① UNIT - II

The metric: Local Intrinsic properties of a surface:

Definition: Surface:

1) A surface often arises as the locus of a point P which satisfies some restrictions as a consequence of which the coordinates x, y, z of P satisfy a relation of the form

$$F(x,y,z) = 0 \longrightarrow ①$$

This is called the implicit or constraint equation of the surface.

Class r:

The parametric or freedom equation of a surface are of the form

$$x = f(u,v),\quad y = g(u,v),\quad z = h(u,v) \longrightarrow ②$$

where u and v are the parameters and are real values varies in some domain D.

The function f, g, h are single valued and continuous function and continuous partial derivative of the r^{th} order.

The surface is said to be of class r.

2) curvilinear co-ordinates:

The class of a surface is not stated explicity it should be assumed that the function f, g, h possess as many derivatives usually 2 or 3.

Parameters u & v are called curvilinear co-ordinates.

The point determined by the pair u, v is referred as the point (u, v).

Ex: 1

consider the surface given by the parametric equation,

$$x = u+v \rightarrow ①,\quad y = u-v \rightarrow ②,\quad z = 4uv \rightarrow ③$$

where u, v are real.

$$① + ② \Rightarrow x+y = 2u+v-v$$

$$x+y = 2u$$

$$\boxed{u = \frac{x+y}{2}} \longrightarrow (4)$$

(1)-(2) $\Rightarrow$

$x - y = 2v + u - v$

$x - y = 2v$

$$\boxed{v = \frac{x-y}{2}} \longrightarrow (5)$$

(3) $\Rightarrow$ $z = 4uv$

$z = 4\left(\frac{x+y}{2}\right)\left(\frac{x-y}{2}\right) \Rightarrow \frac{4}{4}(x+y)(x-y)$

$= (x+y)(x-y)$

$= x^2 - xy + yx - y^2$

$$\boxed{z = x^2 - y^2}$$

which represent the whole of a certain hyperbolic paraboloid.

Ex: 2

consider the surface given by the parametric eqn.

$x = u\cosh v \longrightarrow (1)$

$y = u\sinh v \longrightarrow (2)$

$z = u^2 \longrightarrow (3)$

where u & v are real.

Eliminative u and v we get.

The constraint eqn is..

$$x^2 - y^2 = z$$

which represent the whole of paraboloid.

NOTE:

(i) The parameter transformation of the form

$u' = \phi(u,v)$, $v' = \psi(u,v)$

This transformation is said to be proper. If ϕ & ψ are single valued function have non vanishing Jacobian. $\frac{\partial(\phi,\psi)}{\partial(u,v)} \neq 0$ in some domain D.

(ii) The position vector $\bar{r} = (x,y,z)$ of a point on the surface function of u & v with the same continuity & differentiability f, g, h.

$\bar{r}_1 = \frac{\partial \bar{r}}{\partial u}$, $\bar{r}_2 = \frac{\partial \bar{r}}{\partial v}$

An ordinary point is defined as one for which (3)

$$\bar{r}_1 \times \bar{r}_2 \neq 0$$

$$\text{rank} \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix} = 2$$

$$\bar{r}_1 \times \bar{r}_2 = \left(\frac{\partial \bar{r}}{\partial u'} \phi_1 + \frac{\partial \bar{r}}{\partial v'} \psi_1\right) \times \left(\frac{\partial \bar{r}}{\partial u'} \phi_2 + \frac{\partial \bar{r}}{\partial v'} \psi_2\right)$$

$$\bar{r}_1 \times \bar{r}_2 = \frac{\partial(\phi, \psi)}{\partial(u, v)} \left(\frac{\partial \bar{r}}{\partial u'} \times \frac{\partial \bar{r}}{\partial v'}\right)$$

Since Jacobian is non zero.

