B.W. Curvature & Torsion of a curve given as the intersection of two Surfaces:-

If a curve is given as the intersection of two surfaces,

$$f(x,y,z)=0 \quad , \quad g(x,y,z)=0.$$

If a set of parametric equations for the curve cannot readily be obtained, then the curvature & Torsion of the curve may be calculated as follows:

Let the curve of intersection be represented by the equation $\bar{r}=\bar{r}(u)$ and let the two surfaces be given by $f(\bar{r})=0,\ g(\bar{r})=0$

Now the unit tangent vector $\bar{t}$ to the curve is orthogonal to the normals of both surfaces.

Thus if,

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \ \& \ \nabla g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right)$$

then $\bar{t}$ is parallel to $\nabla f \times \nabla g = \bar{h}$ (say) — (A)

$$\text{(i.e)} \quad \nabla f \times \nabla g = \lambda \bar{r}'$$

$$= \lambda(x'+y'+z')$$

$$= \lambda x' + \lambda y' + \lambda z' \quad \text{— (B)}$$

Comparing (A) & (B) we get,

$$\lambda x' = h_1 \ , \quad \lambda y' = h_2 \ , \quad \lambda z' = h_3.$$

and $\lambda \frac{d}{ds} = \left(h_1 \frac{\partial}{\partial x} + h_2 \frac{\partial}{\partial y} + h_3 \frac{\partial}{\partial z}\right)$ — (1)

This operator can be denoted by Δ.

Then $\Delta \bar{r} = \bar{h}$ — (2)

From the defn of λ & $\bar{h}$ we get,

$\lambda \bar{t} = \bar{h}$ — (3)

Squaring we get, $\lambda^2(\bar{t}\cdot\bar{t}) = h^2$

$$\lambda^2 = h^2 \quad \text{--- (4)}$$

Operating (3) with Δ

$$\Delta(\lambda\bar{t}) = \Delta\bar{h}$$

$$\lambda\frac{d}{ds}(\lambda\bar{t}) = \Delta\bar{h} \qquad \left[\Delta = \frac{d}{ds}\right]$$

where λ is constant. Diff w.r.t 's' we get,

$$\lambda[\lambda\bar{t}' + \lambda'\bar{t}] = \Delta\bar{h}$$

$$\lambda^2\bar{t}' + \lambda\lambda'\bar{t} = \Delta\bar{h}$$

$$\lambda^2 k\bar{n} + \lambda\lambda'\bar{t} = \Delta h. \qquad [\bar{t}' = k\bar{n}] \quad \text{--- (5)}$$

Taking the vector product of (3) & (5) we obtain,

$$\lambda\bar{t}\times(\lambda^2 k\bar{n} + \lambda\lambda'\bar{t}) = \bar{h}\times\Delta h.$$

$$(\lambda\bar{t}\times\lambda^2 k\bar{n}) + (\lambda\bar{t}\times\lambda\lambda'\bar{t}) = \bar{h}\times\Delta h.$$

$$\lambda^3 k(\bar{t}\times\bar{n}) + \lambda^2\lambda'(\bar{t}\times\bar{t}) = \bar{h}\times\Delta h.$$

$$\lambda^3 k\bar{b} + \lambda^2\lambda'(0) = \bar{h}\times\Delta h.$$

$$\lambda^3 k\bar{b} = \bar{k} \quad (\text{say}) \quad \text{--- (6)}$$

Taking modulus we get,

$$|\bar{k}| = \sqrt{(\lambda^3 k\bar{b})^2} = \sqrt{\lambda^6 k^2(\bar{b}\cdot\bar{b})}$$

$$= \sqrt{(\lambda^3 k)^2}.$$

$$|k| = \lambda^3 k. \quad \text{--- (7)}$$

Operating (6) with Δ gives,

$$\lambda\frac{d}{ds}(\lambda^3 k\bar{b}) = \Delta\bar{k}$$

where λ is constant.

$$\lambda[(\lambda^3 k\bar{b}') + b(\lambda^3 k)'] = \Delta\bar{k} \qquad [b' = -\tau\bar{n}]$$

$$\lambda[\lambda^3 k(-\tau\bar{n}) + b(\lambda^3 k)'] = \Delta\bar{k}$$

$$\lambda(\lambda^3 k)'b - \lambda^4 k\tau\bar{n} = \Delta\bar{k} \quad \text{--- (8)}$$

The scalar product of (5) & (8) gives,

$$(\lambda^2 k\bar{n} + \lambda\lambda'\bar{t}) \cdot [\lambda(\lambda^3 k)'\bar{b} - \lambda^4 k\tau\bar{n}] = \Delta\bar{h} \cdot \Delta\bar{k}$$

$$\lambda^2 k(-\lambda^4 k\tau) = \Delta\bar{h} \cdot \Delta\bar{k} \quad \text{—— (9)}$$

From these eqns k & τ are determined in the usual manner. It will be seen that the R.H.S of (5) to (9) are readily expressible in terms of f & g.

