Example 2

Calculate the first fundamental magnitudes for the surface $\bar{r} = (u\cos v, u\sin v, f(u))$. (12)

Soln:

Given $\bar{r} = (u\cos v, u\sin v, f(u)) \longrightarrow (1)$

Diff (1) w.r.t 'u' we get,

$\bar{r_1} = (\cos v, \sin v, f')$

Diff (1) w.r.t 'v' we get

$\bar{r_2} = (-u\sin v, u\cos v, 0)$

$E = \bar{r_1}\cdot\bar{r_1} = (\cos v, \sin v, f')\cdot(\cos v, \sin v, f')$

$= \cos^2 v + \sin^2 v + f'^2$

$\boxed{E = 1+f'^2}$

$F = \bar{r_1}\cdot\bar{r_2} = (\cos v, \sin v, f')\cdot(-u\sin v, u\cos v, 0)$

$= -u\sin v\cos v + u\cos v\sin v + 0$

$\boxed{F = 0}$

$G = \bar{r_2}\cdot\bar{r_2} = (-u\sin v, u\cos v, 0)\cdot(-u\sin v, u\cos v, 0)$

$= u^2\sin^2 v + u^2\cos^2 v$

$\boxed{G = u^2}$

$ds^2 = E\,du^2 + F\,du\,dv + G\,dv^2$

$\boxed{ds^2 = (1+f'^2)du^2 + u^2 dv^2}$

Ex: 3 Calculate the fundamental co-efficients E, F, G and H for the paraboloid $\bar{r} = (u, v, u^2, -v^2)$.

Soln:

Given $\bar{r} = (u, v, u^2, -v^2) \longrightarrow (1)$

Diff (1) w.r.t 'u' we get

$\bar{r_1} = (1, 0, 2u)$

Diff (1) w.r.t 'v' we get

$\bar{r_2} = (0, 1, -2v)$

We know that

$H = (EG - F^2)^{1/2}$

$E = r_1^2 = \bar{r_1}\cdot\bar{r_1} = (1,0,2u)\cdot(1,0,2u)$

$E = 1+4u^2$

$F = \vec{r_1}\cdot\vec{r_2} = (1,0,2u)\cdot(0,1,-2v)$

$= (-4uv)$

$F = -4uv$

$G = \vec{r_2}\cdot\vec{r_2} = (0,1,-2v)\cdot(0,1,-2v)$

$= 1+4v^2$

$G = 1+4v^2$

$H = (EG-F^2)^{1/2}$

$= ((1+4u^2)(1+4v^2)-16u^2v^2)^{1/2}$

$= (1+4v^2+4u^2+16u^2v^2-16u^2v^2)^{1/2}$

$H = (1+4u^2+4v^2)^{1/2}$

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Ex:4

For the anchor ring $\bar{r} = ((b+a\cos u)\cos v, (b+a\cos u)\sin v, a\sin u)$ calculate the area corresponding to the domain $0\le u\le 2\pi$, $0\le v\le 2\pi$.

Soln:

Given $\bar{r} = ((b+a\cos u)\cos v, (b+a\cos u)\sin v, a\sin u) \longrightarrow (1)$

Diff (1) w.r.t 'u' we get

$\bar{r_1} = (-a\sin u\cos v, -a\sin u\sin v, a\cos u)$

Diff (1) w.r.t 'v' we get,

$\bar{r_2} = (-(b+a\cos u)\sin v, (b+a\cos u)\cos v, 0)$

$E = \bar{r_1}^2 = \bar{r_1}\cdot\bar{r_2} = (-a\sin u\cos v, -a\sin u\sin v, a\cos u)(-a\sin u\cos v, -a\sin u\sin v, a\cos u)$

$= a^2(\sin^2u\cos^2v + \sin^2u\sin^2v + \cos^2u)$

$= a^2(\sin^2u(1) + \cos^2u)$

$E = a^2(\sin^2u+\cos^2u)$

$E = a^2(1)$

$E = a^2$.

