Unit - V

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The Second Fundamental Form.

The Second fundamental form :

The normal curvature of a curve at a point P on a surface is given by

$K_n = \bar{N}\cdot r''$ —— (1)

Now $\bar{r}' = \bar{r}_1 u' + \bar{r}_2 v'$

$\bar{r}_1' = \bar{r}_{11}u' + \bar{r}_{12}v'$

$\bar{r}_2' = \bar{r}_{21}u' + \bar{r}_{22}v'$

$\therefore \bar{r}'' = (\bar{r}_1 u' + \bar{r}_2 v')'$

$= \bar{r}_1 u'' + \bar{r}_2 v'' + (\bar{r}_1)' u' + \bar{r}_2' v'$

$= \bar{r}_1 u'' + \bar{r}_2 v'' + \bar{r}_{11} u'^2 + \bar{r}_{12} u'v' + \bar{r}_{21} u'v' + \bar{r}_{22} v'^2$

$= \bar{r}_1 u'' + \bar{r}_2 v'' + \bar{r}_{11} u'^2 + 2\bar{r}_{12} u'v' + \bar{r}_{22} v'^2$

$\therefore$ (1) becomes

$K_n = \bar{N}\cdot(\bar{r}_1 u'' + \bar{r}_2 v'') + \bar{N}\cdot(\bar{r}_{11} u'^2 + 2\bar{r}_{12} u'v' + \bar{r}_{22} v'^2)$

$= 0 + (\bar{N}\cdot\bar{r}_{11}) u'^2 + 2(\bar{N}\cdot\bar{r}_{12}) u'v' + (\bar{N}\cdot\bar{r}_{22}) v'^2$

$[\because \bar{N}$ is normal to both $\bar{r}_1$ & $\bar{r}_2]$

$L = \bar{N}\cdot\bar{r}_{11} \quad M = \bar{N}\cdot\bar{r}_{12} \quad N = N\cdot\bar{r}_{22}$

$= L u'^2 + 2M u'v' + \bar{N} v'^2$ $\quad u' = du/ds$

$\quad v' = \dfrac{dv}{ds}$

$= \dfrac{L du^2 + 2M du dv + N dv^2}{ds^2}$

$ds^2 = E du^2 + 2F du dv + G dv^2$

$= \dfrac{L du^2 + 2M du dv + N dv^2}{E du^2 + 2F du dv + G dv^2}$ —— (2)

Where $L = \bar{N}\cdot\bar{r}_{11}$, $M = \bar{N}\cdot\bar{r}_{12}$, $N = \bar{N}\cdot\bar{r}_{22}$ —— (3)

Alternative expression for L, M, N will now obtained.

(2)

By differentiating $\bar{N}\cdot\bar{r}_1 = 0$

$\bar{N}_1\cdot\bar{r}_1 + N\bar{r}_{11} = 0$ and $\bar{N}_2\cdot\bar{r}_1 + \bar{N}\cdot\bar{r}_{12} = 0$

Similarly, $\bar{N}\cdot\bar{r}_2 = 0$ gives

$\bar{N}_2\cdot\bar{r}_2 + \bar{N}\cdot\bar{r}_{22} = 0$ and $\bar{N}_1\cdot\bar{r}_2 + \bar{N}\cdot\bar{r}_{21} = 0$

Sub in (3)

$$\left.\begin{aligned} L &\Rightarrow \bar{N}_1\cdot\bar{r}_1 + \bar{N}\bar{r}_{11} = -\bar{N}_1\bar{r}_1 \\ M &\Rightarrow \bar{N}_2\cdot\bar{r}_1 + \bar{N}\cdot\bar{r}_{12} = -N_2\bar{r}_1 \\ &\text{and } \bar{N}_1\cdot\bar{r}_2 + \bar{N}\bar{r}_{21} = -\bar{N}_1\bar{r}_2 \\ N &\Rightarrow \bar{N}_2\cdot\bar{r}_2 + \bar{N}\cdot\bar{r}_{22} = -\bar{N}\cdot\bar{r}_{22} \end{aligned}\right\} \quad \text{——— (4)}$$

The quadratic form

$$Ldu^2 + 2Mdudv + Ndv^2 \quad \text{——— (5)}$$

is called the Second fundamental form and the functions of u and v denoted by L, M and N are the Second fundamental co-efficients.

It follows from (2) that all curves having the same direction at P have the same normal curvature, hence normal curvature is a property of a surface and a direction at a point on the surface.

(1) Meusnier's theorem:

If ϕ denote the angle between the Principal normal $\bar{n}$ to a curve on the surface and the surface normal $\bar{N}$, it follows from the relation

(2)

(3)

$$\bar{r}'' = K_n\bar{N} + \lambda r_1 + \mu r_2$$

$$\therefore \bar{N}\cdot\bar{r}'' = K_n$$

$$\bar{r}' = \bar{t}$$

$$\bar{r}'' = \bar{t}' = K\bar{n}$$

$$\bar{N}\cdot K\bar{n} = K_n$$

Since $N\cdot\bar{n} = 1\cdot1\cos\phi$

$$K_n = K\cos\phi$$

a result known as meusnier's theorem.

Since the denominator of the right hand member $K_n = \dfrac{Ldu^2 + 2Mdudv + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2}$ is positive definite.

It follows that the sign of K_n depends only upon the sign of the form.

$$Ldu^2 + 2Mdudv + Ndv^2$$

Elliptic, Parabolic and Hyperbolic point:

1) Elliptic point:

If at a point on the surface this form is definite (i.e) if $LN - M^2 > 0$. Then K_n maintains the same sign for all direction at P. In this case P is called elliptic point.

2) Parabolic point:

when $LN - M^2 = 0$

When K_n retains the same sign for all direction through P except one for which the curvature is zero. P is called a parabolic point

3) Hyperbolic points :

when $LN - M^2 < 0$

i) K_n is positive

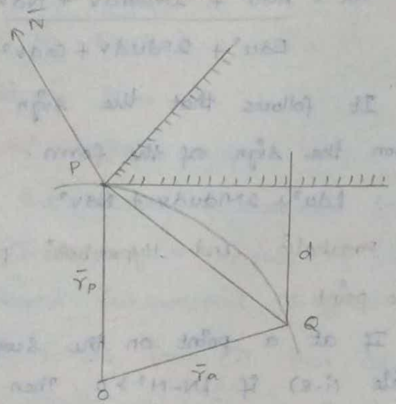
The direction lying within a certain angle

ii) K_n is negative

The direction lying outside a certain angle

iii) $K_n = 0$. Along the direction which fo from the angle p is called a hyperbolic point and critical direction are called the asymptotic direction

<u>Geometrical interpretation :</u>



We now obtain a geometrical interpretation of the second fundamental form.

