dP$$

$$y = -2\left[\frac{P^2}{2} + P + \log(1-P)\right] + C.$$

Here elimination of P from (1) & (2) is very difficult.

Hence $\Rightarrow x = p^2 + y^2$ //

$$y = C - 2\left[\frac{P^2}{2} + P + \log 1-P\right]$$ //

Ex-04. Solve $p^3 - 4xyp + 8y^2 = 0$

Given $p^3 - 4xyp + 8y^2 = 0$

solving for x we get,

$$+4xyp = 8y^2 + p^3$$

$$x = \frac{8y^2 + p^3}{4yp}$$

$$x = \frac{2y}{p} + \frac{p^2}{4y} \longrightarrow (1) \qquad \frac{u}{v} = \frac{vu' - uv'}{v^2}$$

Differentiate we get.

$$\frac{dx}{dy} = \left[\left(\frac{p\cdot 2 - 2y\cdot\frac{dp}{dy}}{p^2}\right) + \left(\frac{4y\cdot 2p\frac{dp}{dy} - p^2\cdot 4}{16y^2}\right)\right]$$

$$\frac{dx}{dy} = \frac{2}{p} - \frac{2y}{p^2}\frac{dp}{dy} + \frac{8py\frac{dp}{dy}}{16y^2} - \frac{4p^2}{16y^2}$$

$$\frac{dx}{dy} = \frac{2}{p} - \frac{2y}{p^2}\frac{dp}{dy} + \frac{p}{2y}\frac{dp}{dy} - \frac{p^2}{4y^2}$$

$$\frac{1}{p} = \left(\frac{2}{p} - \frac{p^2}{4y^2}\right) + \frac{dp}{dy}\left(\frac{p}{2y} - \frac{2y}{p^2}\right)$$

$$\frac{1}{p} - \frac{2}{p} + \frac{p^2}{4y^2} = \frac{dp}{dy}\left(\frac{p}{2y} - \frac{2y}{p^2}\right)$$

$$-\frac{1}{p} + \frac{p^2}{4y^2} = \left(\frac{p^3 - 4y^2}{2yp^2}\right)\frac{dp}{dy}$$

$$\left(\frac{p^3 - 4y^2}{4py^2}\right) = \left(\frac{p^3 - 4y^2}{2yp^2}\right)\frac{dp}{dy}$$

$$\frac{1}{2py}\left(\frac{p^3 - 4y^2}{2y}\right) = \frac{1}{2yp}\left(\frac{p^3 - 4y^2}{p}\right)\frac{dp}{dy}$$

$$\frac{(p^3 - 4y^2)}{2y} - \left(\frac{p^3 - 4y^2}{p}\right)\frac{dp}{dy} = 0$$

$$(p^3 - 4y^2)\left(\frac{1}{2y} - \frac{1}{p}\frac{dp}{dy}\right) = 0$$

$$p^3 - 4y^2 = 0 \qquad \text{(or)} \qquad \frac{1}{2y} - \frac{1}{p}\frac{dp}{dy} = 0 \longrightarrow (3)$$

Case (i)

Substituting (3) in (1) we get.

$$p^3 = 4y^2 \; ; \; p = (4y^2)^{1/3}$$

$$p^3 - 4xyp + 8y^2 = 0$$

$$4y^2 - 4xy(4y^2)^{1/3} + 8y^2 = 0$$

$$12y^2 = 4xy(4y^2)^{1/3}$$

$$\frac{12y^2}{4y} = x(4y^2)^{1/3}$$

$$3y = x(4y^2)^{1/3}$$

$$(3y)^3 = \left[x(4y^2)^{1/3}\right]^3$$

$$27y^2 = 4y^2x^3$$

Which gives the singular solution.

Case (ii)

let $\frac{1}{2y} - \frac{1}{p}\frac{dp}{dy} = 0$

$$\frac{1}{2y} = \frac{1}{p}\frac{dp}{dy}$$

$$\int \frac{dy}{y} = \int \frac{2}{p}dp$$

$$\log y = 2\log p + \log c$$

$$\log y = \log p^2 c$$

$$y = p^2 c$$

$$\boxed{p^2 = \frac{y}{c}} \longrightarrow (4)$$

from (1) we get.