$\bar{r}_1 \times \bar{r}_2 \neq 0$ then $\frac{\partial \bar{r}}{\partial u'} \times \frac{\partial \bar{r}}{\partial v'} \neq 0$

A point is not an ordinary point is called a singularity.

Some singularities are essential, independent of the parametric representation.

The origin of polar co-ordinates in the plane is the simplest example of artificial singularity for it

$$\bar{r} = (u\cos v, u\sin v, 0) \text{ when } u = 0 .$$

Definition 1:

A representation R of a surface S of class r in E_3 is a set of points in E_3 covered by a system of overlapping parts $\{v_j\}$ each part v_j being given by parametric equation of class r. Each point lying in the overlap of two parts $v_i v_j$ is such that the change of parameters from those of one part to those of the other parts is proper and class r.

Definition 2:

Two representation R, R' are said to be r-equivalent if the composite family of parts $\{v_j, v_j'\}$ satisfies the condition that at each point p lying in the overlap of any two parts. The change of parameters from those of one part to those of another is proper and class r.

Definition 2:

Two representation R, R' are said to be r-equivalent if the composite family of parts $\{V_j, V_j'\}$ satisfies the condition that at each point P lying in the overlap of any two parts, the change of parameters from those of one part to those of another is proper and class r.

Definition 3:

A surface S of class r in E_3 is an r-equivalence class of representations.

curves on a surface:

Let $\bar{r} = \bar{r}(u,v)$ be the equation of a surface of class r defined on a domain D.

Let $u = U(t)$, $v = V(t)$ be a curve of class s lying in D.

Then $\bar{r} = \bar{r}(U(t), V(t))$ is a curve lying on the surface with class equal to the smaller of r & s.

The equation.

$u = U(t), \quad v = V(t) \longrightarrow (1)$

is the (curvilinear) equation of a curve.

curves of particular importance are the u (or) v constant.

Let v have a constant value c then u varies the point $\bar{r} = \bar{r}(u,c)$ traces a curve called the parametric curve $v = c$.

orthogonal:-

Two parametric curves through a point P are orthogonal if $\bar{r}_1 \cdot \bar{r}_2 = 0$ at P. If this condition is satisfied at every point (i.e.) for all u, v in the domain D, the two system of parametric curves are orthogonal.

Normal to the surface:-

The normal to the surface at P is the normal to the tangent plane at P and is $\perp$ to $\bar{r}_1$ & $\bar{r}_2$.

$$\boxed{\bar{N} = \frac{\bar{r}_1 \times \bar{r}_2}{H}}$$

where $H = |\bar{r}_1 \times \bar{r}_2| \neq 0$.

NOTE:

$\bar{N}$ and $\bar{N}'$ are therefore the same vector if $\frac{\partial(\phi,\psi)}{\partial(u,v)} > 0$ and are opposite if $\frac{\partial(\phi,\psi)}{\partial(u,v)} < 0$

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3) Surface of Revolution:

(a) (i) Sphere:

When the polar angles (ie) the co-latitude u & the longitude v are taken as parameters on a sphere of centre O and radius a, The position vector is, $\bar{r} = a(\sin u\cos v, \sin u \sin v, \cos u) \longrightarrow (1)$

The poles $u=0$ & $u=\pi$ are singularities and domain of u,v is $0<u<\pi$, $0\le v< 2\pi$.

The parametric curves $v=$ constant are the meridians & $u=$ constant the parallels.

$$\bar{r}_1 = a(\cos u\cos v, \cos u \sin v, -\sin u)$$

$$\bar{r}_2 = a(-\sin u \sin v, \sin u \cos v, 0)$$

clearly $\bar{r}_1 \cdot \bar{r}_2 = 0$ at every point.