Let $P(u,v)$ and $Q(u+h, v+k)$ be near points on a surface, and let d be the perpendicular distance from Q onto the tangent plane to the surface at P :

If r_P, r_Q are the position vectors of P and Q the

$$d = (r_Q - r_P)\bar{N}$$

$$r_P = u\bar{r}_1 + v\bar{r}_2$$

$$r_Q = (u+h)\bar{r}_1 + (v+k)\bar{r}_2$$

$$d = (u+h)\bar{r}_1 + (v+k)\bar{r}_2 - u\bar{r}_1 - v\bar{r}_2$$

$$d = u\bar{r}_1 + h\bar{r}_1 + v\bar{r}_2 + k\bar{r}_2 - u\bar{r}_1 - v\bar{r}_2$$

$$d = h\bar{r}_1 + k\bar{r}_2$$

$$= (h\bar{r}_1 + k\bar{r}_2)\bar{N} + \frac{1}{2}(h^2\bar{r}_{11} + 2hk\bar{r}_{12} + k^2\bar{r}_{22})\bar{N} + O(h^3, k^3)$$

$$= \tfrac{1}{2}[Lh^2 + 2Mhk + Nk^2] + O(h^3, k^3)$$

Thus the second fundamental form at any point p is equal to twice the length of the perpendicular from the neighbouring point Q onto the tangent plane at p.

At an elliptic point d retains the same sign ; and this implies that the surface near P lies on entirely on one side of the tangent plane at P.

At a hyperbolic point the surface crosses over the tangent plane, it follows that any point on an ellipsoidal surface is elliptic, any point on a circular cylinder is parabolic and any point on the hyperboloid is hyperbolic.

Principal curvature: (6)

K for the normal curvature instead of Kn.

The normal curvature at p in a direction specified by direction co-efficients (l, m) is given by

W.K.T $K = Ll^2 + 2Mlm + Nm^2$ —— (1)

$El^2 + 2Flm + Gm^2 = 1$ —— (2)

As l, m vary subject to (2) the normal curvature will vary. Its extreme values may be found by making use of lagranges' multipliers writing.

$K = Ll^2 + 2Mlm + Nm^2 - \lambda(El^2 + 2Flm + Gm^2 - 1)$ —— (A)

K is stationary

(A) $\Rightarrow \frac{\partial K}{\partial l} = 2Ll + 2Mm - 2\lambda El - 2Fm\lambda$

$= 2(Ll + Mm - \lambda El - Fm\lambda)$

$\frac{1}{2}\frac{\partial K}{\partial l} = Ll + Mm - \lambda El - Fm\lambda = 0$ —— (3)

(A) $\Rightarrow \frac{\partial K}{\partial m} = 2Ml + 2mN - 2Fl\lambda - 2G\lambda m$

$= 2(Ml + mN - Fl\lambda - G\lambda m)$

$\frac{1}{2}\frac{\partial K}{\partial m} = Ml + mN - Fl\lambda - G\lambda m$ —— (4)

Multiply (3) by l

$Ll^2 + lMm - \lambda El^2 - Fm\lambda l = 0$ —— (5)

Multiply (4) by m

$Mlm + m^2N - Fl\lambda m - G\lambda m^2 = 0$ —— (6)

(7)

(5) & (6) add we get

$(Ll^2 + lMm - \lambda El^2 - Fm\lambda l) + (Mlm + m^2N - Fl\lambda m - G\lambda m^2) = 0$

$(Ll^2 + lMm + m^2N + Mlm) - \lambda(El^2 - Fml - Flm - Gm^2) = 0$

$(Ll^2 + 2lMm + m^2N) - \lambda(El^2 - 2Fml - Gm^2) = 0$ —— (7)

(A) $\Rightarrow K = Ll^2 + 2Mlm + Nm^2 - \lambda(El^2 + 2Flm + Gm^2 - 1)$

$K = Ll^2 + 2Mlm + Nm^2 - \lambda(ml^2 + 2Flm + Gm^2) + \lambda$

$K - \lambda = Ll^2 + 2Mlm + Nm^2 - \lambda(ml^2 + 2Flm + Gm^2)$ —— (8)

Sub (8) in (7)

$K - \lambda = 0$

$K = \lambda$

From (3) and (4)

(3) $\Rightarrow (L - \lambda E)l + (M - \lambda F)m = 0$

Since $K = \lambda$

$(L - KE)l + (M - KF)m = 0$ —— (9)

(4) $\Rightarrow (M - KF)l + (N - KG)m = 0$

Since $K = \lambda$

$(M - KF)l + (N - KG)m = 0$ —— (10)

Eliminating l and m between (9) and (10)

$(L - KE)l = -(M - KF)m$

$(M - KF)l = -(N - KG)m$

$\frac{(L - KE)l}{(M - KF)K} = \frac{(M - KF)m}{(N - KG)m}$

$(L - KE)(N - KG) = (M - KF)(M - KF)$

$LN - LKG - KEN + K^2EG = Mm - KMF - KMF + K^2F^2$

$LN - LKG - KEN + K^2EG - Mm + KMF + KMF - K^2F^2 = 0$

$K^2(KG - F^2) - K(EN + GL - 2mF) + (LN - Mm) = 0$ —— (11)

The roots K_a, K_b of this equation are called the principal curvature.