$$p^3 - 4xyp + 8y^2 = 0$$

$$p(p^2 - 4xy) = -8y^2$$

squaring we get

$$p^2(p^2 - 4xy)^2 = 64y^4.$$

substitute (4) in (5)

$$p^2\left[p^4 + 16x^2y^2 - 8p^2xy\right] = 64y^4 \longrightarrow (5)$$

$$\frac{y}{c}\left[\frac{y^2}{c^2} + 16x^2y^2 - \frac{8xy^2}{c}\right] = 64y^4$$

$$y^2 + 16c^2x^2y^2 - 8cxy^2 = 64c^3y^3$$

$$y^2\left[1 + 16c^2x^2 - 8cx\right] = 64c^3y^3$$

$$1 + 16c^2x^2 - 8cx = \frac{64c^3y^3}{y^2}$$

$$(1 - 4cx)^2 = 64c^3y$$

Hence the solution //.

Ex.05 Solve $y = 2px + y^2p^3$

Given $y = 2px + y^2p^3 \longrightarrow (1)$

$$2px = y - y^2p^3$$

$$2x = \frac{y - y^2p^3}{p}$$

$$2x = \frac{y}{p} - y^2p^2 \longrightarrow (2)$$

Differentiate w.r.t. y we get.

$$2\frac{dx}{dy} = \frac{p - y\frac{dp}{dy}}{p^2} - \left[y^2 2p\frac{dp}{dy} + p^2 \cdot 2y\right]$$

$$2\frac{dx}{dy} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - 2py^2\frac{dp}{dy} - 2p^2y$$

$$2\frac{1}{p} = -\frac{dp}{dy}\left(\frac{y}{p^2} + 2py^2\right) + \frac{1}{p} - 2p^2y$$

$$\frac{2}{p} - \frac{1}{p} = -\frac{dp}{dy}\left(\frac{y}{p^2} + 2py^2\right) - 2p^2y$$

$$2p^2y + \frac{1}{p} = -\frac{y}{p}\frac{dp}{dy}\left(\frac{1}{p} + 2p^2y\right)$$

$$\left(2p^2y + \frac{1}{p}\right) + \frac{y}{p}\frac{dp}{dy}\left(\frac{1}{p} + 2p^2y\right) = 0$$

$$\left(\frac{1}{p} + 2p^2y\right)\left(1 + \frac{y}{p}\frac{dp}{dy}\right) = 0$$

Equations Solve for y.

Ex. 01. solve $y = 3x + \log p$.

Given $y = 3x + \log p$. $\rightarrow (1)$

Differentiate (1) w.r.t x we get

$$\frac{dy}{dx} = 3 + \frac{1}{p}\frac{dp}{dx}$$

$$p = 3 + \frac{1}{p}\frac{dp}{dx}$$

$$\frac{dp}{dx} = p(p-3)$$

$$\frac{dp}{p(p-3)} = dx$$

Integrating we get

$$\int \frac{dp}{p(p-3)} = \int dx$$

$$\int \left(\frac{A}{p} + \frac{B}{p-3}\right) = \int dx$$

$$\int \left(\frac{-1/3}{p} + \frac{1/3}{p-3}\right) dp = \int dx$$

$$-\frac{1}{3}\log p + \frac{1}{3}\log(p-3) = x + c$$

$$\frac{1}{3}\left(\log(p-3) - \log p\right) = x + c$$

$$\log\left(\frac{p-3}{p}\right) = 3x + 3c$$

$$\frac{A}{p} + \frac{B}{p-3} = A(p-3) + Bp$$

$p = 3$

$1 = 3B$ $\boxed{B = 1/3}$

$$\frac{1}{p(p-3)} = A(p-3) + \frac{1}{3}p$$

$p = 0$

$1 = -3A$ $\boxed{-1/3 = A}$

$$\log\frac{p-3}{p} = 3x + c_1 \qquad \therefore c_1 = 3c$$

$$\frac{p-3}{p} = e^{3x+c_1}$$

$$1 - \frac{3}{p} = e^{3x} \cdot e^{c_1}$$

$$1 - \frac{3}{p} = e^{3x} \cdot c_2 \qquad \therefore c_2 = e^{c_1}$$

$$1 - e^{3x}c_2 = \frac{3}{p}$$

$$p = \frac{3}{1 - c_2 e^{3x}}$$

Substituting (2) in (1)

$$y = 3x + \log\frac{3}{1 - c_2 e^{3x}}.$$

This is the General solution.