$$(\bar{r}_1 \cdot \bar{r}_2) = a[(\cos u\cos v, \cos u \sin v, -\sin u) \cdot (-\sin u \sin v, \sin u \cos v, 0)]$$

$$= a[-\cos u \cos v \sin u \sin v + \cos u \sin v \sin u \cos v + 0]$$

$$= a(0)$$

$$\boxed{\bar{r}_1 \cdot \bar{r}_2 = 0}$$

The two systems of the parametric curve are orthogonal:

$$\bar{r}_1 \times \bar{r}_2 = a^2 \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \cos u \cos v & \cos u \sin v & -\sin u \\ -\sin u \sin v & \sin u \cos v & 0 \end{vmatrix}$$

$$= a^2[\bar{i}(\sin^2 u \cos v) - \bar{j}(-\sin^2 u \sin v) + \bar{k}(\cos u \cos^2 v \sin u + \cos u \sin^2 v \sin u)]$$

$$\bar{r}_1 \times \bar{r}_2 = a^2[\sin^2 u \cos v, \sin^2 u \sin v, \sin u \cos v]$$

$$H = |\bar{r}_1 \times \bar{r}_2| = a^2\sqrt{\sin^4 u \cos^2 v + \sin^4 u \sin^2 v + \sin^2 u \cos^2 u}$$ (6)

$$= a^2 \sqrt{\sin^4 u(\cos^2 v + \sin^2 v) + \sin^2 u \cos^2 u}$$

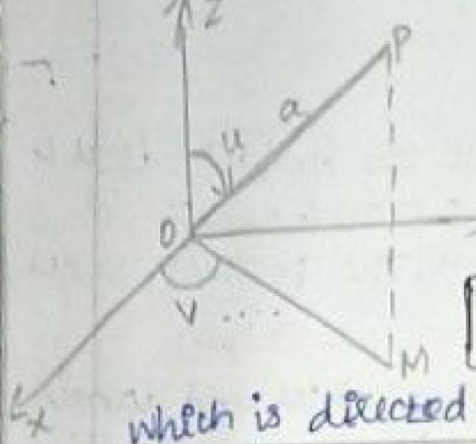
$$= a^2 \sqrt{\sin^4 u + \sin^2 u \cos^2 u}$$

$$= a^2 \sqrt{\sin^2 u(\sin^2 u + \cos^2 u)}$$

$$H = a^2\sqrt{\sin^2 u}$$

$$\boxed{H = a^2 \sin u}$$

The unit Normal $\bar{N} = \dfrac{\bar{r}_1 \times \bar{r}_2}{H} = \dfrac{a^2 (\sin^2 u \cos v, \sin^2 u \sin v, \sin u \cos u)}{a^2 \sin u}$



$$= a^2 \sin u \frac{(\sin u \cos v, \sin u \sin v, \cos u)}{a^2 \sin u}$$

$$= \frac{a \sin u}{a^2 \sin u}(\sin u \cos v, \sin u \sin v, \cos u)$$

$$\boxed{\bar{N} = \frac{1}{a}\bar{r}}$$ [using (1)]

which is directed outwards from the sphere.

(b) The general surface of revolution :

Taking the z-axis for the axis of revolution. Let the generation curve in the xz-plane be given by the parametric equation.

$$x = g(u);\quad y = 0,\quad z = f(u).$$

Then v is the angle of rotation about the z-axis the position vector of the point (u,v) is

$$\bar{r} = (g(u)\cos v,\ g(u)\sin v,\ f(u)) \longrightarrow (2)$$

and the domain of u, v is $0 \le v \le 2\pi$ together with the range of u.

As in the case of sphere $v =$ constant are the meridians given by the various positions of the generating curve and $u =$ constant are parallels (ie) circles in planes parallel to the xy-plane.

Here
$$\bar{r}_1 = (g' \cos v,\ g' \sin v,\ f')$$
$$\bar{r}_2 = (-g \sin v,\ g \cos v,\ 0)$$

$$\bar{r}_1 \cdot \bar{r}_2 = (g' \cos v,\ g' \sin v,\ f') \cdot (-g \sin v;\ g \cos v,\ 0)$$

$= gg' \cos v \sin v + gg' \cos v \sin v + 0$ (7)

$\boxed{\bar{r}_1 \cdot \bar{r}_2 = 0}$ for all u, v.