Associated with these are the mean curvature $\mathcal{U}$ defined by,

(8)

(11) $\Rightarrow$ div $(EG-F^2)$

$$\frac{K^2(EG-F^2)}{(EG-F^2)} - K\frac{(EN - \cancel{GL} - 2FM)}{(EG-F^2)} + \frac{(LN-M^2)}{(EG-F^2)} = 0$$

Sum of root $= \mathcal{U} = 1/2(K_a+K_b) = \dfrac{EN+GL-2FM}{2(EG-F^2)}$ —— (12)

Product of root $= K = K_aK_b = \dfrac{LN-m^2}{EG-F^2}$ —— (13)

Eliminating λ between (3) & (4) so that

(3) $\Rightarrow Ll - Mm - \lambda(El+Fm) = 0$

(4) $\Rightarrow Ml - Nm - \lambda(Fl+Gm) = 0$

$$\frac{Ll+Mm}{Ml+Nm} = \frac{\lambda(El+Fm)}{\lambda(Fl+Gm)}$$

$$(Ll+Mm)(Fl+Gm) = (El+Fm)(Ml+Nm)$$

$$Ll^2F + LlGm + MmFl + Mm^2G = EMl^2 + ElNm + FmMl + Fm^2N$$

$$EMl^2 + ElNm + FMml + Fm^2N - Ll^2F - LlGm - MmFl - Mm^2G = 0$$

$$l^2(EM-FL) + (EN-LG)lm + (FN-MG)m^2 = 0 \quad \text{—— (14)}$$

$(b^2 - 4ac)$

$$(EN-LG)^2 - 4(EM-FL)(FN-MG)$$

$$= (EN-LG)^2 - 4(EM-FL)\left[\frac{F}{N}(EN-GL) - \frac{G}{E}(EM-FL)\right]$$

$$= (EN-LG)^2 - \frac{4F}{E}(EN-GL)(EM-FL) + \frac{4G}{E}(EM-FL)^2$$

add & sub b^2 $\qquad b^2 = \dfrac{4F^2}{E^2}(EM-FL)^2$

$$= (EN-GL)^2 - 4\frac{F}{E}(EN-GL)(EM-FL) + \frac{4F^2}{E^2}(EM-FL)^2 - \frac{4F^2}{E^2}(EM-FL)^2 + \frac{4G}{E}(EM-FL)^2$$

$$= \left[(EN-GL) - \frac{2F}{E}(EM-FL)\right]^2 + \frac{4G}{E}(EM-FL)^2 - \frac{4F^2}{E^2}(EM-FL)^2$$

$$= \left[(EN-GL) - \frac{2F}{G}(EM-FL)\right]^2 + 4(EM-FL)^2\left(\frac{G}{E} - \frac{F^2}{E^2}\right)$$

$$= \left[(EN-GL) - \frac{2F}{E}(EM-FL)\right]^2 + 4(EM-FL)^2\left(\frac{EG-F^2}{E^2}\right)$$

Since $EG-F^2 > 0$ it follows that the roots of eqn (10) are real and distinct, provided the coefficient E, F, G and L, M, N are not proportional.

The values of these coefficients are proportional the principal directions are indeterminate and the normal curvature is the same in all directions such a point where $\frac{L}{E} = \frac{M}{F} = \frac{N}{G}$ is called an umbilic.

At a point which is not an umbilic the two directions determined by eqn (10) are orthogonal. This follows by direct application of the condition of orthogonality

"$ER - 2FQ - GP = 0$"

Line of curvature:

A curve on a surface whose tangent at each point is along a principal direction is called a line of curvature.

Theorem: [Rodrigue's Formula]

The necessary and sufficient condition that a curve on a surface be a line of curvature is that $Kd\bar{r} + d\bar{N} = 0$ at each of its point.

Proof:

Given, Principal of direction

The equation of a line of curvature

$$\left.\begin{aligned}(L-KE)du + (M-KF)dv &= 0\\ (M-KF)du + (N-KG)dv &= 0\end{aligned}\right\} \quad \text{——} (1)$$

where K is one of the principal curvatures

Since,

$E = \bar{r}_1^{\,2}$

$F = \bar{r}_1\bar{r}_2$

$G = \bar{r}_2^{\,2}$

$L = -\bar{N}_1\bar{r}_1$

$M = -\bar{N}_2\cdot\bar{r}_1 = -\bar{N}_1\bar{r}_2$

$N = -\bar{N}_2\bar{r}_2$

Sub values in eqn (1)

$$(-\bar{N}_1\cdot\bar{r}_1 - K\bar{r}_1^{\,2})du + (-\bar{N}_2\cdot\bar{r}_1 - K\bar{r}_1\cdot\bar{r}_2)dv = 0$$

$$(\bar{N}_1\bar{r}_1 + K\bar{r}_1^{\,2})du + (\bar{N}_2\cdot\bar{r}_1 + K\bar{r}_1\cdot\bar{r}_2)dv = 0 \quad \text{——} (A)$$

$$(-\bar{N}_1\bar{r}_2 - K(\bar{r}_1\cdot\bar{r}_2))du + (-\bar{N}_2\cdot\bar{r}_2 - K\bar{r}_2^{\,2})dv = 0$$

$$(\bar{N}_1\cdot\bar{r}_2 + K(\bar{r}_1\cdot\bar{r}_2))du + (\bar{N}_2\bar{r}_2 + K\bar{r}_2^{\,2})dv = 0 \quad \text{——} (B)$$

(A) $\Rightarrow \bar{r}_1[(K\bar{r}_1 + \bar{N}_1)du + (K\bar{r}_2 + \bar{N}_2)dv] = 0$

$\bar{r}_1[K\bar{r}_1 du + \bar{N}_1 du + K\bar{r}_2 dv + \bar{N}_2 dv] = 0$

$\bar{r}_1[K(\bar{r}_1 du + \bar{r}_2 dv)\} + (\bar{N}_1 du + \bar{N}_2 dv)] = 0$

$\bar{r}_1[Kd\bar{r} + d\bar{N}] = 0$

and

(B) $\Rightarrow \bar{r}_2[(K\bar{r}_1 + \bar{N}_1)du + (K\bar{r}_2 + \bar{N}_2)dv] = 0$

$\Rightarrow \bar{r}_2[K\bar{r}_1 du + \bar{N}_1 du + K\bar{r}_2 dv + \bar{N}_2 dv] = 0$

$\Rightarrow \bar{r}_2[K(\bar{r}_1 du + \bar{N}_1 du) + K\bar{r}_2 dv + \bar{N}_2 dv] = 0$

$\Rightarrow \bar{r}_2[K(\bar{r}_1 du + \bar{r}_2 dv) + (\bar{N}_1 du + \bar{N}_2 dv)] = 0$

$\bar{r}_2[Kd\bar{r} + d\bar{N}] = 0$

Then, $(Kd\bar{r} + d\bar{N})\bar{r}_1 = 0$ and

$(Kd\bar{r} + d\bar{N})\bar{r}_2 = 0$

The vector $Kd\bar{r} + d\bar{N}$ is along the surface of Normal. The vector $(Kd\bar{r} + d\bar{N})$ is tangential to the surface. This is possible.

$$Kd\bar{r} + d\bar{N} = 0 \quad \text{——} (2)$$

Conversely,

Along a curve for any function K.

