Ex.9 PT $z=0,\ y=\frac{1}{2}kx^2;\ y=0,\ z=\frac{k\tau}{6}x^3;\ x=0,\ z^2=\frac{2}{9}\left(\frac{\tau^2}{k}\right)y^3$

Soln:

By known results,

$$x = s-\frac{k^2}{6}s^3$$

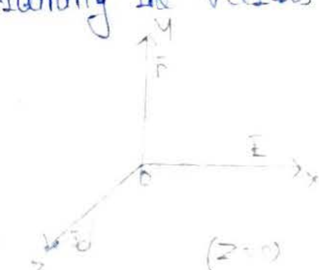
$$y = \frac{k\,s^2}{2}+\frac{k's^3}{6}, \quad z=\frac{k\tau}{6}s^3 \text{ nearly.}$$

Now retaining only the first term in the above, we get $x\simeq s,\ y\simeq \frac{k}{2}s^2,\ z\simeq\frac{k\tau}{6}s^3$

(i) Eliminating 's' between x & y we get,

$$y=\frac{k}{2}x^2, \quad z=0. \quad [x\simeq s, \text{ (i.e) } x=s]$$

This gives the projection of the curve near P on the osculating plane at P (the plane containing the vectors $\bar{t}$ & $\bar{n}$) (i.e) $\bar{t}=x,\ \bar{n}=y$ so $z=0$.



This projection is clearly a parabola.

(ii) The projection of the curve on the rectifying plane (the plane containing the vectors $\bar{b}$ & $\bar{t}$) (i.e) $\bar{b}=z,\ \bar{t}=x$ so $y=0$.

$$z=\frac{k\tau}{6}x^3 \quad [x\simeq s,\ x=s]$$

(iii) The projection of the curve on the normal plane (the plane containing the vectors $\bar{n}$ & $\bar{b}$)

(i.e) $\bar{n}=y,\ \bar{b}=z$ so $x=0$.

$$z=\frac{k\tau}{6}s^3$$

Taking square on both sides,

$$z^2 = \frac{k^2 \tau^2 . s^6}{36} \quad — ①$$

$$y = \frac{k}{2} s^2 \Rightarrow y^3 = \frac{k^3 s^6}{8} \quad — ②$$

So eqn ① mul & div by 'k' we get,

$$z^2 = \frac{k^2 . \tau^2 s^6}{36} \times \frac{k}{k}$$

$$= \frac{k^3 \tau^2 s^6}{k} \left(\frac{2}{9 \times 8}\right) \qquad \left[\because \frac{1}{36} = \frac{2}{9 \times 8}\right]$$

$$= \frac{k^3 . s^6}{8} \times \left(\frac{\tau^2 . 2}{9k}\right)$$

$$= \frac{2}{9}\left(\frac{\tau^2}{k}\right)\left(\frac{k^3 . s^6}{8}\right) \Rightarrow \frac{2}{9}\left(\frac{\tau^2}{k}\right) . y^3 \text{ (From ②)}$$

$$z^2 = \frac{2}{9}\left(\frac{\tau^2}{k}\right) y^3$$

Ex: 10. S.T the length of the common $\perp$r 'd' of the tangents at two near pts distance s, approximately gn by $d = \frac{k \tau s^2}{12}$

Soln:

Let P, Q have parameters '0' & 's' respectively.

The unit tangent vectors at P & Q are $\bar{r}'(0)$, $\bar{r}'(s)$,

So the unit vector of the common perpendicular is along $\bar{r}'(s) \times \bar{r}'(0)$. The projection of the vector $\bar{r}(s) - \bar{r}(0)$ in this direction is equal to 'd' so,

$$\text{(i.e)} \quad d = \bar{r}(s) - \bar{r}(0) \quad — ①$$

$$d = \frac{[\bar{r}(s) - \bar{r}(0),\ \bar{r}'(s) \times \bar{r}'(0)]}{|\bar{r}'(s) \times \bar{r}'(0)|} \quad — ②$$

By known Taylor's Theorem:

$$\bar{r}(s) = \bar{r}(0) + s\bar{r}'(0) + \frac{s^2}{\lfloor 2} \bar{r}''(0) + \frac{s^3}{\lfloor 3} \bar{r}'''(0)$$

$$[\because \bar{r}'(0) = t,\ \bar{r}''(0) = t' = k\bar{n},\ \bar{r}'''(0) = -k^2\bar{t} + k'\bar{n} + k\tau\bar{b}]$$

$$\bar{r}(s) = \bar{r}(0) + s\bar{t} + \frac{s^2}{2}k\bar{n} + \frac{s^3}{6}(-k^2\bar{t} + k'\bar{n} + k\tau\bar{b}] \quad \text{—} (3)$$

Differentiate (3) w.r.t 's' we get,

$$\bar{r}'(s) = \bar{r}'(0) + \bar{t} + \frac{2s}{2}k\bar{n} + \frac{3s^2}{6}(-k^2\bar{t} + k'\bar{n} + k\tau\bar{b})$$

$$\bar{r}'(s) = (0) + \bar{t} + sk\bar{n} + \frac{s^2}{2}(-k^2\bar{t} + k'\bar{n} + k\tau\bar{b})$$