(ie) The parameters are orthogonal.

The unit Normal N is

$$\bar{N} = \frac{\bar{r}_1 \times \bar{r}_2}{H}$$

$$\bar{r}_1 \times \bar{r}_2 = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ g'\cos v & g'\sin v & f' \\ -g\sin v & g\cos v & 0 \end{vmatrix}$$

$$\Rightarrow \bar{i}(-f'g\cos v) - \bar{j}(g\sin v f') + \bar{k}[gg'\cos^2 v + gg'\sin^2 v]$$

$$= -\bar{i} f' g\cos v - \bar{j} g f' \sin v + gg'(\cos^2 v + \sin^2 v)$$

$$= -\bar{i} f' g\cos v - \bar{j} g f' \sin v + gg'(1)$$

$$= -\bar{i} f' g\cos v - \bar{j} g f' \sin v + gg'$$

$$\boxed{\bar{r}_1 \times \bar{r}_2 = (-f'g\cos v, -f'g\sin v, gg')}$$

$$\boxed{\bar{r}_1 \times \bar{r}_2 = (-f'\cos v, -f'\sin v, g')}$$

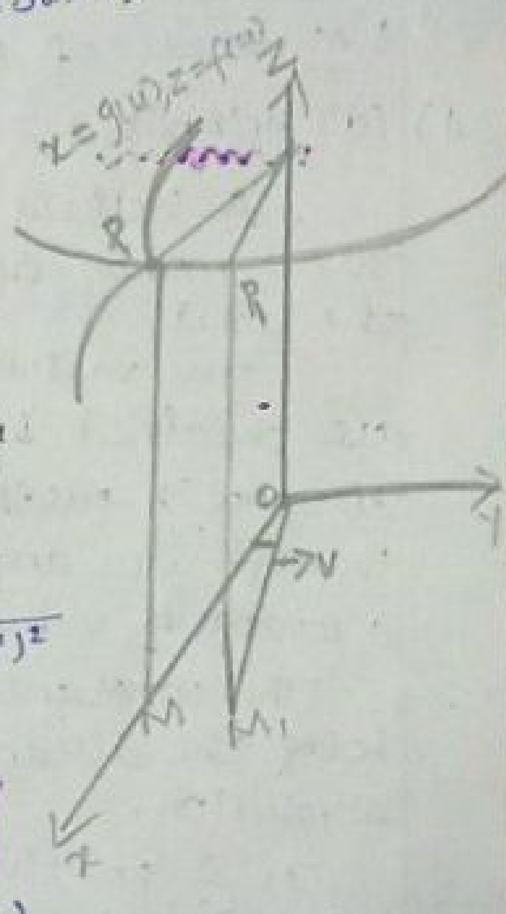
$\div g$ $\boxed{\bar{r}_1 \times \bar{r}_2 = (-f'\cos v, -f'\sin v, g')}$

$$H = |\bar{r}_1 \times \bar{r}_2| = \sqrt{(f')^2\cos^2 v + (f')^2\sin^2 v + (g')^2}$$

For example is given by $g(u) = u$

$f(u) = u \cot\alpha$

$\bar{r} = (u\cos v, u\sin v,$



$$H = |\bar{r}_1 \times \bar{r}_2| = \sqrt{(f')^2(\cos^2 v + \sin^2 v) + (g')^2}$$

$$= \sqrt{(f')^2 + (g')^2}$$

$$\boxed{H = (f'^2 + g'^2)^{1/2}}$$

$$\bar{N} = \frac{\bar{r}_1 \times \bar{r}_2}{H} = \frac{(-f'\cos v, -f'\sin v, g')}{(f'^2 + g'^2)^{1/2}}$$

using the fact that $g \neq 0$ at an ordinary point

It is often convenient to take $g(u) = u$. The right circular cone of semi-vertical angle α.

for example is given by $f(u) = u$, $f(u) = \cot\alpha$.

