UNIT - I

Introduction:

Space Curves:

In a plane Geometry a curve is usually specified either by means of a single equation or $x^2+y^2=a^2$ else by a parametric representation.

$x = at,\ y = 2at$

Example:

The circle with centre at $(0,0)$ and radius 'a' in specified is Cartesian form by the single equation.

$$x^2+y^2=a^2.$$

(or) by the parametric equation.

$$x = a\cos u \ , \quad y = a\sin u \ , \quad 0 \le u \le 2\pi.$$

In three dimensional Euclidean space E_3 a single equation generally represents a surface and equation are needed to specify a curve.

Parametrically a curve may be specified in Cartesian co ordinates by equation.

$$x = x(u), \quad y = y(u), \quad z = z(u).$$

where x, y, z are real valued function of the real parameter u which is restricted to some interval.

In vector notation the curve is specified by a vector valued function.

$$\bar{r} = \bar{R}(u)$$

A curve is defined by a equation

$$F(x, y, z) = 0$$
$$G(x, y, z) = 0$$

and it is required to find parametric equation for curve.

If F and G have continuous first derivatives and if at least one of the Jacobian determinants

$$\frac{\partial(F, G)}{\partial(y, z)}, \quad \frac{\partial(F, G)}{\partial(z, x)}, \quad \frac{\partial(F, G)}{\partial(x, y)} \text{ is } \neq 0$$

Partial-independent

at a point (x_0, y_0, z_0) on the curve

$$F = 0, \quad G = 0.$$

Example:

The first Jacobian is non-zero the variable y and z may be expressed as function of x say,

$$y = y(x), \quad z = z(x).$$

The parametrization $x = u$, $y = y(u)$, $z = z(u)$.

The straight forward method of solving the first equation to obtain $u = f(u)$ and the other two equation.

$$y = y[f(x)], \quad z = z[f(x)].$$

Definitions:

Function of class m:

Let I be a real interval and 'm' a positive integer $(m \geq 1)$. A real valued function f

defined on I is said to be of class 'm' or to be a C^m- function, if f has a mth derivative at every point of I and if this derivative is continuous on I.

In otherwords.

a C^m- function has a continuous mth derivative

C^∞- function:

A function is infinitely differentiable we say that it is of class ∞ or a C^∞ function. and when a function is analytic we say it is of class ω or a C^ω- function.

Note:

A C^m- function of several variables admits all continuous partial derivative of the mth order.

Regular:

The vector equation $\bar{r} = (x, y, z)$ or equivalently by the equation.

$$x = x(u), \quad y = y(u), \quad z = z(u)$$

$$\bar{r} = \bar{r}(u)$$

If the derivative $\dot{\bar{r}} = \frac{d\bar{r}}{du}$ never vanishes on I.

(i.e) if $\dot{x}, \dot{y}, \dot{z}$ do not vanish simultaneously the function is said to be regular.

Path of class m:

A regular vector valued function of class m.

Equivalence Relation:

Two paths $\bar{R}_1, \bar{R}_2$ of the same class m on I_1 and I_2 are called equivalent if there exist a strictly increasing function ϕ of class m which maps I_1 and I_2 is such that $R_1 = R_2 \phi$.

This is equivalent to the three condition.

$$X_1(u) = X_2(\phi(u)), \quad Y_1(u) = Y_2(\phi(u)), \quad Z_1(u) = Z_2(\phi(u))$$

Parametric Representation:

Any equivalence class of path of class m determine a curve of class m. Any path R determine a unique curve and is called a parametric representation of the curve, the variable u being called the parameter.

The mapping ϕ which relates two equivalent paths is called a change of parameter.

Definition:

Curve of class m:

A curve of class m in E_3 is a set of points in E_3 associated with an equivalence class of regular parametric representation of class m involving one parameter.

Example:

Two equivalent representation.