Ex: 7. Curvature & Torsion of 2 quadric surfaces,

$ax^2+by^2+cz^2=1, \quad a'x^2+b'y^2+c'z^2=1.$

Soln:

Given, $ax^2+by^2+cz^2=1, \quad a'x^2+b'y^2+c'z^2=1.$

Let, $f = \frac{1}{2}(ax^2+by^2+cz^2-1), \quad g = \frac{1}{2}(a'x^2+b'y^2+c'z^2-1)$

$$\nabla f = \frac{1}{2}(2ax+2by+2cz) = \frac{2}{2}(ax+by+cz)$$

$$\nabla f = (ax+by+cz)$$

$$\nabla g = \frac{1}{2}(2a'x+2b'y+2c'z) = (a'x+b'y+c'z)$$

$$\therefore \nabla f \times \nabla g = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ ax & by & cz \\ a'x & b'y & c'z \end{vmatrix}$$

$$= \bar{i}[bc'yz - b'cyz] - \bar{j}[ac'xz - a'cxz] + \bar{k}[ab'xy - a'bxy]$$

$$= \bar{i}[yz(bc'-b'c)] - \bar{j}[(ac'-a'c)xz] + \bar{k}[(ab'-a'b)xy]$$

$$= \bar{i}[Ayz] - \bar{j}(Bzx) + \bar{k}(Cxy)$$

Where, $A = bc'-b'c, \quad B = ac'-a'c, \quad C = ab'-a'b$ —— (A)

$\nabla f \times \nabla g$ is parallel to $\bar{t}$ as well as to $\left[\frac{A}{x}, \frac{B}{y}, \frac{C}{z}\right]$ (n_1, n_2, n_3)

$$\lambda\bar{t} = \lambda\bar{r}' = \left(\frac{A}{x}, \frac{B}{y}, \frac{C}{z}\right) \quad \text{—— (1)}$$

Squaring (1) we get,

$$(\lambda\bar{t})^2 = (A/x, B/y, C/z)^2$$

$$\lambda^2 = \Sigma\left(\frac{A^2}{x^2}\right) \quad \text{—— (2)} \quad (\bar{t}\cdot\bar{t}=1)$$

By known result:-

$$\lambda\, d/ds = h_1\, \partial/\partial x + h_2\, \partial/\partial y + h_3\, \partial/\partial z$$

$$= \frac{A}{x}\cdot\frac{\partial}{\partial x} + \frac{B}{y}\cdot\frac{\partial}{\partial y} + \frac{C}{z}\frac{\partial}{\partial z} \quad -(3)$$

Operating (5) on (1),

$$\lambda\frac{d}{ds}(\lambda\bar{t}) = \left[\frac{A}{x}\cdot\frac{\partial}{\partial x} + \frac{B}{y}\cdot\frac{\partial}{\partial y} + \frac{C}{z}\cdot\frac{\partial}{\partial z}\right]\left[\frac{A}{x}, \frac{B}{y}, \frac{C}{z}\right]$$

$$= \frac{A}{x}\cdot\frac{\partial}{\partial x}\left(\frac{A}{x}\right) + \frac{B}{y}\frac{\partial}{\partial y}\left(\frac{B}{y}\right) + \frac{C}{z}\cdot\frac{\partial}{\partial z}\left(\frac{C}{z}\right)$$

Differentiating we get,

$$\lambda[\lambda'\bar{t} + \lambda\bar{t}'] = \frac{A}{x}\left(-\frac{A}{x^2}\right), \frac{B}{y}\left(-\frac{B}{y^2}\right), \frac{C}{z}\left(-\frac{C}{z^2}\right)$$

$$\lambda\lambda'\bar{t} + \lambda^2 k\bar{n} = -\left[\frac{A^2}{x^3}, \frac{B^2}{y^3}, \frac{C^2}{z^3}\right] \quad -(4)$$