Ex. 02. $y = x + a\tan^{-1} p$

Given $y = x + a\tan^{-1} p$ $\longrightarrow (1)$

Differentiate w.r.t. x we get

$$\frac{dy}{dx} = 1 + a\frac{1}{p^2+1}\frac{dp}{dx}$$

$$p = 1 + \frac{a}{1+p^2}\frac{dp}{dx} \Rightarrow p - 1 = \frac{a}{1+p^2}\frac{dp}{dx}$$

$$\cancel{p = \frac{1+p^2+a}{1+p^2}\frac{dp}{dx}}$$

$$\cancel{\frac{dp}{dx} = \frac{(1+p^2)p}{(1+p^2)+a}}$$

$$\frac{dp}{dx} = \frac{(p-1)(p^2+1)}{a}$$

$$\frac{dp}{(p^2+1)(p-1)} = \frac{1}{a}\,dx$$

Integrating we get

$$\int \frac{dp}{(p^2+1)(p-1)} = \int \frac{1}{a}\,dx.$$

$$\int \left[\frac{\frac{1}{2}}{p-1} - \frac{\frac{1}{2}(p+1)}{p^2+1}\right] dp = \frac{x}{a} + c$$

$$\frac{1}{2}\left[\int \frac{dp}{p-1} - \int \frac{p}{p^2+1}\,dp - \int \frac{dp}{p^2+1}\right] = \frac{x}{a} + c$$

$$\frac{1}{2}\left[\log(p-1) - \frac{1}{2}\log(p^2+1) - \tan^{-1}p\right] = \frac{x}{a} + c$$

$$x = \frac{a}{2}\left[\log(p-1) - \frac{1}{2}\log(p^2+1) - \tan^{-1}(p)\right] + c \rightarrow (2)$$

Substitute (2) in (1) we get

$$y = \frac{a}{2}\left[\log(p-1) - \frac{1}{2}\log(p^2+1) - \tan^{-1}(p)\right] + c_1 + a\tan^{-1}p.$$

$$= \frac{a}{2}\left[\left[\log(p-1) - \frac{1}{2}\log(p^2+1) - \tan^{-1}(p)\right] + 2a\tan^{-1}p\right] + c$$

$$= \frac{a}{2}\left[\log(p-1) - \frac{1}{2}\log(p^2+1) - \cancel{a}\tan^{-1}(p)\right] + c. \rightarrow (3)$$

Ex. 3 Solve $4y = x^2 + p^2$

Given $4y = x^2 + p^2 \longrightarrow (1)$

$$y = \frac{x^2+p^2}{4} \longrightarrow (2)$$

Differentiate we get w.r.t. x

$$\frac{dy}{dx} = \frac{1}{4}\left(2x + 2p\frac{dp}{dx}\right)$$

$$p = \frac{1}{2}\left(x + p\frac{dp}{dx}\right)$$

$$p\frac{dp}{dx} = 2p - x.$$

$$\boxed{\frac{dp}{dx} = \frac{2p-x}{p}} \longrightarrow (3)$$

This is the homogeneous differential equation of the first order first degree.

$$p = vx \longrightarrow (4)$$

$$\frac{dp}{dx} = v + x\frac{dv}{dx} \longrightarrow (5)$$

substituting (4) and (5) in (3) we get

$$v + x\frac{dv}{dx} = \frac{2p-x}{p}$$

$$v + x\frac{dv}{dx} = \frac{2vx-x}{vx}$$

$$x\frac{dv}{dx} = \frac{2vx-x}{vx} - v$$

$$x\frac{dv}{dx} = \frac{2v-1}{v} - v$$

$$x\frac{dv}{dx} = \frac{2v-1-v^2}{v}$$

$$\frac{v}{2v-1-v^2}\,dv = \frac{dx}{x}$$

integrating we get.

$$\int \frac{v}{2v-1-v^2}\,dv = \int \frac{dx}{x}$$

multiply (-)

$$\int \frac{v+1-1}{(v-1)^2}\,dv = \int \frac{dx}{x}$$

$$\int\left[\frac{v-1}{(v-1)^2} + \frac{1}{(v-1)^2}\right]dv = \int \frac{dx}{x}$$

$$\int \frac{1}{(v-1)}\,dv + \int \frac{1}{(v-1)^2}\,dv = \int \frac{dx}{x}$$

$$\log(v-1) - \frac{1}{v-1} = \log x - \log c$$

$$\log(v-1) - \log x + \log c = \frac{1}{v-1}$$

$$\log(v-1)c - \log x = \frac{1}{v-1}$$

$$\log \frac{c'(v-1)}{x} = \frac{1}{v-1}$$

Replacing v we get.

$$\log \frac{c\left(\frac{p}{x}-1\right)}{x} = \frac{1}{\frac{p}{x}-1}$$

$$\log\left(\frac{c(p-1)}{x^2}\right) = \frac{x}{p-1} \longrightarrow (6)$$

Here elemination of p from (1) and (6) is very difficult.