Consider the circular helix.

i) $\bar{r} = (a\cos u,\ a\sin u,\ bu)$ $\quad (0 \le u < \pi)$

ii) $\bar{r} = a\left(\frac{1-v^2}{1+v^2},\ \frac{2av}{1+v^2},\ 2b\tan^{-1}v\right)$ $\quad 0 \le v < \infty$.

The change in parameter in this class is,

$$v = \phi(u) = \tan u/2.$$

Arc Length:

The distance between two path points $\bar{r}_1 = (x_1, y_1, z_1)$ $\bar{r}_2 = (x_2, y_2, z_2)$ in Euclidean space is the number.

$$\bar{r}_1 - \bar{r}_2 = \sqrt{(\bar{r}_1 - \bar{r}_2)^2}$$

$$= \sqrt{\{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2\}}$$

This distance in space will be used to define distance along a curve of class $m \ge 1$.

Suppose given a path $\bar{r} = \bar{R}(u)$ and two members $a, b\ (a<b)$ in the range of the parameter, then the path $\bar{r} = \bar{R}(u)$ $(a \le u \le b)$ is an arc of the original path joining the points corresponding to 'a' and 'b'.

To any subdivision Δ of the interval (a, b) by the points $a = u_0 < u_1 < u_2 < \cdots < u_n = b$. there corresponds the length.

$$L_\Delta = \sum_{i=1}^{n} |\bar{R}(u_i) - R(u_{i-1})|$$

of the polygon 'inscribed' to the arc by joining successive points on it.

Hence it is reasonable to define the length of the arc to be the upper bound of l_Δ taken over all possible subdivisions of (a,b). The upper bound is always finite because for any Δ

$$l_\Delta = \sum_{i=1}^{n} \left| \int_{u_{i-1}}^{u_i} \dot{R}(u)\,du \right| \leq \sum_{i=1}^{n} \int_{u_{i-1}}^{u_i} |\dot{R}(u)|\,du = \int_a^b |\dot{R}(u)|\,du$$

$\rightarrow$ (2)

and the R.H.S member is finite & independent of Δ.

We now s.t this upper bound is actually equal to the R.H.S of (2) so that this term gives a formula for the arc length.

The definition of arc length implies that if $a < c < b$, then the arc length from a to b is the sum of the arc lengths from a to c & from c to b.

Ex:1 Obtain the equation of the circular helix $\bar{r} = (a\cos u, a\sin u, bu)$, $-\infty < u < \infty$ where $a > 0$ referred to s as parameter, & s.t the length of one complete turn of the helix is $2\pi c$ where $c = \sqrt{a^2+b^2}$.

Soln:

$$s = \int_0^u \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\,du.$$

$x = a\cos u$, $y = a\sin u$, $z = bu$.

$\dot{x} = -a\sin u$, $\dot{y} = a\cos u$, $\dot{z} = b$.

$$\therefore \quad s = \int_0^u \sqrt{(-a\sin u)^2 + (a\cos u)^2 + b^2}\, du$$

$$= \int_0^u \sqrt{a^2(\sin^2 u + \cos^2 u) + b^2}\, du$$

$$= \int_0^u \sqrt{a^2+b^2}\, du \quad = \int_0^u c\, du$$

$$= c\,[u]_0^u \quad = \quad c\,[u-0]$$

$$s = cu$$

$$u = \frac{s}{c} \qquad \left[\because c = \sqrt{a^2+b^2}\right]$$

The required equations are,

$$\bar{r} = (a\cos u,\ a\sin u,\ bu)$$

$$\bar{r} = [a\cos(s/c),\ a\sin(s/c),\ b(s/c)]$$

The range of u corresponding to one complete turn of the helix is $u_0 \le u \le u_0 + 2\pi$.

So the required length $= \int_{u_0}^{u_0+2\pi} c \cdot du$

$$= c \cdot [u]_{u_0}^{u_0+2\pi}$$

$$= c \cdot [u_0 + 2\pi - u_0]$$

$$= 2\pi c$$

Ex: 2. Find the length of the curve given as the intersection of the surfaces $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $x = a\cosh(z/a)$ from the pt $(a,0,0)$ to the point (x,y,z).