Taking the cross product of (1) & (4) we get,

$$\lambda\bar{t}\times(\lambda^2 k\bar{n} + \lambda\lambda'\bar{t}) = -\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ A/x & B/y & C/z \\ A^2/x^3 & B^2/y^3 & C^2/z^3 \end{vmatrix}$$

$$\lambda^3 k\bar{b} = -\left\{\bar{i}\left[\frac{BC^2}{yz^3} - \frac{B^2C}{y^3z}\right] - \bar{j}\left[\frac{AC^2}{xz^3} - \frac{A^2C}{x^3z}\right] + \bar{k}\left[\frac{AB^2}{xy^3} - \frac{A^2B}{x^3y}\right]\right\}$$

$$= \left\{\bar{i}\left[\frac{B^2C}{y^3z} - \frac{BC^2}{yz^3}\right] + \bar{j}\left[\frac{AC^2}{xz^3} - \frac{A^2C}{x^3z}\right] + \bar{k}\left[\frac{A^2B}{x^3y} - \frac{AB^2}{xy^3}\right]\right\}$$

$$= \bar{i}\left[\frac{B^2Cyz^3 - BC^2y^3z}{y^3z^3\times(yz)}\right] + \bar{j}\left[\frac{AC^2x^3z - A^2Cxz^3}{x^3z^3(xz)}\right] + \bar{k}\left[\frac{A^2Bxy^3 - AB^2x^3y}{x^3y^3(xy)}\right]$$

$$= \bar{i}\,\frac{yzBC}{y^4z^4}[Bz^2 - Cy^2] + \bar{j}\,\frac{ACxz}{x^4z^4}[Cx^2 - Az^2] + \bar{k}\,\frac{ABxy}{x^4y^4}[Ay^2 - Bx^2]$$

$$= \bar{i}\,\frac{BC}{y^3z^3}[Bz^2 - Cy^2] + \bar{j}\,\frac{AC}{x^3z^3}[Cx^2 - Az^2] + \bar{k}\,\frac{AB}{x^3y^3}[Ay^2 - Bx^2]$$

$$\lambda^3 k \bar{b} = \left[\frac{BC}{y^3 z^3} (Bz^2 - Cy^2), \frac{CA}{z^3 x^3} (Cx^2 - Az^2), \frac{AB}{x^3 y^3} (Ay^2 - Bx^2) \right] \quad \text{--- (5)}$$

Now,

$$\begin{aligned} Bz^2 - Cy^2 &= (ca' - c'a) z^2 - (ab' - a'b) y^2 \\ &= a'(cz^2 + by^2) - a(c'z^2 + b'y^2) \\ &= a'(1 - ax^2) - a(1 - a'x^2) \qquad \{ ax^2 + by^2 + cz^2 = 1, \; a'x^2 + b'y^2 + c'z^2 = 1 \} \\ &= a' - a'ax^2 - a + aa'x^2 \\ &= a' - a \end{aligned}$$

Similarly, $Cx^2 - Az^2 = b' - b$ & $Ay^2 - Bx^2 = c' - c$ $\Big\}$ --- (6)

using (6) in (5) we get,

$$\lambda^3 k \bar{b} = \left[\frac{BC}{y^3 z^3} (a' - a), \frac{CA}{z^3 x^3} (b' - b), \frac{AB}{x^3 y^3} (c' - c) \right]$$

$$= \frac{ABC}{x^3 y^3 z^3} \left[\frac{x^3}{A} (a' - a), \frac{y^3}{B} (b' - b), \frac{z^3}{C} (c' - c) \right] \quad \text{--- (7)}$$

Squaring (7),

$$\lambda^6 k^2 = \frac{A^2 B^2 C^2}{x^6 y^6 z^6} \sum \frac{x^6}{A^2} (a' - a)^2 \quad \text{--- (8)}$$

$$k^2 = \frac{A^2 B^2 C^2}{x^6 y^6 z^6} \sum \frac{x^6}{A^2} (a' - a)^2 \times \frac{1}{\left[\sum (A^2/x^2) \right]^3} \quad \text{--- (9)} \qquad \text{(By (5))}$$

Eqn (7) is written as,

$$\mu \bar{b} = \left[\frac{x^3}{A} (a' - a), \frac{y^3}{B} (b' - b), \frac{z^3}{C} (c' - c) \right] \quad \text{--- (10)}$$

where, $\mu = \dfrac{\lambda^3 k . x^3 y^3 z^3}{ABC}$ --- (11)