$r'(0) = t$

$$\bar{r}'(s) \times r'(0) = \begin{bmatrix} \bar{t} & \bar{n} & \bar{b} \\ 1 - \frac{k^2s^2}{2} & sk + \frac{s^2k'}{2} & \frac{s^2}{2}k\tau \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \bar{t}(0) - \bar{n}\left(0 - \frac{s^2}{2}k\tau\right) + \bar{b}\left(0 - \left(sk + \frac{s^2k'}{2}\right)\right]$$

$$= \bar{n}\frac{s^2}{2}k\tau - sk\bar{b} - \frac{s^2k'}{2}\bar{b}$$

$$= -sk\bar{b} + \frac{s^2}{2}\left(k\tau\bar{n} - \frac{k'\bar{b}}{2}\right) \quad \text{—} (4)$$

From (3) we get,

$$\bar{r}(s) - \bar{r}(0) = s\bar{t} + \frac{s^2}{2}k\bar{n} + \frac{s^3}{6}(-k^2\bar{t} + k'\bar{n} + k\tau\bar{b}] \quad \text{—} (5)$$

Eqn (4) & (5) we get,

$$[\bar{r}(s) - \bar{r}(0)] \cdot [\bar{r}'(s) \times \bar{r}'(0)] \quad \text{—} (6)$$

$$= \left[s\bar{t} + \frac{s^2}{2}k\bar{n} + \frac{s^3}{6}(-k^2\bar{t} + k'\bar{n} + k\tau\bar{b})\right] \cdot \left[-sk\bar{b} + \frac{s^2}{2}\left(k\tau\bar{n} - \frac{k'\bar{b}}{2}\right)\right]$$

$$= \left[s\bar{t} - \frac{s^3}{6}k^2\bar{t} + \frac{s^2}{2}k\bar{n} + \frac{s^3}{6}k'\bar{n} + \frac{s^3}{6}k\tau\bar{b}\right] \cdot \left[\frac{s^2}{2}k\tau\bar{n} - sk\bar{b} - \frac{s^2}{2}\frac{k'\bar{b}}{2}\right]$$

$$= \bar{t}(0) + \left(\frac{s^2}{2}k + \frac{s^3}{6}k'\right) \cdot \left(\frac{s^2}{2}k\tau\right) + \frac{s^3}{6}k\tau\left(-sk - \frac{s^2}{2}k'\right)$$

$$= \left(\frac{s^2}{2}k + \frac{s^3}{6}k'\right)\cdot\left(\frac{s^2}{2}k\tau\right) - \frac{s^3}{6}\tau k\left(sk + \frac{s^2}{2}k'\right)$$

$$= \frac{s^4k^2\tau}{4} + \cancel{\frac{s^5kk'\tau}{12}} - \frac{s^4k^2\tau}{6} - \cancel{\frac{s^5kk'\tau}{12}}$$

$$= \frac{s^4k^2\tau}{4} - \frac{s^4k^2\tau}{6} \Rightarrow \frac{6s^4k^2\tau - 4s^4k^2\tau}{24}$$

$$= \frac{2s^4k^2\tau}{24}$$

$$= \frac{s^4k^2\tau}{12} \quad \text{(approximately)} \quad \text{—— (7)}$$

$|a \times b| = \sqrt{(a^2+b^2)}$ $|a \times b|^2 = (\sqrt{(a^2+b^2)})^2$

(4) $\Rightarrow$

$$\bar{\gamma}'(s) \times \bar{\gamma}'(0) = -sk\bar{b} + \frac{s^2}{2}[-k'\bar{b} + k\tau\bar{n}]$$

squaring & taking mod

$$|\bar{\gamma}'(s) \times \bar{\gamma}'(0)|^2 = \left(\frac{s^2}{2}k\tau\right)^2 + \left(sk + \frac{s^2k'}{2}\right)^2$$

$$= \frac{s^4}{4}k^2\tau^2 + s^2k^2 + \frac{s^4k'^2}{4} + \frac{2s^3kk'}{2}$$

$$= s^2k^2 + s^3kk' + \frac{1}{4}s^4k^2\tau^2 + \frac{1}{4}s^4k'^2$$

$$= s^2k^2 + s^3kk'$$

$$|\bar{\gamma}'(s) \times \bar{\gamma}'(0)|^2 = s^2k^2 = (sk)^2$$

$$|\bar{\gamma}'(s) \times \bar{\gamma}'(0)| = sk \quad \text{(approximately)} \text{ —— (8)}$$

Sub eqn (7) & (8) in (5).

$$d = \frac{[\bar{\gamma}(s) \times \bar{\gamma}(0)]\cdot[\bar{\gamma}'(s) \times \bar{\gamma}'(0)]}{|\bar{\gamma}'(s) \times \bar{\gamma}'(0)|}$$

$$= \frac{s^4k^2\tau}{12} \times \frac{1}{sk}$$

$$= \frac{s^3k\tau}{12} \quad \text{(approximately)}$$

Hence, $d = \dfrac{s^3k\tau}{12}$.

To find the curvature & Torsion in terms of the parameter u:.

We have, $\dot{\bar{r}} = \frac{d\bar{r}}{du} = \frac{d\bar{r}}{ds} \cdot \frac{ds}{du} = \bar{r}' \dot{s}$

$\dot{\bar{r}} = \bar{t} \cdot \dot{s}$ $[\bar{r}' = \bar{t}]$ — (1).

$\therefore |\dot{\bar{r}}| = |\bar{t}| \dot{s} = \dot{s}$ $(\because |\bar{t}| = 1)$

Differentiating (1) w.r.t 'u'.