Soln:

The parametric equations are,

$$x = a\cosh u, \quad y = b\sinh u, \quad z = au$$

$$\dot{x} = a\sinh u, \quad \dot{y} = b\cosh u, \quad \dot{z} = a$$

$$\therefore \quad s = \int_0^u \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, du.$$

$$= \int_0^u \sqrt{a^2 \sinh^2 u + b^2 \cosh^2 u + a^2}\, du.$$

$$= \int_0^u \sqrt{a^2(\sinh^2 u + 1) + b^2 \cosh^2 u}\, du$$

$$= \int_0^u \sqrt{a^2 \cosh^2 u + b^2 \cosh^2 u}\, du.$$

$$= \int_0^u \sqrt{(a^2+b^2)\cosh^2 u}\, du.$$

$$= \int_0^u \sqrt{(a^2+b^2)}\sqrt{\cosh^2 u}\, du = \int_0^u \sqrt{a^2+b^2}\cosh u\, du.$$

$$= \sqrt{a^2+b^2} \cdot \int_0^u \cosh u\, du = \sqrt{a^2+b^2}\,[\sinh u]_0^u$$

$$s = \sqrt{a^2+b^2}\,\sinh u. \quad \text{--- (1)}$$

$$y = b \sinh u. \Rightarrow y/b = \sinh u. \quad \text{--- (2)}$$

Sub (2) in (1),

$$s = \sqrt{a^2+b^2}\,(y/b).$$

Ex: 3. S.T if a curve is given in terms of a general Parameter u, then the equation of the osculating Plane is $[\bar{R}-\bar{r}, \dot{\bar{r}}, \ddot{\bar{r}}] = 0$.

Soln:

Let $$\bar{r}' = \frac{d\bar{r}}{ds} = \frac{d\bar{r}}{du} \cdot \frac{du}{ds}$$

$$\bar{r}' = \frac{\dot{\bar{r}}}{\dot{s}} \qquad \left[\because \frac{d\bar{r}}{du} = \dot{\bar{r}},\ \frac{ds}{du} = \dot{s}\right]$$

$$\bar{r}'' = \frac{d}{ds}\left(\frac{\dot{\bar{r}}}{\dot{s}}\right)$$

$$= \frac{d}{du}\left(\frac{\dot{\bar{r}}}{\dot{s}}\right)\frac{du}{ds}.$$

$$= \left(\frac{\dot{s}\ddot{\bar{r}} - \dot{\bar{r}}\ddot{s}}{\dot{s}^2}\right) \cdot \left(\frac{1}{\dot{s}}\right).$$

$$= \frac{\dot{s}\ddot{\bar{r}} - \dot{\bar{r}}\ddot{s}}{\dot{s}^3} = \frac{\dot{s}\ddot{\bar{r}}}{\dot{s}^3} - \frac{\dot{\bar{r}}\ddot{s}}{\dot{s}^3}$$

$$\bar{r}'' = \frac{\ddot{\bar{r}}}{\dot{s}^2} - \frac{\dot{\bar{r}}\,\ddot{s}}{\dot{s}^3}$$

Substituting into the Equation,

$$[\bar{R}-\bar{r},\ \bar{r}',\ \bar{r}''] = 0.$$

we get, $\left[\bar{R}-\bar{r},\ \frac{\dot{\bar{r}}}{\dot{s}},\ \frac{\ddot{\bar{r}}}{\dot{s}^2} - \frac{\dot{\bar{r}}\,\ddot{s}}{\dot{s}^3}\right] = 0.$

(i.e) $[\bar{R}-\bar{r},\ \dot{\bar{r}},\ \ddot{\bar{r}}] = 0.$

Ex: 4. Find the eqn of the osculating plane at the point u on the helix $\bar{r} = [a\cos u,\ a\sin u,\ cu]$.