$F = \bar{r_1}\cdot\bar{r_2} = (-a\sin u\cos v, -a\sin u\sin v, a\cos u)\cdot(-(b+a\cos u)\sin v, (b+a\cos u)\cos v, 0)$

$= (b+a\cos u)a\sin u\cos v\sin v - a(b+a\cos u)\sin u\sin v + 0$

$F = 0$

$G = \bar{r_2}^2 = \bar{r_2}\cdot\bar{r_2} = (-(b+a\cos u)\sin v, (b+a\cos u)\cos v, 0)\cdot((b+a\cos u)\sin v, (b+a\cos u)\cos v, 0)$

$= (b+a\cos u)^2\sin^2v + (b+a\cos u)^2\cos^2v$

$G = (b+a\cos u)^2$

$H = (EG-F)^{1/2} = \sqrt{a^2(b+a\cos u)^2 - 0}$

$H = a(b+a\cos u)$ (4)

Element of area $= H\,du\,dv$

$= a(b+a\cos u)\,du\,dv$

$\therefore$ The total area $= \int_0^{2\pi}\int_0^{2\pi} a(b+a\cos u)\,du\,dv$

$= \int_0^{2\pi} [abu - a^2\sin u]_0^{2\pi}\,dv$

$= \int_0^{2\pi} [ab2\pi - a^2\sin 2\pi - 0]\,dv$

$= \int_0^{2\pi} [2\pi ab]\,dv$ $\boxed{\sin 2\pi = 0}$

$= [2\pi ab]_0^{2\pi}$

$= [2\pi abv]_0^{2\pi}$

The total area $= 4\pi^2 ab$

Ex:5

ST the metric is invariant under a parameter transformation

Soln:

The parameters u & v are transformed by

$u' = \phi(u,v)$

$v' = \psi(u,v)$

The metric takes the form,

$E'du'^2 + 2F'du'dv' + G'dv'^2 = (\bar{r}_1'du' + \bar{r}_2'dv')^2 \longrightarrow (1)$

$\bar{r}_1' = \frac{\partial r}{\partial u'} = \frac{\partial r}{\partial u}\cdot\frac{\partial u}{\partial u'} + \frac{\partial r}{\partial v}\cdot\frac{\partial v}{\partial u'}$

$\bar{r}_1'$ $\cancel{= \bar{r}_1\frac{\partial u}{\partial u'} + \bar{r}_2'\cdot\frac{\partial v}{\partial u'}}$ $\longrightarrow (2)$

$= \bar{r}_1\,\partial u/\partial u' + \bar{r}_2\,\partial v/\partial u'$

$\bar{r}_2' = \frac{\partial r}{\partial u}\frac{\partial u}{\partial v'} + \frac{\partial r}{\partial v}\cdot\frac{\partial v}{\partial v'}$

$\bar{r}_2' = \bar{r}_1\frac{\partial u}{\partial v'} + \bar{r}_2\frac{\partial v}{\partial v'}$

$(1) \Rightarrow E'du'^2 + 2F'du'dv' + G'dv'^2 = (\bar{r}_1 du' + \bar{r}_2 dv')^2$

$= \left\{\left(\bar{r}_1\frac{\partial u}{\partial u'} + \bar{r}_2\frac{\partial v}{\partial u'}\right)du' + \left(\bar{r}_1\frac{\partial u}{\partial v'} + \bar{r}_2\frac{\partial v}{\partial v'}\right)dv'\right\}^2$

$= \left\{\bar{r}_1\frac{\partial u}{\partial u'}du' + \bar{r}_2\frac{\partial v}{\partial u'}\cdot du' + \bar{r}_1\frac{\partial u}{\partial v'}dv' + \bar{r}_2\frac{\partial v}{\partial v'}dv'\right\}^2$

$= \left\{\bar{r}_1\left(\frac{\partial u}{\partial u'}du' + \frac{\partial u}{\partial v'}dv'\right) + \bar{r}_2\left(\frac{\partial v}{\partial u'}du' + \frac{\partial v}{\partial v'}dv'\right)\right\}^2$

$= (\overline{r_1}du + \overline{r_2}dv)^2$

$= Edu^2 + 2Fdudv + Gdv^2$

$E'du'^2 + 2F'du'dv' + G'dv'^2 = Edu^2 + 2Fdudv + Gdv^2$

Thus the metric is invariant under a parameter transformation but the co-efficient E, F, G are not invariant.

b) Direction co-efficients:

At each point P there are three independent vectors $\overline{N}, \overline{r_1}, \overline{r_2}$.

Every vector $\overline{a}$ can be expressed in the form, $\overline{a} = a_n\overline{N} + \lambda\overline{r_1} + \mu\overline{r_2}$ where scalars a_n, λ, μ are uniquely defined.

$\overline{a}$ is the sum of two vectors $a_n\overline{N}$ normal to the surface and $\lambda\overline{r_1} + \mu\overline{r_2}$ in the tangent plane at P.

The scalar a_n is called the normal component of $\overline{a}$ and is given by. $a_n = \overline{a} \cdot \overline{N}$

The vector $\lambda\overline{r_1} + \mu\overline{r_2}$ is called the tangential part of $\overline{a}$ and λ, μ are the tangential component of $\overline{a}$.