Then eqn (1)

$(L-KE)du + (M-KF)dv = 0$

$(M-KF)du + (N-KG)dv = 0$

and curve is thus a line of curvature.

$Kd\bar{r} + d\bar{N} = 0$ characteristics the lines of curvature and is known as Rodrigue's formula.

Hence proved.

Euler's Theorem :

Statement :

If K is a normal curvature in a direction (l, m) making an angle ψ with the principal direction $v =$ constant.

Then $K = K_a \cos^2\psi + K_b \sin^2\psi$

where K_a and K_b are principal curvatures at the point.

Proof :

Take the lines of curvatures as the parametric curves. Then since the principal directions are orthogonal.

we have $F = 0, M = 0$ [$\because$ By using Remark Pg : 4.7]

and Hence the normal curvature in a direction (l, m) is

$$K = Ll^2 + Nm^2 \quad \text{——} \; ①$$

The direction co-efficients of the parametric curves.

$v =$ constant and $u =$ constant are

$\left(\frac{1}{\sqrt{E}}, 0\right)$ and $\left(0, \frac{1}{\sqrt{G}}\right)$ [$\because$ By using Direction coefficients]

$\therefore K_a =$ normal curvature along $v =$ constant

By using eqn ①

At the point $\left(\frac{1}{\sqrt{E}}, 0\right)$

$l = \frac{1}{\sqrt{E}}$, $m = 0$

Sub the points in eqn ①

① $\Rightarrow K = Ll^2 + Nm^2$

$K_a = L\left(\frac{1}{\sqrt{E}}\right)^2 + N(0)^2$

$= L\left(\frac{1}{E}\right) + 0$

$K_a = \frac{L}{E}$

And $K_b =$ normal curvature along $u =$ constant

At the point $\left(0, \frac{1}{\sqrt{G}}\right)$

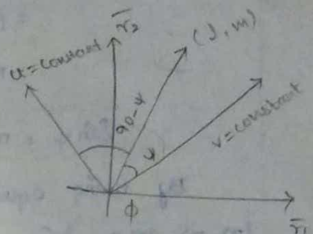
$l = 0$, $m = \frac{1}{\sqrt{G}}$

Sub in eqn ①

$K_b = L(0) + N\left(\frac{1}{\sqrt{G}}\right)^2$

$= 0 + N\left(\frac{1}{\sqrt{G}}\right)^2$

$K_b = \frac{N}{G}$



(i.e) $\boxed{K_a = \frac{L}{E}, \quad K_b = \frac{N}{G}}$ —— ②

Now ψ is the angle between the direction (l, m) And the principal direction $v =$ constant.

At the point $\left(\frac{1}{\sqrt{E}}, 0\right)$

$l' = \frac{1}{\sqrt{E}}$, $m' = 0$

Since $\cos\theta = Ell' + F(lm' + l'm) + Gmm'$

if $F = 0$ $\cos\psi = Ell' + 0 + Gmm'$

$= El\left(\frac{1}{\sqrt{E}}\right) + 0 + Gm(0)$

$= \sqrt{E}\cdot\sqrt{E}\left(\frac{l}{\sqrt{E}}\right) + 0$

$= l\sqrt{E}$

$\boxed{\cos\psi = l\sqrt{E}}$ —— ③

and $\cos(90 - \psi) = \sin\psi$

The principal direction $u =$ constant.

At the point $\left(0, \frac{1}{\sqrt{G}}\right)$

$$l' = 0, \quad m' = \left(\frac{1}{\sqrt{G}}\right)$$

$$\sin\psi = E l l' + 0 + G m m'$$

$$= E l(0) + G m\left(\frac{1}{\sqrt{G}}\right)$$

$$= m\sqrt{G}\cdot\sqrt{G}\left(\frac{1}{\sqrt{G}}\right)$$

$$\boxed{\sin\psi = m\sqrt{G}} \quad \text{——— (4)}$$

By using equation (3) and (4)

(3) $\Rightarrow \cos\psi = l\sqrt{E}$

$$l = \frac{\cos\psi}{\sqrt{E}}$$

(4) $\Rightarrow \sin\psi = m\sqrt{G}$

$$\frac{\sin\psi}{\sqrt{G}} = m$$

$$\therefore m = \frac{\sin\psi}{\sqrt{G}}$$

Sub the values in eqn (1)

Thus $\kappa = L l^2 + N m^2$

$$= L\left(\frac{\cos\psi}{\sqrt{E}}\right)^2 + N\left(\frac{\sin\psi}{\sqrt{G}}\right)^2$$

$$= L\left(\frac{\cos^2\psi}{E}\right) + N\left(\frac{\sin^2\psi}{G}\right)$$

$$= \frac{L}{E}\cos^2\psi + \frac{N}{G}\sin^2\psi$$

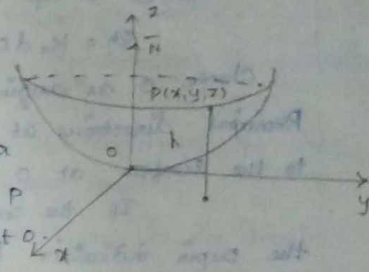
$$= \kappa_a\cos^2\psi + \kappa_b\sin^2\psi \quad [\because \text{By using (2)}]$$

$$\kappa = \kappa_a\cos^2\psi + \kappa_b\sin^2\psi.$$

The Dupin indicatrix:

The Section of a Surface by a plane parallel to the tangent plane at any point O on it and at a small distance from it is called the Dupin indicatrix at O.

Let P be a point on the Dupin indicatrix at O and let h be a perpendicular distance of P from the tangent plane at O.



Then from the Geometrical interpretation of Second we have fundamental form.

$$2h = L du^2 + 2M du dv + N dv^2 \quad \text{——— (1)}$$

Omitting higher order infinitesimals

If we choose the line of curvature as the parametric curves then $F = 0$ and $M = 0$

So that (1) reduces to

$$2h = L du^2 + N dv^2 \quad \text{——— (2)}$$

Also the principal curvature K_a and K_b are given by

$$\left.\begin{aligned} K_a &= \frac{L}{E} \\ K_b &= \frac{N}{G} \end{aligned}\right\} \quad \text{——— (3)}$$

Sub (3) in eqn (2) we get

$$2h = E K_a du^2 + G K_b dv^2 \quad \text{——— (4)}$$

Again the metric along the parametric curves are $ds^2 = Edu^2 + 2Fdudv + Gdv^2$

$$\left.\begin{aligned} ds_1^2 &= Edu^2 \\ ds_2^2 &= Gdv^2 \end{aligned}\right\} \quad \text{——— (5)}$$

Sub (5) in eqn (4) we get

$$2h = K_a ds_1^2 + K_b ds_2^2 \quad \text{——— (6)}$$

Choose O as origin OX and OY along the Principal directions at O, and OZ along the normal to the surface at O.