Soln:

Given, $\bar{r} = [a\cos u,\ a\sin u,\ cu] \longrightarrow (1).$

The eqn of the osculating plane is,

$$[\bar{R}-\bar{r},\ \dot{\bar{r}},\ \ddot{\bar{r}}] = 0.$$

In Cartesian form:

$$\begin{vmatrix} X-x & Y-y & Z-z \\ \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix} = 0.$$

$x = a\cos u$, $\quad y = a\sin u$, $\quad z = cu$

$\dot{x} = -a\sin u$ $\quad \dot{y} = a\cos u$ $\quad \dot{z} = c$

$\ddot{x} = -a\cos u$ $\quad \ddot{y} = -a\sin u$ $\quad \ddot{z} = 0$

$$\begin{vmatrix} X-a\cos u & Y-a\sin u & Z-cu \\ -a\sin u & a\cos u & c \\ -a\cos u & -a\sin u & 0 \end{vmatrix} = 0.$$

Expanding by the last column,

$(Z-cu)\{a^2\sin^2 u + a^2\cos^2 u\} - c\{(X-a\cos u)(-a\sin u)$

$- (Y-a\sin u)(-a\cos u)\} = 0.$

$(z-cu)\{a^2(1)\} - c\{-xa\sin u + a^2 \cancel{\sin u}\cos u$

$+ya\cos u - \cancel{a^2 \sin u}\cos u\} = 0.$

$a^2z - a^2cu + cxa\sin u - cya\cos u = 0.$

$a^2z + ca[x\sin u - y\cos u - au] = 0.$

$a\{az + c[x\sin u - y\cos u - au]\} = 0$

$az + c[x\sin u - y\cos u - au] = 0.$

Ex:5 Find the eqn of the osculating plane at a general pt on the cubic curve given by $r = (u, u^2, u^3)$ & S.T the osculating plane at any 3 pts of the curve meet at a pt lying in the plane det by these 3 points.

Soln:

Given, $\bar{r} = (u, u^2, u^3)$ — ①.

The equation of the osculating plane is,

$$\begin{vmatrix} X-x & Y-y & Z-z \\ \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix} = 0. \quad — ②$$

$x = u, \quad y = u^2, \quad z = u^3.$

$\dot{x} = 1 \quad \dot{y} = 2u \quad \dot{z} = 3u^2$

$\ddot{x} = 0 \quad \ddot{y} = 2 \quad \ddot{z} = 6u.$

② ⇒

$$\begin{vmatrix} X-u & Y-u^2 & Z-u^3 \\ 1 & 2u & 3u^2 \\ 0 & 2 & 6u \end{vmatrix} = 0.$$

$(X-u)\{12u^2 - 6u^2\} - (Y-u^2)\{6u - 0\}$

$+(Z-u^3)\{2-0\} = 0.$

$12xu^2 - 6xu^2 - 12u^3 + 6u^3 - 6yu + 6u^3 +$

$2z - 2u^3 = 0.$

$$6xu^2 - 6u^3 - 6yu + 2z + 4u^3 = 0.$$

$$6xu^2 - 2u^3 - 6yu + 2z = 0.$$

$$3u^2x - 3uy + z - u^3 = 0 \qquad (\div 2) \quad \text{--- (3)}$$

If u_1, u_2, u_3 are three distinct values of the parameter, the osculating planes at these points are linearly independent & the planes meet at a point (x_0, y_0, z_0). The parameters u_1, u_2, u_3 therefore satisfy the condition,

$$(3) \Rightarrow \quad 3u^2x - 3uy + z - u^3 = 0.$$

$$u^3 - 3u^2x + 3uy - z = 0.$$

$$u^3 - 3u^2x_0 + 3uy_0 - z_0 = 0 \quad \text{--- (4).}$$

$$nu^3 - 3nu^2x_0 + 3nuy_0 - nz_0 = 0.$$

If $lx + my + nz + p = 0$ is the equation of the plane passing through the 3 points, then the parameters must also satisfy the condition,

$$lu + mu^2 + nu^3 + p = 0 \quad \text{--- (5).}$$

Since this equation has three distinct roots we have on comparing the co-efficients in the two cubic equation (4) & (5).

$$l = 3ny_0, \quad m = -3nx_0, \quad p = -nz_0.$$

Thus the equation of the plane is,

$$3y_0x - 3x_0y + z - z_0 = 0.$$

This is clearly satisfied by (x_0, y_0, z_0).