$\therefore \bar{r} = (u\cos v, u\sin v, u\cot\alpha)$

(c) The anchor ring:

This is obtained by rotating a circle of radius a about a line in its plane and at a distance b(>a) from its centre. ⑧

It is therefore given by,

$\bar{r} = (g(u)\cos v, g(u)\sin v, f(u)) \longrightarrow ①$

$g(u) = b + a\cos u$

$f(u) = a\sin u$

$① \Rightarrow \bar{r} = [(b+a\cos u)\cos v, (b+a\cos u)\sin v, a\sin u]$

and the domain of u, v is $0 < u < 2\pi$, $0 < v < 2\pi$.

The meridians and parallels are circles and the centre of the meridian circle.

4) (a) Helicoids:

A helicoid is a surface generated by the screw motion of a curve about a fixed line, the axis.

The various position of the generating curve are obtained by first translating it through a distance λ parallel to the axis then rotating it through an angle ν about the axis, where λ/ν has a constant value a.

The constant $2\pi a$ is the pitch of the helicoid, being the distance translated in one complete revolution.

It is positive or negative according as the helicoid is right or left handed and is zero for a surface of revolution.

(b) Right helicoids:

This is a helicoid generated by a straight line which meets the axis at right angle.

Taking the z-axis, the position vector is

$\bar{r} = (u\cos v, u\sin v, av)$

where u is distance from the axis and v is the angle of rotation.

The generator being assumed to be the x-axis where $v=0$.

Here u & v take the real value,

$$\bar{r}_1 = (\cos v, \sin v, 0)$$

$$\bar{r}_2 = (-u\sin v, u\cos v, a)$$

$$\bar{r}_1\cdot\bar{r}_2 = (\cos v, \sin v, 0)\cdot(-u\sin v, u\cos v, a)$$

$$= -u\cos v\sin v + u\cos v\sin v + 0$$

$$= 0$$

$$\boxed{\bar{r}_1\cdot\bar{r}_2 = 0}$$

The curves $v=$ constant are the generator and $u=$ constant are circular helices.

$$\therefore \bar{r}_1\cdot\bar{r}_2 = 0.$$

The helices are orthogonal to the generators.

(c) The general helicoid:

The surface by planes containing the axis are congruent plane curves and the surface is generated by the screw motion of any one of these curves.

Without loss of generality, we may assume the generating curve to be a plane curve given by equations of the form $x = g(u),\ y = 0,\ z = f(u)$

The position vector of a point on the surface is then $\bar{r} = (g(u)\cos v,\ g(u)\sin v,\ f(u)+av) \longrightarrow (1)$

The curve $v=$ constant are the various position of the generating curve.

$u=$ constant are circular helices.

Then $\bar{r}_1 = (g'(u)\cos v,\ g'(u)\sin v,\ f'(u))$

Diff w.r.t v $\bar{r}_2 = (-g(u)\sin v,\ g(u)\cos v,\ a)$

$$\bar{r}_1\cdot\bar{r}_2 = (g'(u)\cos v,\ g'(u)\sin v,\ f'(u))\cdot(-g(u)\sin v,\ g(u)\cos v,\ a)$$

$$= (-g'(u)g(u)\sin v\cos v + g(u)g'(u)\cos v\sin v + af'(u)$$

$$\boxed{\bar{r}_1\cdot\bar{r}_2 = af'(u)}$$

The parametric curve are therefore orthogonal if either $f'(u)=0$ in this case the surface is a right helicoid. (10)

(or)

$\therefore a=0$ in this case surface of revolution.

5) Metric:

* A given surface $\bar{r}=\bar{r}(u,v)$, consider the curve defined by,

$u=u(t), \ v=v(t)$.