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2 \longrightarrow (A)$$

case (i):

If $\overline{a}$ is the vector (λ, μ) then,

$(A) \Rightarrow |\overline{a}| = |\lambda\overline{r_1} + \mu\overline{r_2}| = (E\lambda^2 + 2F\lambda\mu + G\mu^2)^{1/2} \longrightarrow (1)$ $\left[\because du = \lambda, \; dv = \mu\right]$

$|\overline{a}| = (E\lambda^2 + 2F\lambda\mu + G\mu^2)^{1/2}$

This is the formula for the magnitude of tangential vector in terms of its components.

case (ii):

A direction in the tangent plane at P is described by the component of unit vector in this direction.

The components are called the direction co-efficient (l, m).

They are analogous to direction ~~cosines~~ cosines (l, m, n) satisfy $l^2+m^2+n^2=1$ (16)

(A) $\rightarrow$ $El^2+2Flm+Gm^2=1$ $\quad [\because \lambda u = l,\ \lambda v = m]$.

The formula for the angle between two direction at the same point.

case(iii):

If (l, m) and (l', m') are co-efficients of two direction direction at the same point, the corresponding unit vectors are,

$$\bar{a} = l\bar{r}_1 + m\bar{r}_2$$
$$\bar{a}' = l'\bar{r}_1 + m'\bar{r}_2.$$

The angle between these direction measured in the sense, $\cos\theta = \bar{a}\cdot\bar{a}'$

$$\sin\theta = \bar{a}\times\bar{a}'$$

$$\bar{a}\cdot\bar{a}' = (l\bar{r}_1+m\bar{r}_2)\cdot(l'\bar{r}_1+m'\bar{r}_2)$$
$$= ll'\bar{r}_1^{\,2} + lm'\bar{r}_1\cdot\bar{r}_2 + ml'\bar{r}_2\cdot\bar{r}_1 + mm'\bar{r}_2^{\,2}$$

$$\cos\theta = Ell' + F(lm'+ml') + Gmm'$$

$$\bar{a}\times\bar{a}' = (l\bar{r}_1+m\bar{r}_2)\times(l'\bar{r}_1+m'\bar{r}_2)$$
$$= lm'(\bar{r}_1\times\bar{r}_2) + ml'(\bar{r}_2\times\bar{r}_1)$$
$$= (lm'-l'm)(\bar{r}_1\times\bar{r}_2)$$

$$\sin\theta = (lm'-l'm)H$$

$$\left.\begin{aligned} \cos\theta &= Ell' + F(lm'+ml') + Gmm' \\ \sin\theta &= (lm'-l'm)H \end{aligned}\right\} \rightarrow (3)$$

Case (iv):

From the definition of direction co-efficient l, m it follows that the direction (l, m) is $(-l, -m)$

$$\lambda/l = \mu/m = k.$$

$$1 = El^2 + 2Flm + Gm^2$$

$$\lambda = lk, \quad \mu = mk.$$

$$l = \lambda/k, \quad \mu = \mu/k.$$

$K^2 = E\lambda^2 + 2F\lambda\mu + G\mu^2$ (17)

$K = \sqrt{E\lambda^2 + 2F\lambda\mu + G\mu^2}$

$(l, m) = \frac{(\lambda, \mu)}{K} = \frac{(\lambda, \mu)}{\sqrt{E\lambda^2 + 2F\lambda\mu + G\mu^2}}$

The condition is orthogonal direction is $\cos\theta = 0$

(3) $\Rightarrow E\lambda\lambda' + F(\lambda\mu' + \lambda'\mu) + G\mu\mu' = 0 \longrightarrow$ (4)

The vectors $\bar{r}_1$ & $\bar{r}_2$ have components $(1,0)$ & $(0,1)$

(A) $\Rightarrow \frac{(1,0)}{(E(1)^2 + 2F \cdot 1 \cdot 0 + G0^2)^{1/2}}$ & $\frac{(0,1)}{(E0^2 + 2F \cdot 0 \cdot 1 + G \cdot 1^2)^{1/2}}$

(i.e.) $\left(\frac{1}{\sqrt{E}}, 0\right)$ and $\left(0, 1/\sqrt{G}\right)$

Case (v):

$u = u(t)$, $v = v(t)$, the position vector is -

$\bar{r} = \bar{r}(u, v) = \bar{r}(t)$, $\dot{\bar{r}}$ is the tangent vector.