$\frac{d}{du}(\dot{\bar{r}}) = \ddot{\bar{r}} = \frac{d}{du}(\bar{t} \cdot \dot{s})$

$= \frac{d}{du}(\bar{t}) \cdot \dot{s} + \bar{t} \cdot \frac{d}{du}(\dot{s})$

$= \frac{d}{ds}(\bar{t}) \frac{ds}{du} \dot{s} + \bar{t} \ddot{s}$

$\ddot{\bar{r}} = \bar{t}' \dot{s} \dot{s} + \bar{t} \ddot{s} \Rightarrow \bar{t}' \dot{s}^2 + \bar{t} \ddot{s}$ — (2)

$= k\bar{n}\dot{s}^2 + \bar{t}\ddot{s}$ — (2)

$\dot{\bar{r}} \times \ddot{\bar{r}} = \bar{r}' \dot{s} \times (k\bar{n}\dot{s}^2 + \bar{t}\ddot{s})$

$= \bar{t}\dot{s} \times (k\bar{n}\dot{s}^2 + \bar{t}\ddot{s})$

$= (\bar{t}\dot{s} \times k\bar{n}\dot{s}^2) + (\bar{t}\dot{s} \times \bar{t}\ddot{s})$

$= \bar{b}\dot{s}^3 k + 0$ $[\because \bar{t} \times \bar{n} = \bar{b},\ \bar{t} \times \bar{t} = 0]$

$\dot{\bar{r}} \times \ddot{\bar{r}} = \dot{s}^3 \bar{b} k$ — (3)

Differentiating (3) w.r.t 'u',

$\dot{\bar{r}} \times \dddot{\bar{r}} + \ddot{\bar{r}} \times \ddot{\bar{r}} = \dot{s}^3 k \frac{d}{ds}(\bar{b}) \frac{ds}{du} + \bar{b} \frac{d}{du}(\dot{s}^3 k)$

$= \dot{s}^3 k \bar{b}' (\dot{s}) + \bar{b}\, d/du(\dot{s}^3 k)$

$= \dot{s}^4 k(-\tau\bar{n}) + \bar{b}\, d/du(\dot{s}^3 k)$

$= -\dot{s}^4 k \tau \bar{n} + \bar{b}\, d/du(\dot{s}^3 k)$ — (4)

Taking the dot product of (2) & (4) we obtain,

$\ddot{\bar{r}} \cdot [\dot{\bar{r}} \times \dddot{\bar{r}} + \ddot{\bar{r}} \times \ddot{\bar{r}}] = (k\bar{n}\dot{s}^2 + \bar{t}\ddot{s}) \cdot (-\dot{s}^4 k\tau\bar{n} + \bar{b}\, d/du(\dot{s}^3 k)]$

$$[\dot{\bar{r}}\ \ddot{\bar{r}}\ \dddot{\bar{r}}] = (k\dot{s}^2)(-\dot{s}^4 k\tau) = -\dot{s}^6 k^2 \tau \quad —— (5)$$

$[n \cdot n = 1,\ n \cdot b = 0$
$t \cdot n = 0 \cdot t \cdot b = 0]$

From (3)

$$\dot{\bar{r}} \times \ddot{\bar{r}} = \dot{s}^3 \bar{b} k$$

Taking mod on both sides,

$$|\dot{\bar{r}} \times \ddot{\bar{r}}| = \dot{s}^3 |\bar{b}| k = \dot{s}^3 k \qquad [\because |b| = 1]$$

$$k = \frac{|\dot{\bar{r}} \times \ddot{\bar{r}}|}{\dot{s}^3}$$

$$k = \frac{|\dot{\bar{r}} \times \ddot{\bar{r}}|}{|\dot{r}|^3} \qquad [\because |\dot{r}| = \dot{s}] \quad —— (6)$$

From (5) & (6) we get,

$$\tau = \frac{[\dot{\bar{r}}\ \ddot{\bar{r}}\ \dddot{\bar{r}}]}{-\dot{s}^6 k^2}$$

$$= \frac{[\dot{\bar{r}}\ \ddot{\bar{r}}\ \dddot{\bar{r}}]}{|\dot{r}|^6} \times \frac{|\dot{r}|^6}{|\dot{\bar{r}} \times \ddot{\bar{r}}|^2} \qquad (-1)' = (1)$$

$$\tau = \frac{[\dot{\bar{r}}\ \ddot{\bar{r}}\ \dddot{\bar{r}}]}{|\dot{\bar{r}} \times \ddot{\bar{r}}|^2} \quad —— (7)$$

Ex: 1 S.T $[\dot{r}\ \ddot{r}\ \dddot{r}] = 0$ is a necessary & sufficient condn that the curve be plane.