Hence proved.

Theorem 1:

A necessary & sufficient condition that a curve be a straight line is that $k=0$ at all points.

Proof:

Let us assume that the curve be a straight line.

To prove that $k=0$ at all points.

Any straight line has equation of the form

$$\bar{r} = \bar{a}s + \bar{b}$$

where $\bar{a}$ & $\bar{b}$ are constant vectors.

$$\bar{r} = \bar{a}s + \bar{b}$$

$$\bar{t} = \bar{r}' = \bar{a}$$

$$\bar{t}' = \bar{r}'' = 0.$$

$\therefore$ $k=0$ is necessary.

Conversely,

If $k=0$ at all points.

To prove that the curve be a straight line.

If $k=0$ identically then,

$$\bar{r}'' = 0.$$

Integrating we obtain, $\bar{r} = \bar{a}s + \bar{b}$, where $\bar{a}, \bar{b}$ are constant.

$\bar{r} = \bar{a}s + \bar{b}$ which is the equation of a st. line.

Serret - Frenet Formulae:

The relations,

1. $\bar{t}' = k\bar{n}$
2. $\bar{n}' = \tau\bar{b} - k\bar{t}$
3. $\bar{b}' = -\tau\bar{n}$

are known as the Serret - Frenet Formulae.

Ex: 6. S.T i) $\frac{d\bar{t}}{ds} = \bar{w} \times \bar{t}$, ii) $\frac{d\bar{n}}{ds} = \bar{w} \times \bar{n}$ & iii) $\frac{d\bar{b}}{ds} = \bar{w} \times \bar{b}$

where $\bar{w}$ is called the Darboure vector & $\bar{w} = \tau\bar{t} + k\bar{b}$

Solution:

i) $\frac{d\bar{t}}{ds} = k\bar{n}$.

$$\bar{t}' = k\bar{n}$$
$$= 0 + k\bar{n}$$
$$= \tau(0) + k\bar{n}$$
$$= \tau(\bar{t} \times \bar{t}) + k(\bar{b} \times \bar{t}) \quad [\because \bar{t} \times \bar{t} = 0, \ \bar{b} \times \bar{t} = \bar{n}]$$
$$= (\tau\bar{t} + k\bar{b}) \times \bar{t}$$
$$\frac{d\bar{t}}{ds} = \bar{w} \times \bar{t}$$

ii) $$\frac{d\bar{n}}{ds} = \bar{n}' = \tau\bar{b} - k\bar{t}. \quad [\because \bar{t} = \bar{n} \times \bar{b}]$$
$$= \tau(\bar{t} \times \bar{n}) - k(-\bar{b} \times \bar{n})$$
$$= (\tau\bar{t} + k\bar{b}) \times \bar{n}$$
$$= \bar{w} \times \bar{n}.$$

iii) $$\frac{d\bar{b}}{ds} = \bar{b}' = -\tau\bar{n}$$
$$= -\tau(\bar{b} \times \bar{t}) = \tau(\bar{t} \times \bar{b})$$
$$= \tau(\bar{t} \times \bar{b}) + k(\bar{b} \times \bar{b})$$
$$= (\tau\bar{t} + k\bar{b}) \times \bar{b} \quad \therefore [\bar{b} \times \bar{b} = 0]$$
$$= \bar{w} \times \bar{b}$$

Ex: 7 P.T (i) $[\bar{t}' \ \bar{t}'' \ \bar{t}'''] = k^5 \, d/ds\,(\tau/k)$.

ii) $[\bar{b}' \ \bar{b}'' \ \bar{b}'''] = \tau^3 (k'\tau - k\tau') = \tau^5 \, d/ds\,(k/\tau)$.