$\dot{\bar{r}} = \bar{r}_1\dot{u} + \bar{r}_2\dot{v}$ is the components of $\dot{\bar{r}}$ are $(\dot{u}, \dot{v})$

The magnitude $|\dot{\bar{r}}|$ is is calculated from the metric:

$$\frac{d\bar{r}}{ds} = \frac{du}{ds}\bar{r}_1 + \frac{dv}{ds}\bar{r}_2$$

The direction co-efficients are,

$$l = \frac{du}{ds}, \quad m = dv/ds.$$

$\phi(u,v) = 0$, $\phi_1 du + \phi_2 dv = 0$.

Families of curves:

A family of curve on a surface we mean a system given by an implicit eqn $\phi(u,v) = c$ where ϕ is single valued continuous function derivatives ϕ_1, ϕ_2 which do not vanish. and c is a real parameter.

$\phi_1 du + \phi_2 dv = 0$ It follows that at any point (u,v) the tangent to the curve through the direction ratios $(-\phi_2, \phi_1)$.

...is is constant on the ... (17)
differential eqn

Proof:

Let μ be the magnitude of the vector $(-\phi_2, \phi_1)$ and θ the angle between the direction with coefficients (l, m) the vector $(-\phi_2, \phi_1)$

Then from the sine formula,

$\mu\sin\theta = \mu(l\phi_1 + m\phi_2)$

$= \mu\left(\frac{du}{ds}\phi_1 + \frac{dv}{ds}\phi_2\right)$

$\mu\sin\theta = \mu\frac{d\phi}{ds}$

$\because l = \frac{du}{ds}, m = \frac{dv}{ds}$ for the differential du, dv in the direction (l, m).

μ & μ are independent of (l, m).

$\left|\frac{d\phi}{ds}\right|$ has its maximum values μ/μ when $\theta = \pm\pi/2$

(i.e.) in a direction is orthogonal to ϕ = constant.

$\because \mu > 0$ & $\mu > 0$ the orthogonal direction for which $\frac{d\phi}{ds} > 0$ is such that $\theta = \pi/2$.

The curves of the family ϕ = constant are the solution of differential eqn $\phi_1 du + \phi_2 dv = 0$.

Conversely,

The first order differential eqn of the form

$P(u,v)du + Q(u,v)dv = 0 \longrightarrow (1)$

where P & Q are C^1 functions which do not vanish simultaneously always defines a family of curve $\lambda(u,v) \neq 0$ & $\phi(u,v)$ can be found so that $\lambda P = \phi_1$ & $\lambda Q = \phi_2$, the soln of the eqn are the curves ϕ = constant.

Hence Proved.

Orthogonal trajectories

P.T $\frac{\partial(\phi,\psi)}{\partial(u,v)} = \frac{1}{\lambda\mu}[EQ^2 ...]$

Proof:

For a given family of curve there always exists a second family, the orthogonal trajectories such that at every point the two curves one from each family are orthogonal.

Let the given family be defined by

$Pdu + Qdv = 0$

The tangent at the point (u,v) is in the direction $(-Q, P)$ and if du, dv are now taken to be differential in an orthogonal direction,

The condition for orthogonality, w.k.T

$E\lambda\lambda' + F(\lambda\mu' + \lambda'\mu) + G\mu\mu' = 0$ we get $(\lambda,\mu) \to (du, dv)$, $(\lambda', \mu') \to (-Q, P)$

$E(-Q)du + F(-Qdv + Pdu) + GPdv = 0$

$E(-Q)du + F(-Qdv + Pdu) + GPdv = 0$

$(FP - EQ)du + (GP - FQ)dv = 0 \longrightarrow (2)$

(i.e.) $(FP - EQ)du + (GP - FQ)dv = 0$

The coefficient du & dv are continuous and do not vanish together since $EG \neq F^2$ and P & Q do not vanish together.

If (2) is $\psi(u,v)$ = constant, then $FP - EQ = \mu\psi_1$ & $GP - FQ = \mu\psi_2$ for some $\mu \neq 0$.

If also the given family is ϕ = constant such $P = \lambda\phi_1$ & $Q = \lambda\phi_2$ $(\lambda \neq 0)$.

$$\frac{\partial(\phi,\psi)}{\partial(u,v)} = \frac{1}{\lambda\mu}\begin{vmatrix} P & Q \\ FP-EQ & GP-FQ \end{vmatrix}$$

$$= \frac{1}{\lambda\mu}[P(GP - FQ) - Q(FP - EQ)]$$

$$= \frac{1}{\lambda\mu}[GP^2 - FPQ - FPQ + EQ^2]$$

$$\frac{\partial(\phi,\psi)}{\partial(u,v)} = \frac{1}{\lambda\mu}[EQ^2 - 2FPQ + GP^2]$$

$\therefore \frac{\partial(\phi,\psi)}{\partial(u,v)} \neq 0$, P, Q do not vanish together.