Squaring (10),

$$\mu^2 = \sum \frac{x^6}{A^2} (a' - a)^2 \quad \text{--- (12)}$$

operating by (3) on (10) we get,

$$\lambda . d/ds (\mu \bar{b}) = \left(A/x . \partial/\partial x + B/y . \partial/\partial y + C/z . \partial/\partial z \right) \left[\frac{x^3}{A} (a' - a), \frac{y^3}{B} (b' - b), z^3/C \, (c' - c) \right]$$

$$\lambda[\mu' b + \mu \bar{b}'] = \left(\frac{A}{x}\frac{\partial}{\partial x}\right)\left(\frac{x^3}{A}(a'-a)\right), \left(\frac{B}{y}\frac{\partial}{\partial y}\right)\left(\frac{y^3}{B}(b'-b)\right),$$

$$\left(\frac{C}{z}\,\partial/\partial z\right)\left(z^3/c\,(c'-c)\right)$$

$$\lambda[\mu' b + \mu(-\tau \bar{n})] = \left[\frac{3x^2}{x}(a'-a), \frac{3y^2}{y}(b'-b), \frac{3z^2}{z}(c'-c)\right]$$

$$\lambda\mu' b - \lambda\mu\tau\bar{n} = 3[x(a'-a), y(b'-b), z(c'-c)] \quad \text{— (13)}$$

Taking dot product of (4) & (13)

$$[\lambda^2 k\bar{n} + \lambda\lambda'\bar{t}] \cdot [\lambda\mu'\bar{b} - \lambda\mu\tau\bar{n}] = \left\{-\left(\frac{A^2}{x^3}, \frac{B^2}{y^3}, \frac{C^2}{z^3}\right)\cdot\right.$$

$$\left. 3[x(a'-a), y(b'-b), z(c'-c)]\right\}$$

$$-\lambda^2 k(-\lambda\mu\tau) = +\left\{\frac{A^2}{x^3}\cdot 3x(a'-a) + \frac{B^2}{y^3}\cdot 3y(b'-b) + \frac{C^2}{z^3}\cdot 3z(c'-c)\right\}$$

$$\lambda^3 k\mu\tau = 3\left\{\frac{A^2}{x^2}(a'-a) + \frac{B^2}{y^2}(b'-b) + \frac{C^2}{z^2}(c'-c)\right\}$$

$$\lambda^2 k\mu\tau = 3\sum\frac{A^2}{x^2}(a'-a) \quad \text{—— (14)}$$

Dividing (12) by (11),

$$\frac{\mu^2}{\mu} = \sum\frac{x^6}{A^2}(a'-a)^2 \times \frac{ABC}{\lambda^3 k\, x^3 y^3 z^3}$$

$$\mu\lambda^3 k = \frac{ABC}{x^3y^3z^3}\sum\frac{x^6}{A^2}(a'-a)^2 \quad \text{— (15)}$$

Dividing (14) by (15)

$$\frac{\lambda^2 k\mu\tau}{\mu\lambda^3 k} = \frac{3\sum A^2/x^2\,(a'-a)}{\sum x^6/A^2\,(a'-a)^2} \times \frac{x^3y^3z^3}{ABC}$$

$$\tau = \frac{3x^3y^3z^3}{ABC}\cdot\frac{\sum A^2/x^2\,(a'-a)}{\sum x^6/A^2\,(a'-a)^2} \quad \text{— (16)}$$

Eqns (9) & (16) give the curvature & torsion of the given curve of intersection of the two quadric surfaces.

Ex:6.3 Fd the eqn of the osculating sphere (i) osculating circle at $(1,2,3)$ on the curve, $\bar{r} = (2u+1, 3u^2+2, 4u^3+3)$.

Soln:-

The point $(1,2,3)$ on the curve corresponds to the parameter value $u=0$

Given: $\bar{r} = (2u+1, 3u^2+2, 4u^3+3)$ — (1)

Differentiate w.r.t 'u' we get,

$$\dot{\bar{r}} = (2, 6u, 12u^2) = (2,0,0) \text{ at } u=0.$$

$$\ddot{\bar{r}} = (0, 6, 24u) = (0,6,0) \text{ at } u=0.$$

$$\dddot{\bar{r}} = (0,0,24) \quad \text{at } u=0.$$

Case (i):-

The equation of the osculating sphere is,

$$(\bar{c}-\bar{R})^2 = R^2.$$

$$\therefore (\bar{r}-\bar{c})^2 = \rho^2 \quad \text{— (2)}.$$

Where $\bar{c} = x_1\bar{i} + y_1\bar{j} + z_1\bar{k}$ is the centre and ρ is the radius.