Soln:

$$\bar{r}' = \frac{dr}{ds} = \frac{dr}{du} \cdot \frac{du}{ds} = \dot{r} u'$$

$$\bar{r}'' = \dot{r} u'' + u' \frac{d}{ds}(\dot{r})$$

$$= \dot{r} u'' + u' \frac{d\dot{r}}{du} \cdot \frac{du}{ds} = \dot{r} u'' + u' \ddot{r} u'$$

$$\bar{r}'' = \dot{r} u'' + \ddot{r} u'^2$$

$$\bar{r}''' = \frac{d}{ds} \dot{r} u'' + \dot{r} u''' + \frac{d}{ds} \ddot{r} u'^2 + \ddot{r}\, 2u' u''$$

$$= \frac{d\dot{r}}{du} \cdot \frac{du}{ds} u'' + \dot{r} u''' + \frac{d\ddot{r}}{du} \cdot \frac{du}{ds} u'^2 + \ddot{r}\, 2u'u''$$

$$= \dddot{r} u'u'' + \dot{r} u''' + \dddot{r} u' u'^2 + \ddot{r}\, 2u'u''$$

$$= 3\ddot{r} u'u'' + \dot{r} u''' + \dddot{r} u'u'^2$$

$$r''' = 3\ddot{r} u' u'' + \dot{r} u''' + \dddot{r} u'^3.$$

Where $u' = \frac{du}{ds}$.

$$[r' \; r'' \; r'''] = [\dot{r} u',\; \ddot{r} u'^2 + \dot{r} u'',\; \dddot{r} u'^3 + 3\ddot{r} u'u'' + \dot{r} u''']$$

$$= \begin{array}{ccc} \dot{r} & \ddot{r} & \dddot{r} \end{array}$$
$$\begin{vmatrix} u' & 0 & 0 \\ u'' & u'^2 & 0 \\ u''' & 3u'u'' & u'^3 \end{vmatrix}$$

$$= u' [u'^5 - 0] = u'(u'^5)$$

$$[r' \; r'' \; r'''] = u'^6 [\dot{r} \; \ddot{r} \; \dddot{r}]$$

$$[\dot{r} \; \ddot{r} \; \dddot{r}] = [r' \; r'' \; r'''] (u')^{-6}.$$

$$[\dot{r} \; \ddot{r} \; \dddot{r}] = k^2 \tau (u')^{-6}. \qquad [\because [r' \; r'' \; r'''] = k^2\tau]$$

If the L.H.S is zero then either $k=0$ (or) $\tau=0$.

We P.T $\tau=0$ always as follows:-

Suppose $\tau \neq 0$ at some point, then there is a neighbourhood of this point where $\tau \neq 0$. Hence $k=0$ in this neighbourhood and the arc is a straight line

Then $\tau=0$ on this line, contrary to the hypothesis, and hence $\tau=0$ at all points and the curve is therefore plane.

Conversely,

If the curve is plane then $\tau=0$ and so $[\dot{r} \; \ddot{r} \; \dddot{r}] = 0$.

The condition is thus necessary and sufficient.

Ex: 2 Cal the curvature & torsion of the curve $\bar{r} = (u, u^2, u^3)$

Soln:

Given $\bar{r} = (u, u^2, u^3)$ —— (1).

Differentiate (1) w.r.t 'u',

$$\dot{\bar{r}} = (1, 2u, 3u^2) \quad \text{—— (2)}$$

$$\ddot{\bar{r}} = (0, 2, 6u) \quad \text{—— (3)}$$

$$\dddot{\bar{r}} = (0, 0, 6) \quad \text{—— (4)}$$

$$\dot{\bar{r}} \times \ddot{\bar{r}} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 2u & 3u^2 \\ 0 & 2 & 6u \end{vmatrix}$$

$$= \bar{i}[12u^2 - 6u^2] - \bar{j}[6u - 0] + \bar{k}[2 - 0]$$

$$\dot{\bar{r}} \times \ddot{\bar{r}} = 6u^2\bar{i} - 6u\bar{j} + 2\bar{k} \quad \text{—— (5)}$$

$$|\dot{\bar{r}} \times \ddot{\bar{r}}| = \sqrt{36u^4 + 36u^2 + 4}$$

$$= \sqrt{4(9u^4 + 9u^2 + 1)}$$

$$= 2\sqrt{9u^4 + 9u^2 + 1} \quad \text{—— (6)}$$

$$[\dot{\bar{r}}\ \ddot{\bar{r}}\ \dddot{\bar{r}}] = \begin{vmatrix} 1 & 2u & 3u^2 \\ 0 & 2 & 6u \\ 0 & 0 & 6 \end{vmatrix}$$

$$= 1(12 - 0) - 2u(0 - 0) + 3u^2(0)$$

$$= 12. \quad \text{—— (7)}$$

$$|\dot{\bar{r}}| = \sqrt{1 + 4u^2 + 9u^4}$$

$$|\dot{\bar{r}}|^2 = 1 + 4u^2 + 9u^4 \quad \text{—— (8)}$$

Curvature:

$$k = \frac{|\dot{\bar{r}} \times \ddot{\bar{r}}|}{|\dot{\bar{r}}|^3}$$

$$k^2 = \frac{|\dot{\bar{r}} \times \ddot{\bar{r}}|^2}{|\dot{\bar{r}}|^6} = \frac{|\dot{\bar{r}} \times \ddot{\bar{r}}|^2}{(|\dot{\bar{r}}|^2)^3}$$

From (6) & (8) we get,

$$K^2 = \frac{(2\sqrt{9u^4+9u^2+1})^2}{(1+4u^2+9u^4)^3} = \frac{4(9u^4+9u^2+1)}{(1+4u^2+9u^4)^3}$$

(or) $$K = \frac{2\sqrt{9u^4+9u^2+1}}{(1+4u^2+9u^4)^{3/2}}$$

Torsion: $\tau = [\dot{\bar{r}}\ \ddot{\bar{r}}\ \dddot{\bar{r}}] / |\dot{\bar{r}} \times \ddot{\bar{r}}|^2$