Double family of curves:

If P, Q and R are continuous functions of u & v which do not vanish together.

The quadratic diff eqn.

$$P du^2 + 2Q du dv + R dv^2 = 0 \longrightarrow (1)$$

has two families of curves provided that $Q^2 - PR > 0$.

The diff eqn for the separate family are found by solving the quadratic, for the ratio du, dv.

The direction co-efficient for the two tangents at a point are (l, m) and (l', m') then l/m & l'/m' are two values of du/dv obtained from the quadratic eqn.

$$P\left(\frac{du}{dv}\right)^2 + 2Q\left(\frac{du}{dv}\right) + R = 0$$

Thus $\frac{l}{m} + \frac{l'}{m'} = \frac{-2Q}{P}$

$$\frac{l}{m} \cdot \frac{l'}{m'} = R/P$$

$$\frac{lm' + l'm}{mm'} = \frac{-2Q}{P} \Rightarrow \frac{lm' + l'm}{-2Q} = \frac{mm'}{P} \longrightarrow (2)$$

$$\frac{ll'}{R} = \frac{mm'}{P} \longrightarrow (3)$$

From (2) & (3)

$$\frac{ll'}{R} = \frac{mm'}{P} = \frac{lm' + l'm}{-2Q}$$

The condition for orthogonality of the two families is therefore

$$Ell' + F(lm' + l'm) + Gmm' = 0$$

$$E(R) + F(-2Q) + G(P) = 0$$

$$E(R) - 2FQ + GP = 0 \longrightarrow (4)$$

If $P = R = 0$, (1) becomes $du dv = 0$ giving the two families of parametric curves.

∴ The condition for orthogonality is then $F = 0$.

Ex 11

On the paraboloid $x^2 - y^2 = z$, find the orthogonal trajectories of the sections by the planes $z =$ constant.

soln:

Given $x^2 - y^2 = z$

Let $x = u$, $y = v$

$z = u^2 - v^2$ as the parametric eqns on the given surface.

Let $\bar{r} = (x, y, z) = (u, v, u^2 - v^2) \longrightarrow (A)$

Then the given curves are

$$u^2 - v^2 = z = \text{constant}$$

$$2u du - 2v dv = 0$$

$$u du - v dv = 0$$

$$u du = v dv$$

$$\frac{du}{dv} = v/u$$

The direction of the tangent at the point (u, v) is therefore that of (v, u).

If (du, dv) is now orthogonal to this direction

then $E v du + F(v dv + u du) + G u dv = 0 \longrightarrow (1)$

Diff (A) w.r.t 'u' we get.

$$\bar{r}_1 = (1, 0, 2u)$$

Diff (A) w.r.t 'v' we get

$$\bar{r}_2 = (0, 1, -2v)$$

$$\left.\begin{aligned} E &= r_1^2 = \bar{r}_1 \cdot \bar{r}_1 = (1,0,2u)\cdot(1,0,2u) = 1 + 4u^2 \\ F &= \bar{r}_1 \cdot \bar{r}_2 = (1,0,2u)\cdot(0,1,-2v) = -4uv \\ G &= r_2^2 = \bar{r}_2 \cdot \bar{r}_2 = (0,1,-2v)\cdot(0,1,-2v) = 1 + 4v^2 \end{aligned}\right\} \longrightarrow (2)$$

Sub (2) in (1) we get

(1) $\Rightarrow (1 + 4u^2) v du - 4uv(v dv + u du) + (1 + 4v^2) u dv = 0$

$$v du + 4u^2 v du - 4uv^2 dv - 4u^2 v du + u dv + 4uv^2 dv = 0$$

$$v du + u dv = 0$$

$$d(uv) = 0$$

∴ $uv =$ constant.

The orthogonal trajectories are therefore given by $xy = c$.

These are the sections of the paraboloid by the hyperbolic cylinders

10) A helicoid is generated by a screw motion of a straight line which meets the axis at an angle α. Find the orthogonal trajectories of the generators. Find also the metric of the surface referred to the generators and their orthogonal trajectories as parametric curves.

(93)

Soln:

case(i):

The eqn of the helicoid is given by,

$\bar{r} = (g(u)\cos v, g(u)\sin v, f(u) + av) \longrightarrow (1)$

where $g(u) = u\sin\alpha$, $f(u) = u\cos\alpha$.