If the coordinates of the point P on the Dupin indicatrix be (x, y, z)

Then $x = ds_1$, $y = ds_2$, $z = h$

Hence the eqn to the dupin indicatrix

Sub in (6) we get

$$x^2 K_a + y^2 K_b = 2h$$

Hence $z = h$.

$$\frac{x^2}{R_a} + \frac{y^2}{R_b} = 2z$$

where $R_a = \frac{1}{K_a}$, $R_b = \frac{1}{K_b}$

$K_a = \frac{1}{R_a}$, $K_b = \frac{1}{R_b}$

Thus Dupin indicatrix is a conic section.

Conjugate Directions :

Two direction at P are said to be conjugate if the corresponding diameters of the Dupin indicatrix are conjugate.

In terms of general curvilinear co-ordinates the equation of the indicatrix are

$z = 2h$, $2h = Lx^2 + 2Mxy + Ny^2$

It follows that the direction (l_1, m_1) (l_2, m_2) will be conjugate.

$$Ll_1l_2 + M(l_1m_2 + l_2m_1) + Nm_1m_2 = 0 \quad \text{——— (A)}$$

In particular the direction of the parametric curve will be conjugate if $M = 0$

Thus the line of curvature are along conjugate direction at every point.

$\frac{d\bar{r}}{ds}$ is a tangent vector in the direction (l_1, m_1) and $\frac{\delta \bar{N}}{\delta s}$ is the rate of change of surface normal $\bar{N}$ with the arc length in the direction (l_2, m_2) then eqn (4)

$$\frac{d\bar{r}}{ds} \cdot \frac{\delta \bar{N}}{\delta s} = 0$$

Asyptotic line :

An asymptotic line is a curve whose direction at every point is asymptotic.

The equation of such a line is

$$\frac{d\bar{r}}{ds} \cdot \frac{d\bar{N}}{ds} = 0.$$

$$Ldu^2 + 2Mdudv + Ndv^2 = 0$$

it follows that asymptotic lines are self - conjugate.

Example

1. Show that Gaussian curvature of the surface given by the monge's form.

$z = f(x,y)$ is $(rt - s^2)(1+p^2+q^2)^{-2}$

Given $z = f(x,y)$

Let $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, $r = \frac{\partial^2 z}{\partial x^2}$, $t = \frac{\partial^2 z}{\partial y^2}$

and $s = \frac{\partial^2 z}{\partial x \partial y}$

Let us take x, y as parameters then the position vector of any point on the given surface is

$\bar{r} = \{x, y, f(x,y)\}$

$\bar{r}_1 = \{1, 0, p\}$

$= \bar{i} + p\bar{k}$

$\bar{r}_2 = (0, 1, q)$

$= \bar{j} + q\bar{k}$

$\bar{r}_{11} = (0, 0, \frac{\partial z}{\partial x}) \cdot (0, 0, \frac{\partial z}{\partial x})$

$\Rightarrow \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 r}{\partial x^2}$

$= r\bar{t}$

$\bar{r}_{12} = s\bar{k}$ & $\bar{r}_{22} = t\bar{k}$

$E = \bar{r}_1^2$

$= (1, 0, p) \cdot (1, 0, p)$

$= 1 + p^2$

$F = \bar{r}_1 \cdot \bar{r}_2$

$= (0, 1, q) \cdot (0, 1, q)$

$= 1 + q^2$

$G = \bar{r}_2^2$

$= (0, 1, q)(0, 1, q)$

$= 1 + q^2$

$H^2 = EG - F^2$

$= (1+p^2)(1+q^2) - p^2q^2$

$= 1 + p^2 + q^2 + p^2q^2 - p^2q^2$

$= 1 + p^2 + q^2$

$$\bar{r}_1 \times \bar{r}_2 = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 0 & p \\ 0 & 1 & q \end{vmatrix}$$

$= -p\bar{i} - \bar{j}q + \bar{k}$

$|\bar{r}_1 \times \bar{r}_2| = \sqrt{1 + p^2 + q^2}$

$\bar{N} = \frac{\bar{r}_1 \times \bar{r}_2}{|\bar{r}_1 \times \bar{r}_2|}$

$= \frac{-p\bar{i} - q\bar{j} + \bar{k}}{\sqrt{1+p^2+q^2}}$

$L = \bar{N} \cdot \bar{r}_{11}$

$= \frac{-p\bar{i} + q\bar{j} + \bar{k}}{\sqrt{1+p^2+q^2}} \cdot r\bar{k}$

$= \frac{r}{\sqrt{1+p^2+q^2}}$

$M = \bar{N} \cdot \bar{r}_{12}$

$= \frac{-p\bar{i} - q\bar{j} + \bar{k}}{\sqrt{1+p^2+q^2}} \cdot s\bar{k}$

$= \frac{s}{\sqrt{1+p^2+q^2}}$

$N = \bar{N} \cdot \bar{r}_{22}$

$= \frac{-p\bar{i} - q\bar{j} + \bar{k}}{\sqrt{1+p^2+q^2}} \cdot t\bar{k}$

$$= \frac{t}{\sqrt{1+p^2+q^2}}$$

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Gaussian curvature :

$$K = \frac{LN - M^2}{EG - F^2}$$

$$= \frac{\dfrac{rt}{\sqrt{1+p^2+q^2}\sqrt{1+p^2+q^2}} - \dfrac{s^2}{1+p^2+q^2}}{1+p^2+q^2}$$

$$= \frac{\dfrac{rt - s^2}{(1+p^2+q^2)}}{(1+p^2+q^2)}$$

$$= \frac{rt - s^2}{(1+p^2+q^2)^2}$$

$$= rt - s^2 \cdot (1+p^2+q^2)^{-2}$$

Hence proved.