Hence the solution is given by

$$\log\left(\frac{c(p-x)}{x^2}\right) = \frac{x}{p-x}$$

$$y = \frac{x^2+p^2}{4}$$

EX-04. Solve $y = 2px + p^4x^2$

Given $y = 2px + p^4x^2$

Differentiate (1) w.r.t x we get

$$\frac{dy}{dx} = 2\left(p + x\frac{dp}{dx}\right) + p^4 2x + x^2 4p^3$$

$$\frac{dy}{dx} = \frac{dp}{dx}(2x + 4x^2p^3) + 2p + 2xp^4.$$

$$-2p^4x - 2p + p = \frac{dp}{dx}(2x + 4x^2p^3)$$

$$-p - 2xp^4 = \frac{dp}{dx}(2x + 4x^2p^3)$$

$$-p(1+2xp^3) = \frac{dp}{dx}\,2x(1+2xp^3)$$

$$0 = 2x\frac{dp}{dx}(1+2xp^3) + p(1+2xp^3)$$

$$(1+2xp^3)\left(2x\frac{dp}{dx} + p\right) = 0.$$

Case (i)

Let $2xp^3 = -1 \Rightarrow p^3 = -1/2x.$

$P = \left(\frac{-1}{2x}\right)^{1/3} \longrightarrow (2)$

Substituting (2) in (1) we get

$$y = 2px + p^4x^2$$

$$y = p(2x + p^3x^2)$$

$$y = \left(\frac{-1}{2x}\right)^{1/3}\left(2x - \frac{1}{2\cancel{x}}\cdot \cancel{x^2}^{x}\right)$$

$$y = \left(\frac{-1}{2x}\right)^{1/3}\left(2x - \frac{x}{2}\right)$$

$$y = \left(-\tfrac{1}{2x}\right)^{1/3}\left(\frac{3x}{2}\right)$$

$$y^3 = \left[\left(-\tfrac{1}{2x}\right)^{1/3}\right]^3\left(\frac{3x}{2}\right)^3$$

$$y^3 = \frac{-1}{2x}\cdot\frac{27x^3}{8}$$

$$y^3 = \frac{-27x^{\cancel{3}2}}{16\cancel{x}}$$

$$\boxed{16y^3 = -27x^2}$$

This gives the singular solution.

Case(ii)

$$2x\frac{dp}{dx} + p = 0$$

$$\frac{2x\,dp}{dx} = -p$$

$$2\frac{dp}{p} = -\frac{dx}{x}$$

Integrating we get

$$2\int\frac{dp}{p} = -\int\frac{dx}{x}$$

$$2\log p = -\log x + \log c$$

$$\log p^2 + \log x = \log c$$

$$\log xp^2 = \log c$$

$$xp^2 = c$$

$$p^2 = c/x$$

$$\boxed{p = \sqrt{c/x}} \longrightarrow (3)$$

Substitute (3) in (1) we get

$$y = 2\sqrt{\frac{c}{x}}\cdot x + \left(\sqrt{c/x}\right)^4 x^2$$

$$y = 2\sqrt{cx} + \frac{c^{2}}{\cancel{x^{2}}}\cancel{x^2}$$

$$y = 2\sqrt{cx} + c^2$$

This gives the general solution.

Ex. 5. Solve $y = -px + x^4p^2$

Given $y = -px + x^4p^2 \longrightarrow (1)$

Differentiate we get

$$\frac{dy}{dx} = -\left(p + x\frac{dp}{dx}\right) + x^4\cdot 2p\frac{dp}{dx} + p^2\cdot 4x^3$$

$$p = -p - x\frac{dp}{dx} + 2px^4\frac{dp}{dx} + 4p^2x^3$$

$$p = \frac{dp}{dx}\left(-x + 2px^4\right) - p + 4p^2x^3.$$

$$2p = x\frac{dp}{dx}\left(-1 + 2px^3\right) + 4p^2x^3$$

$$x\frac{dp}{dx}(1-2px^3)+2p-4p^2x^3 = 0$$

$$x\frac{dp}{dx}(1-2px^3)+2p(1-2px^3)=0$$

$$\left(x\frac{dp}{dx}+2p\right)(1-2px^3) = 0$$

case (i)