$(1) \Rightarrow \bar{r} = (u\sin\alpha\cos v, u\sin\alpha\sin v, u\cos\alpha + av) \longrightarrow (2)$

$\bar{r}_1 = (\sin\alpha\cos v, \sin\alpha\sin v, \cos\alpha)$

$\bar{r}_2 = (-u\sin\alpha\sin v, u\sin\alpha\cos v, a)$.

$E = r_1^2 = \bar{r}_1\cdot\bar{r}_1 = \sin^2\alpha\cos^2 v + \sin^2\alpha\sin^2 v + \cos^2\alpha$.

$= \sin^2\alpha(\cos^2 v + \sin^2 v) + \cos^2\alpha$

$= \sin^2\alpha + \cos^2\alpha$

$\boxed{E = 1}$

$F = \bar{r}_1\cdot\bar{r}_2 = (\sin\alpha\cos v, \sin\alpha\sin v, \cos\alpha)\cdot(-u\sin\alpha\sin v, u\sin\alpha\cos v, a)$

$\boxed{F = a\cos\alpha}$

$G = r_2^2 = \bar{r}_2\cdot\bar{r}_2 \Rightarrow u^2\sin^2\alpha\sin^2 v + u^2\sin^2\alpha\cos^2 v + a^2$

$= u^2\sin^2\alpha(\sin^2 v + \cos^2 v) + a^2$

$\boxed{G = u^2\sin^2\alpha + a^2}$

The generators are given by v = constant and have direction ratio $(1,0)$

The direction (du, dv) is orthogonal to $(1,0)$ if,

$E v\,du + F(v\,dv + u\,du) + G u\,dv = 0 \longrightarrow (3)$

$(3) \Rightarrow E(1)du + F(1dv + 0) + G(0)dv = 0$

$E\,du + F\,dv = 0$

$\Rightarrow du + a\cos\alpha\,dv = 0$

$d(u + av\cos\alpha) = 0$

The orthogonal trajectories of the generators are therefore given by, $u + av\cos\alpha$ = constant.

(94)

case(ii):

To examine these trajectories that $u = 0$ for some value of v on every curve so that every trajectory meets the axis of the helicoid.

For a particular curve there is no loss of generality in taking its intersection with the axis to be the origin.

Then $u = -av\cos\alpha$

The curve is given by

$(2) \Rightarrow \bar{r} = a\sin\alpha(-v\cos\alpha\cos v, -v\cos\alpha\sin v, v\sin\alpha)$

with v as parameter.

It is the intersection of the cone

$x^2 + y^2 = (v^2\cos^2\alpha\cos^2 v + v^2\cos^2\alpha\sin^2 v)a^2\sin^2\alpha$

$= a^2v^2\sin^2\alpha\cos^2\alpha$

$x^2 + y^2 = z^2\cot^2\alpha$

The cylinder whose cross section by the xy plane is the spiral

$r = -\frac{1}{2}a\cos\alpha$ [illegible]

A transformation which takes the generators and their orthogonal trajectories into parametric curve is

$u' = u + av\cos\alpha$; $v' = v$

$du' = du + a\cos\alpha\,dv$; $dv' = dv$

$du = du' - a\cos\alpha\,dv'$; $dv = dv' \longrightarrow (4)$

The metric is

$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$.

$= (1)du^2 + 2a\cos\alpha\,du\,dv + (a^2 + u^2\sin^2\alpha)dv^2$

$ds^2 = (du' - a\cos\alpha\,dv')^2 + 2a\cos\alpha(du' - a\cos\alpha\,dv')dv' + [a^2 + \sin^2\alpha(u' - av'\cos\alpha)^2]dv'^2$

$= du'^2 + a^2\cos^2\alpha\,dv'^2 - 2a\cos\alpha\,dv'\,du' + 2a\cos\alpha\,du'\,dv' - 2a^2\cos^2\alpha\,dv'^2 + a^2dv'^2 + \sin^2\alpha(u' - av'\cos\alpha)^2dv'^2$

$= du'^2 - a^2\cos^2\alpha\,dv'^2 + a^2dv'^2 + \sin^2\alpha(u' - av'\cos\alpha)^2dv'^2$

$ds^2 = du'^2 + a^2dv'^2(1 - \cos^2\alpha) + \sin^2\alpha(u' - av'\cos\alpha)^2dv'^2$

$ds^2 = du^2 + \sin^2\alpha\,[a^2 + (u - av\cos\alpha)^2]\,dv^2$

The new co-eff are,

$E = 1,\ F = 0,\ G = \sin^2\alpha\,[a^2 + (u - av\cos\alpha)^2]$

Hence proved.