Example 2 :

Obtain the differential equation of the lines of curvature on a surface $z = f(x,y)$ and deduce that at an umblic $\frac{1+p^2}{r} = \frac{1+q^2}{t} = \frac{pq}{s}$

Soln:

we know that,

$E = 1+p^2$, $F = pq$, $G = 1+q^2$, $L = \frac{r}{\sqrt{1+p^2+q^2}}$, $M = \frac{s}{\sqrt{1+p^2+q^2}}$

$$N = \frac{t}{\sqrt{1+p^2+q^2}}$$

The differential equation of the lines of curvature is

$$(EM - FL)du^2 + (EN - GL)du\,dv + (FN - GM)dv^2 = 0$$

Given curvature on the surface $z = f(x,y)$

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So, we can write the equation is

$$(EM - FL)dx^2 + (EN - GL)dx\,dy + (FN - GM)dy^2 = 0$$

$$\left((1+p^2)\cdot\frac{s}{\sqrt{1+p^2+q^2}} - pq\cdot\frac{r}{\sqrt{1+p^2+q^2}}\right)dx^2 +$$

$$\left((1+p^2)\cdot\frac{t}{\sqrt{1+p^2+q^2}} - (1+q^2)\cdot\frac{r}{\sqrt{1+p^2+q^2}}\right)dx\,dy + \left((pq)\cdot\frac{t}{\sqrt{1+p^2+q^2}} - \frac{(1+q^2)s}{\sqrt{1+p^2+q^2}}\right)dy^2 = 0$$

$$\frac{1}{\sqrt{1+p^2+q^2}}\left[\left\{(1+p^2)s - pqr\right\}dx^2 + \left\{(1+p^2)t - (1+q^2)r\right\}dx\,dy + \left\{pqt - (1+q^2)s\right\}dy^2\right] = 0$$

$$\therefore \left[(1+p^2)s - pqr\right]dx^2 + \left[(1+p^2)t - (1+q^2)r\right]dx\,dy + \left[pqt - (1+q^2)s\right]dy^2 = 0$$

We know that, the umbilic equation is given by,

$$\frac{L}{E} = \frac{M}{F} = \frac{N}{G}$$

Above equation take 1^{st} two terms we get,

$$\frac{L}{E} = \frac{M}{F}$$

$$LF = ME \Rightarrow \frac{F}{M} = \frac{E}{L} \quad \text{——— (A)}$$

Similarly, get last

$$\frac{M}{F} = \frac{N}{G}$$

$$MG = FN \Rightarrow \frac{F}{M} = \frac{G}{N} \quad \text{——— (B)}$$

$$\frac{(1+p^2)}{r/\sqrt{1+p^2+q^2}} = \frac{pq}{s/\sqrt{1+p^2+q^2}} = \frac{1+q^2}{t/\sqrt{1+p^2+q^2}} = 0.$$

$$\sqrt{1+p^2+q^2}\left[\frac{1+p^2}{r}=\frac{pq}{s}=\frac{1+q^2}{t}\right]=0$$

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$$\frac{1+p^2}{r}=\frac{pq}{s}=\frac{1+q^2}{t}$$

Hence proved.

Developables:

A developable is surface enveloped by a one-parameter family of planes such a family is given by the equation

$$\bar{r}\cdot\bar{a}=p \quad \text{--------} (1)$$

Where $\bar{a}$ and p are functions of a real parameter u. It will be convenient to refer to that plane determined by the value of u of the parameter as the plane u.

The planes u, v $(u<v)$ will intersect in a line provided they are not parallel.

If $f(u)=\bar{r}\cdot\bar{a}(u)-p(u)$.

The equation of the lines are $f(u)=0$, $f(v)=0$

From Rolle's theorem, it follow that

"A function f which is differentiable with respect to (a,b), and continuous with respect to $[a,b]$ such that $f(a)=f(b)$, then there exist atleast a point an $c\in(a,b)$. So that $f'(c)=0$"

if there is a value u_1, $u<u_1<v$ such that $f(u_1)=0$

As $v\to u$, $u_1\to u$ and the equations of the limiting position of the line become

$$\left.\begin{aligned}\bar{r}\cdot\bar{a}&=p\\ \bar{r}\cdot\dot{\bar{a}}&=\dot{p}\end{aligned}\right\} \text{--------} (2)$$

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This line is called the characteristic line corresponding to the plane u. It can be regarded as the line of intersection of the plane u with an infinitesimally near plane.

In a similar way, the three planes u, v, w $(u<v<w)$ will generally intersect in one point and the limiting of this point as $v\to u$ and $w\to u$ independently is called the characteristic point corresponding to u.

By using Rolle's theorem, the eqns which determine the point are

$$\left.\begin{aligned}\bar{r}\cdot\bar{a}&=p\\ \bar{r}\cdot\dot{\bar{a}}&=\dot{p}\\ \bar{r}\cdot\ddot{\bar{a}}&=\ddot{p}\end{aligned}\right\} \text{--------} (3)$$

If $\bar{a}, \dot{\bar{a}}, \ddot{\bar{a}}$ are linearly dependent, these equations either have no solution or else the solution is indeterminate.

For example, when the developable is a cylinder with generating lines all parallel to $\bar{a}$.

When the family of planes forms a pencil, the developable degenerates to the axis of the pencil, and ~~the~~ hence the characteristic points are indeterminate.

When the family of planes envelope a cone, all planes have the same characteristic point which is the vertex of the cone.

1) Let us show that the tangent to the edge of regression are the characteristic lines. (24)

By known result,

The edge of regression is given by

$\bar{r}.\bar{a} = p$ ——— ①

$\bar{r}.\dot{\bar{a}} = \dot{p}$ ——— ②

$\bar{r}.\ddot{\bar{a}} = \ddot{p}$ ——— ③

where $\bar{r}, \bar{a}, p$ are all function of parameter 'u'

Differentiate w.r.t 'u' in ①

$$\frac{d}{du}(\bar{r}.\bar{a}) = \frac{d}{du}(p)$$

$$\bar{a}\frac{d}{du}\bar{r} + \bar{r}\frac{d}{du}\bar{a} = \frac{d}{du}(p)$$

$$\bar{a}.\frac{d\bar{r}}{ds}\frac{ds}{du} + \bar{r}\dot{\bar{a}} = \dot{p}$$

$$\bar{a}.\bar{r}'\dot{s} + \bar{r}\dot{\bar{a}} = \dot{p}$$

$$\dot{s}\bar{t}.\bar{a} + \bar{r}\dot{\bar{a}} = \dot{p} \qquad (\because \bar{r}' = \bar{t}) \quad \text{——— ④}$$