Proof:

(i) $$\bar{r}' = \frac{d\bar{r}}{ds} = \bar{t} \quad [\because \bar{t} = \bar{r}']$$
$$\bar{r}'' = \bar{t}' = \frac{d\bar{t}}{ds} = k\bar{n} \quad [\because \bar{t}' = k\bar{n}]$$
$$\bar{r}''' = \frac{d}{ds}(k\bar{n})$$

$= k'\bar{n} + k\bar{n}'$

$= k'\bar{n} + k(-k\bar{t} + \tau\bar{b}) \quad [\because \bar{n}' = \tau\bar{b} - k\bar{t}]$

$\bar{\gamma}''' = \bar{t}'' = k'\bar{n} - k^2\bar{t} + k\tau\bar{b}$

$\bar{\gamma}^{(iv)} = \bar{t}''' = k''\bar{n} + \bar{n}'k' - (2kk'\bar{t} + k^2\bar{t}') + k'\tau\bar{b}$
$+ k\tau'\bar{b} + k\tau\bar{b}'$

$= k''\bar{n} + (\tau\bar{b} - k\bar{t})k' - 2kk'(\bar{t}) - k^2(k\bar{n}) +$
$k'\tau(+\tau\bar{b}) + k\tau'(+\tau\bar{b}) + k\tau(-\tau\bar{n})$

$= (k''\bar{n} - k^3\bar{n} - k\tau^2\bar{n}) + \tau\bar{b}k' - kk'\bar{t} - 2kk'\bar{t}$
$+ k'\tau\bar{b} + k\tau'\bar{b}$

$\bar{t}''' = (k'' - k^3 - k\tau^2)\bar{n} - 3kk'\bar{t} + (2k'\tau + k\tau')\bar{b}$

$$\therefore [\bar{t}' \ \bar{t}'' \ \bar{t}'''] = \begin{array}{c} \begin{array}{ccc} \bar{t} & \bar{n} & \bar{b} \end{array} \\ \begin{vmatrix} 0 & k & 0 \\ -k^2 & k' & k\tau \\ -3kk' & k''-k^3-k\tau^2 & 2k'\tau+k\tau' \end{vmatrix} \end{array}$$

$= 0 - k[-k^2(2k'\tau + k\tau') + 3k^2k'\tau]$

$= (-k)(-k^2)[2k'\tau + k\tau' - 3k'\tau]$

$= k^3 [k\tau' - k'\tau]$

Multiply & divide by k^2.

$= k^3 [k\tau' - k'\tau] \times \dfrac{k^2}{k^2}$

$= \dfrac{k^5 [k\tau' - k'\tau]}{k^2}$

$= k^5 \dfrac{d}{ds}\left(\dfrac{\tau}{k}\right)$

Hence the Proof.

ii)
$$\bar{b}' = -\tau\bar{n}$$
$$\bar{b}'' = -[\tau'\bar{n} + \tau\bar{n}'] = -\tau'\bar{n} - \tau\bar{n}'$$
$$= -\tau'\bar{n} - \tau(-k\bar{t} + \tau\bar{b})$$
$$= \tau k\bar{t} - \tau'\bar{n} - \tau^2\bar{b}$$
$$\bar{b}''' = \tau k\bar{t}' + \tau k'\bar{t} + \tau' k\bar{t} - \tau'\bar{n}' - \tau''\bar{n} - \tau^2\bar{b}' - 2\tau\tau'\bar{b}$$
$$= \tau k(k\bar{n}) + \tau k'\bar{t} + \tau' k\bar{t} - \tau'(\tau\bar{b} - k\bar{t}) - \tau''\bar{n} - \tau^2(-\tau\bar{n}) - 2\tau\tau'\bar{b}$$
$$= \tau k^2\bar{n} + \tau k'\bar{t} + \tau' k\bar{t} - \tau'\tau\bar{b} + \tau' k\bar{t} - \tau''\bar{n} + \tau^3\bar{n} - 2\tau\tau'\bar{b}$$
$$= (2k\tau' + k'\tau)\bar{t} + (k^2\tau - \tau'' + \tau^3)\bar{n} - 3\tau\tau'\bar{b}$$
$$[\bar{b}'\ \bar{b}''\ \bar{b}'''] = \begin{vmatrix} 0 & -\tau & 0 \\ \tau k & -\tau' & -\tau^2 \\ 2k\tau' + k'\tau & k^2\tau - \tau'' + \tau^3 & -3\tau\tau' \end{vmatrix}$$
$$= 0 + \tau[\tau k(-3\tau\tau') + \tau^2(2k\tau' + k'\tau)]$$
$$= \tau[-3\tau^2 k\tau' + 2k\tau^2\tau' + k'\tau\tau^2]$$
$$= \tau[-k\tau^2\tau' + k'\tau^3]$$
$$= \tau^3[-k\tau' + k'\tau] \times \frac{\tau^2}{\tau^2} \qquad [\times \text{ \& } \div\ \tau^2]$$
$$= \frac{\tau^5[k'\tau - k\tau']}{\tau^2}$$
$$= \tau^5 \frac{d}{ds}\left(\frac{k}{\tau}\right)$$