From (6) & (7) we get,

$$\tau = \frac{12}{(2\sqrt{9u^4+9u^2+1})^2} = \frac{12}{4(9u^4+9u^2+1)}$$

$$\tau = \frac{3}{9u^4+9u^2+1}$$

Ex:3 For the curve $x = 3u, y = 3u^2, z = 2u^3$ S.T $\rho = \sigma = 3/2\,(1+2u^2)^2$

Soln:

Given the curve,

$$x = 3u,\ y = 3u^2,\ z = 2u^3$$

$$\bar{r} = (x, y, z)$$

(i.e) $$\bar{r} = (3u, 3u^2, 2u^3) \quad \text{—} \quad (1)$$

Differentiate w.r.t 'u',

$$\dot{\bar{r}} = (3, 6u, 6u^2)$$

$$\ddot{\bar{r}} = (0, 6, 12u)$$

$$\dddot{\bar{r}} = (0, 0, 12)$$

$$\dot{\bar{r}} \times \ddot{\bar{r}} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 3 & 6u & 6u^2 \\ 0 & 6 & 12u \end{vmatrix}$$

$$= \bar{i}[72u^2 - 36u^2] - \bar{j}[36u] + \bar{k}(18)$$

$$= 36u^2\bar{i} - 36u\bar{j} + 18\bar{k}$$

$$|\dot{\bar{r}} \times \ddot{\bar{r}}| = \sqrt{(36)^2u^4 + (36)^2u^2 + (18)^2}$$

$$= \sqrt{(18\times 2)^2 u^4 + (18\times 2)^2 u^2 + 18^2}$$

$$= 18\sqrt{4u^4 + 4u^2 + 1}$$

$$= 18\sqrt{(2u^2+1)^2}$$

$$|\dot{\bar{r}} \times \ddot{\bar{r}}| = 18(2u^2+1) \quad \text{—— (2)}$$

$$|\dot{\bar{r}}| = \sqrt{9 + 36u^2 + 36u^4} \Rightarrow \sqrt{9(1+4u^2+4u^4)}$$

$$= 3\sqrt{1+4u^2+4u^4} \quad = 3\sqrt{(2u^2+1)^2}$$

$$|\dot{\bar{r}}| = 3(2u^2+1) \quad \text{—— (3)}$$

$$[\dot{\bar{r}} \ \ddot{\bar{r}} \ \dddot{\bar{r}}] = \begin{vmatrix} 3 & 6u & 6u^2 \\ 0 & 6 & 12u \\ 0 & 0 & 12 \end{vmatrix}$$

$$= 3(72) - 6u(0) + 6u^2(0)$$

$$= 216 \quad \text{—— (4)}$$

To Find ρ & σ:

Radius of curvature $\rho = 1/k$

$$k = \frac{|\dot{\bar{r}} \times \ddot{\bar{r}}|}{|\dot{\bar{r}}|^3}$$

$$\rho = \frac{1}{\frac{|\dot{\bar{r}} \times \ddot{\bar{r}}|}{|\dot{\bar{r}}|^3}} = \frac{|\dot{\bar{r}}|^3}{|\dot{\bar{r}} \times \ddot{\bar{r}}|}$$

$$= \frac{[3(2u^2+1)]^3}{18(2u^2+1)}$$

$$= \frac{27(2u^2+1)^3}{18(2u^2+1)}$$

$$\rho = \frac{3}{2}(2u^2+1)^2 \quad \text{—— (5)}$$

Radius of Torsion $\sigma = 1/\tau$

$$\tau = \frac{[\dot{\bar{r}}\ \ddot{\bar{r}}\ \dddot{\bar{r}}]}{|\dot{\bar{r}} \times \ddot{\bar{r}}|^2}$$

$$\sigma = \frac{|\dot{\bar{r}} \times \ddot{\bar{r}}|^2}{[\dot{\bar{r}}\ \ddot{\bar{r}}\ \dddot{\bar{r}}]} = \frac{[18(2u^2+1)]^2}{216}$$

$$= \frac{324(2u^2+1)^2}{216}$$

$$\sigma = 3/2\,(2u^2+1)^2 \quad — ⑥$$

$$\therefore \quad \rho = \sigma = 3/2\,(2u^2+1)^2 \qquad [\text{from ⑤ \& ⑥}]$$

Ex: P.T $\rho = -\sigma = \dfrac{2\sqrt{2}\,a}{\sin^2 2\theta}$

Soln:

Given: $x = a\tan\theta,\ y = a\cot\theta,\ z = a\sqrt{2}\log\tan\theta$

TPT: $\rho = -\sigma = \dfrac{2\sqrt{2}\,.\,a}{\sin^2 2\theta}$

Case (i):

$$\bar{r} = (x, y, z)$$

$$\bar{r} = (a\tan\theta,\ a\cot\theta,\ a\sqrt{2}\log\tan\theta)$$

$$= a(\tan\theta,\ \cot\theta,\ \sqrt{2}\log\tan\theta) \quad — Ⓐ$$

Differentiate w.r.t 's',

$$\frac{dr}{ds} = \bar{r}' = \bar{t} = a\left(\sec^2\theta,\ -\operatorname{cosec}^2\theta,\ \sqrt{2}\frac{1}{\tan\theta}\sec^2\theta\right)\frac{d\theta}{ds}$$

$$= a\left(\sec^2\theta,\ -\operatorname{cosec}^2\theta,\ \sqrt{2}\ \frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\cos^2\theta}\right)\frac{d\theta}{ds}$$

$$= a\left(\sec^2\theta,\ -\operatorname{cosec}^2\theta,\ \sqrt{2}\ \frac{1}{\sin\theta\cos\theta}\right)\frac{d\theta}{ds} \quad — ①$$