let $1-2px^3 = 0$

$$\cancel{-}2x^3p = \cancel{-}1$$

$$\boxed{p = \frac{1}{2x^3}} \longrightarrow (2)$$

substitute (2) in (1) we get

$$y=-px+x^4p^2$$

$$y=-\left(\frac{1}{2x^{\cancel{3}2}}\right)\cancel{x}+x^4\left(\frac{1}{2x^3}\right)^2$$

$$y=-\frac{1}{2x^2}+\cancel{x^4}\cdot\frac{1}{4x^{\cancel{6}2}}$$

$$y=-\frac{1\times 2}{2x^2\times 2}+\frac{1}{4x^2}$$

$$y=-\frac{1}{4x^2}$$

$$\boxed{4xy^2+1=0}$$

This gives the singular solution.

Case (ii)

let $x\frac{dp}{dx}+2p=0$

$$x\frac{dp}{dx} = -2p$$

$$\frac{dp}{p} = -2\frac{dx}{x}$$

Integrating we get

$$\log p = -2\log x+\log c$$

$$\log p+\log x^2 = \log c$$

$$\log px^2 = \log c$$

$$px^2 = c$$

$$\boxed{p = \frac{c}{x^2}} \longrightarrow (3)$$

Substitude (3) in (1) we get

$$y = -\frac{c}{x^2}\cdot x+x^4\cdot\frac{c^2}{x^4}$$

$$y=-\frac{c}{x}+c^2$$

$$\boxed{xy = c^2y-c.}$$

This gives general solution

Ex. 6. Solve $x-yp = ap^2$

from the given ~~eduation~~ we can find y explicitly as

$$-y = \frac{ap^2-x}{p}$$

$$y = \frac{x-ap^2}{p} \longrightarrow (1)$$

Differentiate we get

$$\frac{dy}{dx} = \frac{p\left(1-2ap\frac{dp}{dx}\right)-(x-ap^2)\frac{dp}{dx}}{p^2}$$

solution.

Clairaut's equation

Ex.01. Find the general solution and singular solution of the Clairaut's equation $y = xp + p^2$.

Soln: Given $y = xp + p^2 \longrightarrow (1)$

The general solution is obtained by putting $\boxed{p = c}$ (constant) in (1)

$$y = cx + c^2 \longrightarrow (2)$$

To find singular solution

S.1 ⇒ The general solution is $y = cx + c^2$

S.2 ⇒ Differentiate we get

$$0 = x + 2c \longrightarrow (3)$$

S.3 ⇒ we have to eliminate c between eqn (2) & (3)

from (3) we get

$$c = -\frac{x}{2} \longrightarrow (4)$$

substituting (4) in (2) we get

$$y = \left(-\frac{x}{2}\right)x + \left(\frac{x}{2}\right)^2$$

$$y = \frac{-2x^2}{2\times 2} + \frac{x^2}{4}$$

$$y = \frac{-x^2}{4}$$

$x^2 + 4y = 0$. // which gives singular solution.

Also $x^2 + 4y = 0$ gives the envelope of the family of straight lines given by eqn (2).

Ex.02. Solve $y = px + p^{-1}a$

Given $y = px + ap^{-1} \longrightarrow (1)$

The given equation obtained by replacing p by c. we get

$$y = cx + ac^{-1} \longrightarrow (2)$$

To find singular solution.

S.1 ⇒ The general solution is

$$y = cx + ac^{-1}$$

S.2 ⇒ Differentiate w.r.t c we get

$$0 = x - ac^{-2}$$

$$0 = x - \frac{a}{c^2} \longrightarrow (3)$$

S.3 ⇒ we have to eliminate c between eqn (2) & (3).

from (3) we get

$$\boxed{x = \frac{a}{c^2}} \longrightarrow (4)$$

Substituting (4) in (2) we get

$$y = \cancel{c} \cdot \frac{a}{c^{\cancel{2}}} + ac^{-1}$$

$$y = \frac{a}{c} + \frac{a}{c}$$

$$y = \frac{2a}{c}$$

$$\boxed{c = \frac{2a}{y}} \longrightarrow (5)$$

Substitute (5) in (3) we get

$$x - \frac{a}{c^2} = 0$$

$$x - \frac{a}{(2a/y)^2} = 0$$

$$x - \frac{\cancel{a} y^2}{4a^{\cancel{2}}} = 0$$

$$\frac{4ax - y^2}{4a} = 0$$

$$\boxed{y^2 = 4ax}$$

which gives singular solution

Ex.03. solve $y = (x-a)p - p^2$

Given $y = (x-a)p - p^2$ $\longrightarrow$ (1)

The given equation can be obtained by replacing p by c. we get

$$y = (x-a)c - c^2 \longrightarrow (2)$$

which gives general solution.