Differentiate w.r.t 'u' in ②

$$\frac{d}{du}(\bar{r}.\dot{\bar{a}}) = \frac{d}{du}\dot{p}$$

$$\frac{d\bar{r}}{ds}\frac{ds}{du}.\dot{\bar{a}} + \bar{r}\frac{d}{du}\dot{\bar{a}} = \frac{d}{du}\dot{p}$$

$$\bar{r}'\dot{s}.\dot{\bar{a}} + \bar{r}\ddot{\bar{a}} = \ddot{p}$$

$$\dot{s}\bar{t}.\dot{\bar{a}} + \bar{r}\ddot{\bar{a}} = \ddot{p} \qquad [\because \bar{r}' = \bar{t}] \quad \text{——— ⑤}$$

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Sub ② in ④

$$\dot{s}\bar{t}.\bar{a} + \bar{r}.\dot{\bar{a}} = \bar{r}.\dot{\bar{a}}$$

$$\dot{s}\bar{t}.\bar{a} = 0$$

$$\bar{t}.\bar{a} = 0 \quad \text{——— ⑥}$$

Sub ③ in ⑤

$$\dot{s}\bar{t}.\dot{\bar{a}} + \bar{r}.\ddot{\bar{a}} = r.\ddot{\bar{a}}$$

$$\dot{s}\bar{t}.\dot{\bar{a}} = 0$$

$$\bar{t}.\dot{\bar{a}} = 0 \quad \text{——— ⑦}$$

Therefore ⑥ and ⑦ show that the edge of regression is perpendicular to both $\bar{a}$ and $\dot{\bar{a}}$

Therefore parallel to $\bar{a}\times\dot{\bar{a}}$

But the characteristic line through the point is also parallel to $\bar{a}\times\dot{\bar{a}}$

Hence it follows the tangent to the edge of regression is the characteristic line.

<u>Edge of Regression</u>:

The locus of ultimate intersection of consecutive characteristic lines is called the edge of regression which is a curve lying on the developable.

(B.W)

2) It will be shown that, in general, a developable consists of two sheets which are tangent to the edge of regression along a sharp edge, a property which justifies the terminology used.

Proof:

Let O be the point $S=0$ on the edge of regression C and let Ox, Oy, Oz be the Set of rectangular cartesian axes choosen respectively along $\bar{t}, \bar{n}$ and $\bar{b}$ at O. Then any point on the developable has position Vector given by.

$$\bar{R} = \bar{r} + v\bar{t}$$

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Expanding $\bar{R}$ in Powers of S

$$\bar{R}(s) = \bar{r}(s) + v\bar{t}(s) \quad \text{——— (1)}$$

Since $\bar{r}(s) = s\bar{t} + \frac{s^2}{2}\kappa\bar{n} + \frac{s^3}{6}(\kappa'\bar{n} - \kappa^2\bar{t} + \kappa\tau\bar{b}) + O(s^4)$ ——— (2)

eqn (2) D.w.r.t 's' we get

$$\bar{r}'(s) = \bar{t}(s) = \bar{t} + s\kappa\bar{n} + \frac{1}{2}s^2(\kappa'\bar{n} + \kappa\tau\bar{b} - \kappa^2\bar{t}) + O(s^3) \quad \text{——— (3)}$$

Eqn (2) Sub in (1)

$$\bar{R}(s) = s\bar{t} + \frac{s^2}{2}\kappa\bar{n} + \frac{s^3}{6}(\kappa'\bar{n} - \kappa^2\bar{t} + \kappa\tau b + O(s^4)) + v\bar{t}(s) \quad \text{——— (4)}$$

Since $x\bar{t} + y\bar{n} + z\bar{b} = \bar{r}(s)$ ——— (5)

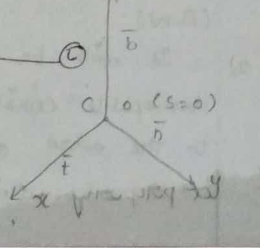
Compare eqn (4) & (5) we get $\bar{t}$ co-efficient

$$x = s - \frac{s^3}{6}\kappa^2 + v$$

The normal plane $x=0$ meets the surface so

$$0 = s + v - \frac{s^3}{6}\kappa^2$$

$$v = -s + \frac{s^3}{6}\kappa^2 + O(s^4) \quad \text{——— (6)}$$



eqn (3) sub in (4) we get

$$\bar{R}(s) = s\bar{t} + \frac{1}{2}s^2\kappa\bar{n} + \frac{s^3}{6}(\kappa'\bar{n} + \kappa\tau\bar{b} - \kappa^2\bar{t}) + O(s^4)$$
$$+ v\{\bar{t} + s\kappa\bar{n} + \tfrac{1}{2}s^2(\kappa'\bar{n} + \kappa\tau\bar{b} - \kappa^2\bar{t}) + O(s^3) \quad \text{——— (7)}$$

compare eqn (5) and (7) we get

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$\bar{n}$ coefficient

$$\Rightarrow y = \frac{1}{2}s^2\kappa + \frac{s^3}{6}\kappa' + s\kappa v + \frac{1}{2}s^2\kappa' v$$

$$= \frac{1}{2}s^2\kappa + \frac{s^3}{6}\kappa' + s\kappa\left(-s + \frac{1}{6}s^3\kappa^2\right) + \frac{1}{2}s^2\kappa'\left(-s - \frac{1}{6}s^3\kappa^2\right)$$

[From (6)]

$$= \frac{1}{2}s^2\kappa + \frac{s^3}{6}\kappa' - s^2\kappa + \frac{1}{6}s^4\kappa^3 - \frac{1}{2}s^3\kappa' - \frac{1}{12}s^5\kappa^2\kappa'$$

$$= -\frac{1}{2}s^2\kappa - \frac{1}{3}s^3\kappa' + \frac{1}{6}s^4\kappa^3 - \frac{1}{12}s^5\kappa^2\kappa'$$

$$y = -\frac{1}{2}s^2\kappa + O(s^3) \quad \text{——— (8)}$$

$\bar{b}$ coefficients

$$\Rightarrow z = \frac{s^3}{6}\kappa\tau + \frac{1}{2}s^2\kappa\tau v$$

$$= \frac{s^3}{6}\kappa\tau + \frac{1}{2}s^2\kappa\tau\left(-s + \frac{1}{6}s^3\kappa^2\right)$$

$$= \frac{s^3}{6}\kappa\tau - \frac{1}{2}s^3\kappa\tau + \frac{1}{12}s^5\kappa^3\tau$$

$$= -\frac{1}{3}s^3\kappa\tau + O(s^4) \quad \text{——— (9)}$$

eqn (9) squaring we get

$$z^2 = \frac{1}{9}s^6\kappa^2\tau^2 \quad \text{——— (10)}$$

and (8) $\Rightarrow$ $s^2 = -\frac{2y}{\kappa}$ ——— (11)

eqn (11) sub in (10) we get.