Squaring ① we get,

$$\bar{t}\cdot\bar{t} = 1 \Rightarrow a^2\left(\sec^4\theta,\ \operatorname{cosec}^4\theta,\ \frac{2}{\sin^2\theta\cos^2\theta}\right)\left(\frac{d\theta}{ds}\right)^2$$

$$\left(\frac{ds}{d\theta}\right)^2 = a^2\left[\frac{1}{\cos^4\theta} + \frac{1}{\sin^4\theta} + 2\,\frac{1}{\sin^2\theta\cos^2\theta}\right]$$

$$= a^2\left[\frac{\sin^4\theta + \cos^4\theta + 2\sin^2\theta\cos^2\theta}{\sin^4\theta\cos^4\theta}\right]$$

$$= a^2\left[\frac{(\sin^2\theta + \cos^2\theta)^2}{\sin^4\theta\cos^4\theta}\right]$$

$$\left(\frac{ds}{d\theta}\right)^2 = \frac{a^2}{\sin^4\theta\cos^4\theta} \qquad [\because \sin^2\theta + \cos^2\theta = 1]$$

$$\frac{ds}{d\theta} = \frac{a}{\sin^2\theta\cos^2\theta} \quad \text{—— (2)}$$

Sub (2) in (1) we get,

$$\bar{r}' = \bar{t} = a\left(\sec^2\theta, -\operatorname{cosec}^2\theta, \frac{\sqrt{2}}{\sin\theta\cos\theta}\right)\frac{\sin^2\theta\cos^2\theta}{a}$$

$$= \left(\sec^2\theta, -\operatorname{cosec}^2\theta, \frac{\sqrt{2}}{\sin\theta\cos\theta}\right)\sin^2\theta\cos^2\theta$$

$$= \Big(\sec^2\theta\,\sin^2\theta\cos^2\theta, -\operatorname{cosec}^2\theta\,\sin^2\theta\cos^2\theta,$$
$$\frac{\sqrt{2}}{\sin\theta\cos\theta}\sin^2\theta\cos^2\theta\Big)$$

$$= \Big(\sec^2\theta\,\sin^2\theta\,\frac{1}{\sec^2\theta}, -\operatorname{cosec}^2\theta\,\frac{1}{\operatorname{cosec}^2\theta}\cos^2\theta,$$
$$\sqrt{2}\sin\theta\cos\theta\Big)$$

$$= (\sin^2\theta, -\cos^2\theta, \sqrt{2}\sin\theta\cos\theta) \quad \text{—— (B)}$$

$$= (\sin^2\theta, -\cos^2\theta, \tfrac{\sqrt{2}}{2}\sin 2\theta)$$

Differentiating w.r.t 's',

$$t' = \left(2\sin\theta\cos\theta, 2\cos\theta\sin\theta, \frac{\sqrt{2}}{2}\cos 2\theta \times 2\right)\cdot\frac{d\theta}{ds}$$

$$k\bar{n} = (\sin 2\theta, \sin 2\theta, \sqrt{2}\cos 2\theta)\,\frac{\sin^2\theta\cos^2\theta}{a} \quad \text{—— (3)}$$

$[\because \sin 2\theta = 2\sin\theta\cos\theta]$

Squaring (3) we get,

$$(k\bar{n})\cdot(k\bar{n}) = (\sin^2 2\theta, \sin^2 2\theta, 2\cos^2 2\theta)\,\frac{\sin^4\theta\cos^4\theta}{a^2}$$

$$K^2 = \frac{\sin^4\theta \cos^4\theta}{a^2}[2\sin^2 2\theta, 2\cos^2 2\theta]$$

$$= \frac{2\sin^4\theta \cos^4\theta}{a^2}[\sin^2 2\theta + \cos^2 2\theta]$$

$$K^2 = \frac{2\sin^4\theta \cos^4\theta}{a^2}$$

$$k = \sqrt{2}\frac{\sin^2\theta \cos^2\theta}{a} \quad \text{--- (4)}$$

Hence,

$$\rho = \frac{1}{k} = \frac{a}{\sqrt{2}\sin^2\theta\cos^2\theta} \times \frac{4}{4} = \frac{4a}{\sqrt{2}\, 4\sin^2\theta\cos^2\theta}$$

$$= \frac{4a}{\sqrt{2}\sin^2 2\theta} \quad [\because \sin^2 2\theta = 4\sin^2\theta\cos^2\theta]$$

$$\rho = \frac{2\sqrt{2}\,a}{\sin^2 2\theta}$$

Case (ii):

Sub (4) in (3) we get,

$$\sqrt{2}\frac{\sin^2\theta\cos^2\theta}{a}\bar{n} = \frac{\sin^2\theta\cos^2\theta}{a}(\sin 2\theta, \sin 2\theta, \sqrt{2}\cos 2\theta)$$

$$\bar{n} = \frac{1}{\sqrt{2}}(\sin 2\theta, \sin 2\theta, \sqrt{2}\cos 2\theta)$$