To find singular solution.

S.1 ⇒ The general solution is

$$y = (x-a)c - c^2$$

S.2 ⇒ Differentiate we get

$$0 = (x-a) - 2c$$

$$\boxed{c = \frac{x-a}{2}} \longrightarrow (3)$$

Substitute (3) in (2) we get

$$y = (x-a)\left(\frac{x-a}{2}\right) - \left(\frac{x-a}{2}\right)^2$$

$$y = \frac{2(x-a)^2}{2\times 2} - \frac{(x-a)^2}{4}$$

$$y = \frac{(x-a)^2}{4}$$

$$\boxed{4y = (x-a)^2}$$

This gives the singular solution

Ex.04. Find the general and singular solution of $y = xp + \sqrt{p^2+1}$.

Given $y = xp + \sqrt{p^2+1}$ $\longrightarrow$ (1)

This given equation canbe obtained by replacing p by c we get

$$\boxed{y = xc + \sqrt{c^2+1}} \longrightarrow (2)$$

which gives general solution,

To find singular solution.

S.1 ⇒ The general solution is

$$y = xc + \sqrt{c^2+1}$$

S.2 ⇒ Differentiate we get

$$0 = x + \frac{1}{\cancel{2}\sqrt{c^2+1}} \cdot \cancel{2}c.$$

$$x + \frac{c}{\sqrt{c^2+1}} = 0 \longrightarrow (3)$$

S.3 ⇒ we have eliminate c between (2) & (3), from (3) we get

$$x = -\frac{c}{\sqrt{c^2+1}} \longrightarrow (4)$$

substitute (4) in (2) we get

$$y = \frac{-c}{\sqrt{c^2+1}} \cdot c + \sqrt{c^2+1}$$

$$y = \frac{-c^2}{\sqrt{c^2+1}} + \sqrt{c^2+1}$$

$$y = \frac{\cancel{-c^2} + \cancel{c^2} + 1}{\sqrt{c^2+1}}$$

$$\boxed{y = \frac{1}{\sqrt{c^2+1}}} \longrightarrow (5)$$

squaring (4) we get $x^2 = \frac{c^2}{c^2+1}$

Squaring (5) we get $y^2 = \frac{1}{c^2+1}$

Adding we get

$$x^2+y^2 = \frac{c^2}{c^2+1} + \frac{1}{c^2+1} = \frac{\cancel{c^2+1}}{\cancel{c^2+1}} = 1$$

$$x^2+y^2 = 1 \;//$$

Therefore the singular solution is $x^2+y^2=1$ which cannot be obtained by giving particular values to c in the general solution.

Ex-05 Find the general solution and singular solution of the equation $y = 2px + y^2p^3$.

Given $y = 2px + y^2p^3$

~~This equation can be obtained by replacing p by c we get~~

$$\cancel{y = 2cx + y^2c^3}$$

The given equation is not of Clairaut's type. But we can reduce the given eqn to a clairaut's form as in following way.

Multiply the given equation by y we get

$$y^2 = 2pxy + y^3p^3 \longrightarrow (1)$$

$$\text{Put } y^2 = u \longrightarrow (2)$$

$$2y\frac{dy}{dx} = \frac{du}{dx}$$

$$2yp = \frac{du}{dx} \longrightarrow (3)$$

Substitute (2) and (3) in (1) we get

$$u = x\frac{du}{dx} + y^3\left(\frac{1}{2y}\frac{du}{dx}\right)^3$$

$$u = x\frac{du}{dx} + \left(\frac{1}{2}\frac{du}{dx}\right)^3$$

$$u = xP_1 + \frac{1}{8}P_1^3 \quad \text{where } P_1 = \frac{du}{dx} \longrightarrow (4)$$

equation (4) is clairauts equation in u & x. Replacing p by c we get

The general solution is

$$u = xc + c^3$$

Hence the General solution of (4) is

$$y^2 = xc + c^3 \qquad \boxed{\therefore u = y^2}$$

To find singular solution.