Ex: 3

S.T the curves $du^2 - (u^2 + a^2)dv^2 = 0$ on the right helicoid form an orthogonal net.

Soln:

Let $P du^2 + 2Q du dv + R dv^2 = 0 \longrightarrow (1)$

Given $du^2 - (u^2 + a^2)dv^2 = 0 \longrightarrow (2)$

comparing (1) & (2) we get.

$P = 1,\ Q = 0,\ R = (-(u^2 + a^2))$

The eqn of the right helicoid is,

$\bar{r} = (u\cos v, u\sin v, av) \longrightarrow (3)$

diff (3) w.r.t 'u' we get.

$\bar{r}_1 = (\cos v, \sin v, 0)$

diff (3) w.r.t 'v' we get

$\bar{r}_2 = (-u\sin v, u\cos v, a)$

$E = r_1^2 = \bar{r}_1 \cdot \bar{r}_1 = (\cos v, \sin v)\cdot(\cos v, \sin v) = \cos^2 v + \sin^2 v = 1$

$F = \bar{r}_1 \cdot \bar{r}_2 = (\cos v, \sin v, 0)\cdot(-u\sin v, u\cos v, a)$

$= -\cos v\, u\sin v + u\cos v\sin v + 0$

$F = 0$

$G = r_2^2 = \bar{r}_2 \cdot \bar{r}_2 = u^2\sin^2 v + u^2\cos^2 v + a^2$

$= u^2(\sin^2 v + \cos^2 v) + a^2$

$= u^2(1) + a^2$

$G = u^2 + a^2$

We know that

$ER - 2FQ + GP = 1[-(u^2 + a^2)] - 2(0) + (u^2 + a^2)(1)$

$= -u^2 - a^2 + u^2 + a^2$

$= 0$

$ER - 2FQ + GP = 0$

$\therefore$ The given curves form an orthogonal net.

Ex: 4 S.T on the right helicoid, the family of curves orthogonal to the curves $u\cos v$ = constant are the family $(u^2 + a^2)\sin^2 v$ = constant

Soln:

The right helicoid is

$\bar{r} = (u\cos v, u\sin v, av) \longrightarrow (1)$

diff (1) w.r.t 'u' we get

$\bar{r}_1 = (\cos v, \sin v, 0)$

diff (1) w.r.t 'v' we get

$\bar{r}_2 = (-u\sin v, u\cos v, a)$

$E = \bar{r}_1^2 = 1,\ F = \bar{r}_1 \cdot \bar{r}_2 = 0,\ G = \bar{r}_2^2 = a^2 + u^2$

Family of curves given by $u\cos v$ = constant has the diff equation.

$\cos v\, du - u\sin v\, dv = 0$

$\frac{du}{dv} = \frac{u\sin v}{\cos v}$

Hence the direction ratios of the tangent to the family of curves $u\cos v$ = constant at the point (u, v) are $(u\sin v, \cos v)$.

If (du, dv) be a direction orthogonal to this direction we get.

$Eu\sin v\, du + F(u\sin v\, dv + \cos v\, du) + G\cos v\, dv = 0$

$(1)u\sin v\, du + 0 + (u^2 + a^2)\cos v\, dv$

$u\sin v\, du = -(u^2 + a^2)\cos v\, dv$

$u\, du = -\frac{(u^2 + a^2)\cos v\, dv}{\sin v}$

$\frac{u\, du}{u^2 + a^2} = \frac{-\cos v\, dv}{\sin v}$

$\frac{u\, du}{u^2 + a^2} + \frac{\cos v}{\sin v}\, dv = 0$

$\times$ (2)

$\frac{2u\, du}{u^2 + a^2} + \frac{2\cos v}{\sin v}\, dv = 0$

Integrating we get $\log(u^2 + a^2) + 2\log\sin v = \log c$

log $(u^2+a^2)\sin^2 v = \log c$

$(u^2+a^2)\sin^2 v = c$

Hence proved.

Ex:15

Show that the curves bisecting the angle between the parametric curves are given by $E du^2 - G dv^2 = 0$.

Soln:

The direction co-efficients of the parametric curves $v =$ constant & $u =$ constant are respectively w.r.t

$$\left(\frac{1}{\sqrt{E}}, 0\right) \text{ and } \left(0, \frac{1}{\sqrt{G}}\right)$$

Let $\left(\frac{du}{ds}, \frac{dv}{ds}\right)$ be the direction co-efficient which bisects the angle between the parametric curves.