$$z^2 = \frac{1}{9}\left(-\frac{2y}{\kappa}\right)^3\kappa^2\tau^2$$

$$z^2 = -\frac{8y^3}{9K^3} K^2 T^2$$

$$= -\frac{8}{9} \frac{T^2}{K} y^3 \quad \text{——— (12)}$$

From which it follows that the developable cuts the normal plane to the edge of regression in a cusp whose tangent is along the principal normal. The two sheets of the developable are thus tangent to the edge of regression along a sharp edge.

5. Developable associated with space curves:

At each point of a curve we have the three planes, viz osculating plane, normal plane and the rectifying plane. Each of these planes contains only one parameter viz, the arc lengths. The envelope of these planes are respectively called the osculating developable, polar developable and rectifying developable.

5.1 Osculating developable:

This is the envelope of family of osculating plane of a space curve. Its characteristic lines are the tangents to the curve and hence this developable is also referred to as the tangential developable.

3) Let us prove that its edge of regression is the curve itself.

Proof:

Consider the osculating plane at any point p with position vector $\bar{r}$ on a space curve $\bar{r} = \bar{r}(s)$

If $\bar{R}$ is the position vector of any point on the osculating plane then the vector $\bar{R} - \bar{r}$ lies in the osculating plane.

Hence the family of osculating planes has the equation

$$[\bar{R} - \bar{r}(s)] \cdot \bar{b}(s) = 0 \quad \text{——— (1)}$$

Diff w.r.t 's' we get

$$(\bar{R} - \bar{r})(-\tau\bar{n}) + \bar{b}(-\bar{t}) = 0$$

$$-\bar{t} \cdot \bar{b} + (\bar{R} - \bar{r})(-\tau\bar{n}) = 0$$

$$\bar{t} \cdot \bar{b} + (\bar{R} - \bar{r})(\tau\bar{n}) = 0$$

$$(\bar{R} - \bar{r})\tau\bar{n} = 0$$

$$(\bar{R} - \bar{r}) \cdot \bar{n} = 0 \quad \text{——— (2)}$$

The characteristic lines are the lines of intersection of (1) and (2) which represent the osculating plane and rectifying plane respectively.

Hence their intersection is the tangent to the curve at $p(\bar{r})$.

Diff (2) w.r.t 's' we get.

$$-\bar{t} \cdot \bar{n} + (\bar{R} - \bar{r})(\tau\bar{b} - K\bar{t}) = 0$$

$$(\bar{R} - \bar{r})(\tau\bar{b}) - (\bar{R} - \bar{r})(K\bar{t}) = 0$$

$$(\bar{R} - \bar{r})K\bar{t} = 0$$

$$(\bar{R} - \bar{r}) \cdot \bar{t} = 0 \quad \text{——— (3)}$$

From (1) and (2), (3) we get

$$(\bar{R} - \bar{r}) = 0$$

(i.e) $\bar{R} = \bar{r}$

Thus the characteristic point which is the intersection of the plane (1) (2) (3) is $P(\bar{r})$ itself.

The edge of regression which is the locus of the characteristic point is therefore curve itself.

Polar developable :

This is the surface enveloped by the normal plane of a space curve.

4) To S.T the edge of regression of the polar develople is the locus of the centre of spherical curvature of the given curve.

Proof :

The normal plane $\bar{r} = \bar{r}(s)$ is $(\bar{R}-\bar{r})\bar{t} = 0$ —— (1)

Diff w.r.t 's' we get

$$-\bar{t}\cdot\bar{t} + (\bar{R}-\bar{r})k\bar{n} = 0$$

$$(\bar{R}-\bar{r})k\bar{n} = 1$$

$$(\bar{R}-\bar{r})\bar{n} = \frac{1}{k}$$

$$(\bar{R}-\bar{r})\bar{n} = \rho \quad \text{—— (2)}$$

Diff (2) w.r.t s we get

$$-\bar{t}\cdot\bar{n} + (\bar{R}-\bar{r})(\tau\bar{b}-k\bar{t}) = \rho'$$

$$(\bar{R}-\bar{r})(\tau\bar{b}) - (\bar{R}-\bar{r})(k\bar{t}) = \rho'$$

$$(\bar{R}-\bar{r})(\tau\bar{b}) = \rho'$$

$$(\bar{R}-\bar{r})(\bar{b}) = \frac{1}{\tau}\rho'$$

$$(\bar{R}-\bar{r})(\bar{b}) = \sigma\rho' \quad \text{—— (3)}$$

From (1) (2) and (3) we find the characteristic point is $\bar{R} = \bar{r} + \rho\bar{n} + \sigma\rho'\bar{b}$

But this is the centre of osculating sphere (i.e) The centre of spherical curvature at $P(\bar{r})$ on the curve.

Thus the edge of regression of the polar developable is the locus of the spherical curvature of the given curve.

24 Theorem :

A necessary and sufficient condition for a surface to be developable is that its Gaussian curvature shall be zero.

Proof :

If the developable is a cylinder or cone the Gaussian curvature is evidently zero. If these cases are excluded the developable may be regarded as the osculating developable of its edge of regression and its equation may be written as

$$\bar{R} = \bar{r}(s) + v\bar{t}(s)$$

Denote by suffixes 1 and 2 the differentiations w.r.t the parameter s & v.

$$\bar{R}_1 = \bar{t} + v\cdot k\bar{n}$$

$$\bar{R}_2 = \bar{t}$$

$$\bar{R}_{11} = k\bar{n} + vk'\bar{n} + vk(-k\bar{t}+\tau\bar{b})$$

$$\bar{R}_{12} = \bar{R}_{21} = k\bar{n}$$

$$\bar{R}_{22} = 0$$

$$\bar{R}_1 \times \bar{R}_2 = \begin{vmatrix} \bar{t} & \bar{n} & \bar{b} \\ 1 & vk & 0 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \bar{t}(0) - \bar{n}(0) + \bar{b}(-vk)$$

$$= -vk\bar{b}$$