Then $\bar{r}$ is a function of t along the curve and the arc length is related to the parameter t by,

$$\left(\frac{ds}{dt}\right)^2=\left(\frac{d\bar{r}}{dt}\right)^2=\left(\bar{r}_1\frac{du}{dt}+\bar{r}_2\frac{dv}{dt}\right)^2$$

$$=\bar{r}_1^2\left(\frac{du}{dt}\right)^2+2\bar{r}_1\bar{r}_2\frac{du}{dt}\frac{dv}{dt}+\bar{r}_2^2\left(\frac{dv}{dt}\right)^2$$

$$=E\left(\frac{du}{dt}\right)^2+2F\frac{du}{dt}\cdot\frac{dv}{dt}+G\left(\frac{dv}{dt}\right)^2$$

where $E=\bar{r}_1^2, F=\bar{r}_1\cdot\bar{r}_2, G=\bar{r}_2^2 \longrightarrow (1)$.

This equation can be expressed in the differential form. $ds^2=Edu^2+2Fdudv+Gdv^2 \longrightarrow (2)$

The R.H.S of (2) does not involve the parameter t except in so far as u & v depend on t.

Geometrically ds can be interpreted as the infinitesimal distance from the point (u,v) to the point (u,v) to the point $(u+du, v+dv)$.

Now, $(\bar{r}_1\times\bar{r}_2)^2=(\bar{r}_1\times\bar{r}_2)\cdot(\bar{r}_1\times\bar{r}_2)$

$=(\bar{r}_1^2\bar{r}_2^2)-(\bar{r}_1\cdot\bar{r}_2)^2$

$(\bar{r}_1\times\bar{r}_2)^2=EG-F^2=H^2$

where $H=|\bar{r}_1\times\bar{r}_2|=+\sqrt{EG-F^2}$

The co-efficient E, G & H^2 satisfy $E>0, G>0, H^2=EG-F^2>0$.

These inequalities S.T the metric is a +ve finite quadratic form in du, dv.

we have defined

$H=|\bar{r}_1\times\bar{r}_2|$. so that it is the positive square $EG-F^2$.

Metric form

The quadratic differential form $ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$ in du and dv is called metric or first fundamental form of the surface and the quantities E, F, G are called first fundamental coefficients or fundamental magnitudes of first order.

Note:

The metric is mostly used for the calculation of arc lengths on surface but the coefficients E, F, G also play important parts in other ways.

For example $F = \bar{r}_1 \cdot \bar{r}_2 = 0$ is the condition for the parametric curves to be orthogonal.

Angle between parametric curves:

The parametric directions are given by $\bar{r}_1$ and $\bar{r}_2$ the angle $\omega (0 < \omega < \pi)$ between them are given by

$$\cos\omega = \frac{\bar{r}_1 \cdot \bar{r}_2}{|\bar{r}_1||\bar{r}_2|} = \frac{F}{\sqrt{EG}}$$

$$\text{and } \sin\omega = \frac{|\bar{r}_1 \times \bar{r}_2|}{|\bar{r}_1||\bar{r}_2|} = \frac{H}{\sqrt{EG}}$$

In general the angle between the parametric direction vary from point to point

Element of area:

consider the figure shown with vertices (u,v), $(u+\delta u, v)$, $(u+\delta u, v+\delta v)$ & $(u, v+\delta v)$ joined by parametric curves when δu & δv are small and +ve.

This figure is approximately a parallelogram with adjacent side given by the vector $\bar{r}_1 \delta u$, $\bar{r}_2 \delta v$ and area $|\bar{r}_1 \delta u \times \bar{r}_2 \delta v| = |\bar{r}_1 \times \bar{r}_2| \delta u \delta v = H \delta u \delta v$.

This form $\boxed{ds = H\,du\,dv}$ with du, dv in the place of $\delta u, \delta v$ is taken to be the element of area ds for the surface so that $\boxed{ds = H\,du\,dv}$