Let α be the angle which these direction makes with $(1/\sqrt{E}, 0)$ and β be the angle which their direction makes with $(0, 1/\sqrt{G})$

$$\cos\alpha = \frac{E}{\sqrt{E}}\frac{du}{ds} + \frac{F}{\sqrt{E}}\frac{dv}{ds}$$

$$\cos\beta = \frac{F}{\sqrt{G}}\frac{du}{ds} + \frac{G}{\sqrt{G}}\frac{dv}{ds}$$

Since the direction $\left(\frac{du}{ds}, \frac{dv}{ds}\right)$ bisects the angle between the parametric curves $\cos\alpha = \pm\cos\beta$

$$\frac{1}{\sqrt{E}}\left(E\frac{du}{ds} + F\frac{dv}{ds}\right) = \pm\frac{1}{\sqrt{G}}\left(F\frac{du}{ds} + G\frac{dv}{ds}\right)$$

$$\frac{E\sqrt{G}\,du + F\sqrt{G}\,dv}{ds} = \pm\frac{F\sqrt{E}\,du + G\sqrt{E}\,dv}{ds}$$

$$E\sqrt{G}\,du + F\sqrt{G}\,dv = F\sqrt{E}\,du + G\sqrt{E}\,dv$$

Squaring on both sides we get,

$$(E\sqrt{G}\,du + F\sqrt{G}\,dv)^2 = (F\sqrt{E}\,du + G\sqrt{E}\,dv)^2$$

$$E^2G\,du^2 + F^2G\,dv^2 + 2EFG\,du\,dv = F^2E\,du^2 + G^2E\,dv^2 + 2EFG\,du\,dv$$

$$E^2G\,du^2 - F^2E\,du^2 = G^2E\,dv^2 - F^2G\,dv^2$$

$$E(EG-F^2)du^2 = G(EG-F^2)dv^2$$

$\because EG - F^2 \neq 0$

we get, $E\,du^2 = G\,dv^2$

$E\,du^2 - G\,dv^2 = 0$.

Hence proved.

Ex:16

P.T if θ is the angle at the point (u,v) between the two direction given by $P\,du^2 + 2Q\,du\,dv + R\,dv^2 = 0$ then

$$\tan\theta = \frac{2H(Q^2-PR)^{1/2}}{ER-2FQ+GP}$$

Soln:

Given θ is the angle at the point (u,v).

If (λ,μ) and (λ',μ') be ratio of the two directions given by.

$$P\,du^2 + 2Q\,du\,dv + R\,dv^2 = 0$$

$$\div dv^2 \Rightarrow P\left(\frac{du}{dv}\right)^2 + 2Q\left(\frac{du}{dv}\right) + R = 0 \longrightarrow (1)$$

then $$\lambda/\mu + \lambda'/\mu' = \frac{-2Q}{P} \longrightarrow (2)$$

$$\frac{\lambda\lambda'}{\mu\mu'} = \frac{R}{P} \longrightarrow (3)$$

$$\tan\theta = \frac{H(\lambda\mu' - \lambda'\mu)}{E\lambda\lambda' + F(\lambda'\mu + \mu'\lambda) + G\mu\mu'}$$

$$= \frac{H(\lambda\mu' - \lambda'\mu)}{E\lambda\lambda' + F(\lambda'\mu + \mu'\lambda) + G\mu\mu'}$$

$$= \frac{H\left(\lambda/\mu - \frac{\lambda'}{\mu'}\right)}{E\left(\frac{\lambda\lambda'}{\mu\mu'}\right) + F\left(\lambda/\mu + \lambda'/\mu'\right) + G}$$

$$= \frac{H\left[\left(\lambda/\mu + \lambda'/\mu'\right)^2 - 4\frac{\lambda\lambda'}{\mu\mu'}\right]^{1/2}}{E\left(\frac{\lambda\lambda'}{\mu\mu'}\right) + F\left(\lambda/\mu + \lambda'/\mu'\right) + G}$$

$$= \frac{H\left[\left(\frac{-2Q}{P}\right)^2 - 4R/P\right]^{1/2}}{E(R/P) + F\left(\frac{-2Q}{P}\right) + G}$$

$$= \frac{2H\left(\frac{Q^2}{P^2} - R/P\right)^{1/2}}{ER/(-2Q)+GP} \Rightarrow \frac{\frac{2H}{P}(Q^2-PR)^{1/2}}{\frac{ER-2FQ+GP}{P}}$$

$$\tan\theta = \frac{2H(Q^2-PR)^{1/2}}{ER-2FQ+GP}$$