Differentiate w.r.t 's' we get,

$$\bar{n}' = -k\bar{t} + \tau\bar{b} = \frac{1}{\sqrt{2}}(2\cos 2\theta, 2\cos 2\theta, -\sqrt{2}\times 2\sin 2\theta)\frac{d\theta}{ds}$$

$$\tau\bar{b} - k\bar{t} = \frac{2}{\sqrt{2}}(\cos 2\theta, \cos 2\theta, -\sqrt{2}\sin 2\theta) \propto \frac{\sin^2\theta\cos^2\theta}{a}$$

$$= \frac{2\sin^2\theta\cos^2\theta}{a\sqrt{2}}(\cos 2\theta, \cos 2\theta, -\sqrt{2}\sin 2\theta)$$

Squaring both sides,

$$\tau^2 + k^2 = \frac{4\sin^4\theta\cos^4\theta}{a^2(2)}(\cos^2 2\theta, \cos^2 2\theta, 2\sin^2 2\theta)$$

$$= \frac{2\sin^4\theta\cos^4\theta}{a^2}(2\cos^2 2\theta, 2\sin^2 2\theta)$$

$$= \frac{4 \sin^4\theta \cos^4\theta}{a^2} \left(\cos^2 2\theta + \sin^2 2\theta \right)$$

$$\tau^2 + k^2 = \frac{4 \sin^4\theta \cos^4\theta}{a^2}$$

$$\tau^2 = \frac{4 \sin^4\theta \cos^4\theta}{a^2} - k^2$$

$$= \frac{4 \sin^4\theta \cos^4\theta}{a^2} - \frac{2 \sin^4\theta \cos^4\theta}{a^2}$$

$$= \frac{2 \sin^4\theta \cos^4\theta}{a^2}$$

$$\therefore \tau = - \frac{\sqrt{2} \sin^2\theta \cos^2\theta}{a}$$

Here negative sign is taken as τ is negative for a left hand system.

$$\sigma = + \frac{1}{\tau} = \frac{a}{-\sqrt{2} \sin^2\theta \cos^2\theta}$$

mul & div $\sqrt{2}$ & (2) by the above eqn we get,

$$\sigma = \frac{1}{\tau} = - \frac{2\sqrt{2}\, a}{\sqrt{2} \cdot \sqrt{2} \cdot 2 \sin^2\theta \cos^2\theta}$$

$$= - \frac{2\sqrt{2}\, a}{4 \sin^2\theta \cos^2\theta}$$

$$\sigma = \frac{1}{\tau} = - \frac{2\sqrt{2}\, a}{\sin^2 2\theta}$$

$$\rho = -\sigma = \frac{2\sqrt{2}\, a}{\sin^2 2\theta}$$

Ex 5. Fd the curvature & torsion $\bar{r} = \{ a(3u - u^3), 3au^2, a(3u + u^3) \}$

Soln:-

Given:

$$\bar{r} = \{ a(3u - u^3), 3au^2, a(3u + u^3) \}$$

$$\dot{\bar{r}} = \{a(3-3u^2),\ 6au,\ a(3+3u^2)\}$$
$$= a\{(3-3u^2),\ 6u,\ (3+3u^2)\}$$
$$\ddot{\bar{r}} = a\{-6u,\ 6,\ 6u\}$$
$$\dddot{\bar{r}} = a\{-6, 0, 6\}$$
$$|\dot{\bar{r}}| = a\sqrt{(3-3u^2)^2+36u^2+(3+3u^2)^2}$$
$$= a\sqrt{9+9u^4-18u^2+36u^2+9+9u^4+18u^2}$$
$$= a\sqrt{18+18u^4+36u^2} \Rightarrow a\sqrt{(9\times 2)(1+2u^2+u^4)}$$
$$= 3a\sqrt{2(1+u^2)^2}$$
$$|\dot{\bar{r}}| = 3a\sqrt{2}\sqrt{(1+u^2)^2} = 3\sqrt{2}\,a(1+u^2) \quad \text{— (1)}$$
$$\dot{\bar{r}}\times\ddot{\bar{r}} = a^2 \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 3-3u^2 & 6u & 3+3u^2 \\ -6u & 6 & 6u \end{vmatrix}$$
$$= a^2[\bar{i}(36u^2-18-18u^2) - \bar{j}(18u-18u^3+18u+18u^3)$$
$$+\bar{k}(18-18u^2+36u^2)]$$
$$= a^2[\bar{i}(18u^2-18) - \bar{j}(36u) + \bar{k}(18u^2+18)]$$
$$|\dot{\bar{r}}\times\ddot{\bar{r}}| = a^2\sqrt{(18u^2-18)^2+(36u)^2+(18u^2+18)^2}$$
$$= a^2\sqrt{18^2(u^2-1)^2 + 18^2\times(2u)^2 + 18^2(u^2+1)^2}$$
$$= a^2\, 18\sqrt{u^4+1-2u^2+4u^2+u^4+1+2u^2}$$
$$= 18a^2\sqrt{2u^4+2+4u^2} \Rightarrow 18a^2\sqrt{2(u^4+1+2u^2)}$$
$$= 18a^2\sqrt{(u^2+1)^2}\cdot\sqrt{2}$$
$$= 18a^2\sqrt{2}\,(u^2+1) \quad \text{—— (2)}$$