S.1 ⇒ The general solution is

$$y^2 = xc + c^3 \longrightarrow (5)$$

S.2 ⇒ Differenciate we get w.r.t c.

$$0 = x + 3c^2 \longrightarrow (6)$$

S.3 ⇒ we have to eliminate c between (5) & (6), from (6) we get

$$\boxed{x = -3c^2}$$

$$c^2 = -\frac{x}{3} \longrightarrow (7)$$

Squaring (5) we get

$$y^4 = (xc + c^3)^2$$

$$y^4 = x^2c^2 + c^6 + 2xc^4 \longrightarrow (8)$$

Substitute (7) in (8) we get

$$y^4 = x^2\left(-\frac{x}{3}\right) + \left(\frac{-x}{3}\right)^3 + 2x\left(\frac{-x}{3}\right)^2$$

$$y^4 = -\frac{x^3 \times 9}{3\times 9} - \frac{x^3}{27} + \frac{2x^3 \times 3}{9\times 3}$$

$$y^4 = \frac{-9x^3 - x^3 + 6x^3}{27}$$

$$y^4 = \frac{-4x^3}{27}$$

$$\boxed{27y^4 = -4x^3}$$

which gives singular solution.

Further this singular solution gives the envelope of the family of curves $y^2 = cx + c^3$.

Ex.06. Find the G.S & S.S of the eqn

$$y = x\frac{dy}{dx} + \frac{a\,dy/dx}{\sqrt{1 + (dy/dx)^2}}.$$

The given equation can be written us

$$y = xP + \frac{aP}{\sqrt{1+P^2}} \quad \text{where } P = dy/dx \longrightarrow (1)$$

The general solution obtained by replacing p by c. we get

$$y = xc + \frac{ac}{\sqrt{1+c^2}} \longrightarrow (2)$$

To find singular solution.

S.1 ⇒ The general solution is

$$y = xc + \frac{ac}{\sqrt{1+c^2}}$$

S.2 ⇒ Differentiate w.r.t c we get

$$0 = x + a\left[\frac{\sqrt{1+c^2}\cdot 1 - c\,\frac{1\cdot \cancel{2}c}{\cancel{2}\sqrt{1+c^2}}}{1+c^2}\right]$$

$$0 = x + a\left[\frac{1+\cancel{c^2}-\cancel{c^2}}{\sqrt{1+c^2}} \Big/ 1+c^2\right]$$

$$0 = x + \left(\frac{a}{\sqrt{1+c^2}}\right)\cdot\frac{1}{1+c^2}$$

$$0 = x + \frac{a}{(1+c^2)^{3/2}} \longrightarrow (3)$$

S.3 ⇒ we have to eliminate c between (3) and (2) ∴ from (3) we get

$$x = -\frac{a}{(1+c^2)^{3/2}} \longrightarrow (4)$$

Substitute (4) in (2) we get

$$y = \frac{-ac}{(1+c^2)^{3/2}} + \frac{ac}{\sqrt{1+c^2}}$$

$$y = \frac{-ac + ac(1+c^2)}{(1+c^2)^{3/2}}$$

$$y = \frac{\cancel{-ac} + \cancel{ac} + ac^3}{(1+c^2)^{3/2}}$$

$$y = \frac{ac^3}{(1+c^2)^{3/2}} \longrightarrow (5)$$

From (4) & (5) by raising to the power $\frac{2}{3}$ on both sides we get

$$(4) \Rightarrow \quad x^{2/3} = \frac{+a^{2/3}}{(1+c^2)} \longrightarrow (6)$$

$$(5) \Rightarrow \quad y^{2/3} = \frac{a^{2/3}c^2}{(1+c^2)} \longrightarrow (7)$$

Adding (6) & (7) we get

$$x^{2/3} + y^{2/3} = \frac{+a^{2/3}}{(1+c^2)} + \frac{a^{2/3}c^2}{(1+c^2)}$$

$$x^{2/3} + y^{2/3} = \frac{a^{2/3}\cancel{(1+c^2)}}{\cancel{(1+c^2)}}$$

$x^{2/3} + y^{2/3} = a^{2/3}$ which is the singular solution and this gives an astroid. Thus the envelope of the family of straight lines.