Ex:8 S.T the Principal normals at consecutive points do not intersect unless $\tau = 0$.

Soln:-

Let P & Q be two consecutive points with position vectors $\bar{r}$ and $\bar{r}+d\bar{r}$ on the curve C.

Let the principal normals at these points be $\bar{n}$ & $\bar{n}+d\bar{n}$.

The principal normals will intersect if the three vectors $\bar{n}$, $\bar{n}+d\bar{n}$ and $d\bar{r}$ are coplanar.

The coplanar tripple product $[a, b, c] = 0$.

$$\therefore [\bar{n}, \bar{n}+d\bar{n}, d\bar{r}] = 0.$$

By known result;

$$\text{"} [\bar{a}, \bar{b}, \bar{c}+\bar{d}] = [\bar{a}, \bar{b}, \bar{c}] + [\bar{a}, \bar{b}, \bar{d}] \text{"}$$

$$\therefore [\bar{n}, \bar{n}+d\bar{n}, d\bar{r}] = 0.$$

(i.e) $[\bar{n}, \bar{n}, d\bar{r}] + [\bar{n}, d\bar{n}, d\bar{r}] = 0.$

$[\bar{n}, d\bar{n}, d\bar{r}] = 0.$ $\quad$ $\left[\begin{array}{c}\bar{n}, \bar{n} = 0 \\ \therefore [\bar{n}, \bar{n}, d\bar{r}] = 0\end{array}\right]$

(i.e) $\left[\bar{n}, \frac{d\bar{n}}{ds}, \frac{d\bar{r}}{ds}\right] = 0.$ (∵ in the first bracket of the components are equal).

$$[\bar{n}, \bar{n}', \bar{r}'] = 0.$$

By Serret - Frenet Formulae:

$$\text{"} \bar{n}' = \tau\bar{b} - k\bar{t} \text{"}$$

(i.e) $[\bar{n}, (-k\bar{t}+\tau\bar{b}), \bar{r}'] = 0.$

$[\bar{n}, -k\bar{t}, \bar{r}'] + [\bar{n}, \tau\bar{b}, \bar{r}'] = 0.$ $\quad$ $[\bar{r}' = \bar{t}]$

$[\bar{n}, -k\bar{t}, \bar{t}] + [\bar{n}, \tau\bar{b}, \bar{t}] = 0.$

$[\bar{n}, \tau\bar{b}, \bar{t}] = 0.$ $\quad$ $\left[\begin{array}{c}\bar{t}, \bar{t} = 0 \\ [\bar{n}, -k\bar{t}, \bar{t}] = 0\end{array}\right]$

$\tau [\bar{n}, \bar{b}, \bar{t}] = 0.$

$\tau = 0 \quad \therefore [\bar{n}, \bar{b}, \bar{t}] = 1$

Hence the principal normals at consecutive points do not intersect unless $\tau = 0$.