Curvature, $k = \dfrac{|\dot{\bar{r}}\times\ddot{\bar{r}}|}{|\dot{\bar{r}}|^3}$

From (1) & (2) we get,

$$k = \frac{18a^2\sqrt{2}(1+u^2)}{[3\sqrt{2}a(1+u^2)]^3}$$

$$= \frac{18a^2\sqrt{2}(1+u^2)}{27\times 2\sqrt{2}\times a^3(1+u^2)^3}$$

$$= \frac{18a^2\sqrt{2}(1+u^2)}{54\sqrt{2}a^3(1+u^2)^3}$$

$$k. = \frac{1}{3a\cdot(1+u^2)^2} \quad \text{—— (3)}$$

$$[\dot{r}\ \ddot{r}\ \dddot{r}] = a^3\begin{vmatrix} 3-3u^2 & 6u & 3+3u^2 \\ -6u & 6 & 6u \\ -6 & 0 & 6 \end{vmatrix}$$

$$= a^3\begin{vmatrix} 6 & 6u & 3+3u^2 \\ 0 & 6 & 6u \\ 0 & 0 & 6 \end{vmatrix} \quad C_1 \to C_1 + C_3$$

$$= a^3[6(36)-0] = 216a^3 \quad \text{—— (4)}$$

$$\text{torsion } \tau = \frac{[\dot{r}\ \ddot{r}\ \dddot{r}]}{|\dot{r}\times\ddot{r}|^2}$$

$$= \frac{216a^3}{[18a^2\sqrt{2}(1+u^2)]^2} \quad [\text{From (2) \& (4)}]$$

$$= \frac{216a^3}{18^2a^4\times 2(1+u^2)^2}$$

$$= \frac{a^3}{3a^4(1+u^2)^2}$$

$$\tau = \frac{1}{3a(1+u^2)^2}$$

$$k = \tau = \frac{1}{3a(1+u^2)^2}.$$

Ex: 6. For a pt on the curve of intersection of the surfaces, $x^2 - y^2 = c^2, y = x \tanh z/c$. P.T $\rho = -\sigma = 2x^2/c$

Soln:

The parametric representation is;

$$x = c \cosh u$$
$$y = c \sinh u$$
$$z = cu.$$

$\therefore \bar{r} = (x, y, z) \Rightarrow \bar{r} = (c\cosh u, c\sinh u, cu)$

$$\dot{\bar{r}} = c(\sinh u, \cosh u, 1)$$
$$\ddot{\bar{r}} = c(\cosh u, \sinh u, 0)$$
$$\dddot{\bar{r}} = c(\sinh u, \cosh u, 0)$$

$$\dot{\bar{r}} \times \ddot{\bar{r}} = c^2 \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \sinh u & \cosh u & 1 \\ \cosh u & \sinh u & 0 \end{vmatrix}$$

$$= c^2 [\bar{i}(0 - \sinh u) - \bar{j}(0 - \cosh u) + \bar{k}(\sinh^2 u - \cosh^2 u)]$$

$$= c^2 [-\sinh u\, \bar{i} + \cosh u\, \bar{j} + (\sinh^2 u - \cosh^2 u)\bar{k}]$$

$$= c^2 [-\sinh u, \cosh u, -1]$$

$$|\dot{\bar{r}} \times \ddot{\bar{r}}| = c^2 \sqrt{\sinh^2 u + \cosh^2 u + 1}$$

$$= c^2 \sqrt{\cosh^2 u + [\sinh^2 u + 1]} \Rightarrow c^2 \sqrt{\cosh^2 u + \cosh^2 u}$$

$$= c^2 \sqrt{2\cosh^2 u}.$$

$$= c^2 \sqrt{2} \cosh u. \quad \text{——— (1)}$$

$$|\dot{\bar{r}}| = c\sqrt{\sinh^2 u + \cosh^2 u + 1}$$

$$= c\sqrt{\cosh^2 u + \cosh^2 u}.$$

$$= c\sqrt{2\cosh^2 u}.$$

$$= c\sqrt{2} \cosh u. \quad \text{——— (2)}$$

$$[\dot{r}\ \ddot{r}\ \dddot{r}] = c^3 \begin{vmatrix} \sinh u & \cosh u & 1 \\ \cosh u & \sinh u & 0 \\ \sinh u & \cosh u & 0 \end{vmatrix}$$

$$= c^3 \{ 1 [\cosh^2 u - \sinh^2 u] \} = c^3 \{1(1)\}$$

$$= c^3 \quad \text{——} \quad (3)$$

$$\rho = 1/k = \frac{|\dot{r}|^3}{|\dot{r} \times \ddot{r}|} \qquad \left[\because k = \frac{|\dot{r} \times \ddot{r}|}{|\dot{r}|^3}\right]$$

$$= \frac{[c\sqrt{2} \cosh u]^3}{c^2 \sqrt{2} \cosh u}$$

$$= \frac{c^3 \, 2\sqrt{2} \cosh^3 u}{c^2 \sqrt{2} \cosh u}$$

$$= c \cdot 2 \cosh^2 u. \qquad [x = c \cosh u]$$

$$= 2 \cdot c \, (x^2/c^2)$$

$$\rho = \frac{2x^2}{c} \quad \text{——} \quad (4)$$

$$\sigma = \frac{1}{\tau} = \frac{|\dot{r} \times \ddot{r}|^2}{[\dot{r}\ \ddot{r}\ \dddot{r}]}$$

$$= \frac{[c^2 \sqrt{2} \cosh u]^2}{c^3} = \frac{2c^4 \cosh^2 u}{c^3}$$

$$= 2c \cosh^2 u. = 2c\,(x^2/c^2)$$

$$\sigma = \frac{2x^2}{c} \quad \text{——} \quad (5)$$

$$\rho = -\sigma = -\frac{2x^2}{c}$$

for a left handed System.