Differentiating (2) w.r.t 'u' thrice,

$$\left.\begin{aligned} &2(\bar{r}-\bar{c})\cdot\dot{\bar{r}} = 0. \Rightarrow (\bar{r}-\bar{c})\dot{\bar{r}} = 0. \\ &(\bar{r}-\bar{c})\ddot{\bar{r}} + \dot{\bar{r}}\cdot\dot{\bar{r}} \Rightarrow (\bar{r}-\bar{c})\ddot{\bar{r}} + \dot{\bar{r}}^2 = 0. \\ &(r-c)\dddot{\bar{r}} + \ddot{\bar{r}}(\dot{\bar{r}}) + \dot{\bar{r}}\ddot{\bar{r}}(2) \Rightarrow (\bar{r}-\bar{c})\dddot{\bar{r}} + 3\dot{\bar{r}}\ddot{\bar{r}} = 0. \end{aligned}\right\} \text{— (3)}$$

At $u=0$ the eqn (3) becomes,

$$(\bar{r}-\bar{c})\dot{\bar{r}} = 0.$$

$$[(\bar{i}+2\bar{j}+3\bar{k}) - (x_1\bar{i}+y_1\bar{j}+z_1\bar{k})]\cdot 2\bar{i} = 0.$$

$$(\bar{i} - x_1\bar{i})2\bar{i} = 0.$$

$$(1-x_1)\bar{i}\cdot 2\bar{i} = 0$$

$$(1-x_1)\cdot 2 = 0 \quad \Rightarrow 1-x_1 = 0$$

$$\therefore \boxed{x_1 = 1}$$

$$(\bar{r}-\bar{c})\ddot{\bar{r}} + \dot{\bar{r}}^2 = 0.$$

$$\{(\bar{i}+2\bar{j}+3\bar{k})-(x_1\bar{i}+y_1\bar{j}+z_1\bar{k})\}\,6\bar{j}+(2\bar{i})^2=0$$

$$(2\bar{j}-y_1\bar{j})\,6\bar{j}+4=0$$

$$(2-y_1)\,6+4=0$$

$$(2-y_1)6=-4 \Rightarrow (2-y_1)=-4/6=-2/3$$

$$-y_1=-2/3-2=\frac{-2-6}{3}=\frac{-8}{3}$$

$$\boxed{y_1=\frac{8}{3}}$$

and

$$(\bar{r}-\bar{c})\dddot{\bar{r}}+3\dot{\bar{r}}\ddot{\bar{r}}=0.$$

$$\{(\bar{i}+2\bar{j}+3\bar{k})-(x_1\bar{i}+y_1\bar{j}+z_1\bar{k})\}(24\bar{k})+3(2\bar{i})(6\bar{j})=0.$$

$$(3\bar{k}-z_1\bar{k})\,24\bar{k}=0.$$

$$(3-z_1)(24)=0 \qquad 3-z_1=0\times\frac{1}{24}=0.$$

$$3-z_1=0.$$

$$\boxed{z_1=3.}$$

To find ρ:-

(2) $\Rightarrow$ $(\bar{r}-\bar{c})^2=\rho^2$ where $\bar{c}=x_1\bar{i}+y_1\bar{j}+z_1\bar{k}$.

$$\{(\bar{i}+2\bar{j}+3\bar{k})-(x_1\bar{i}+y_1\bar{j}+z_1\bar{k})\}^2=\rho^2$$

$$\{(\bar{i}+2\bar{j}+3\bar{k})-(\bar{i}+\tfrac{8}{3}\bar{j}+3\bar{k})\}^2=\rho^2$$

$$\{(\bar{i}-\bar{i})+(2\bar{j}-\tfrac{8}{3}\bar{j})+(3\bar{k}-3\bar{k})\}^2=\rho^2$$

$$(2\bar{j}-\tfrac{8}{3}\bar{j})^2=\rho^2 \Rightarrow (2-\tfrac{8}{3})^2\,\bar{j}^2=\rho^2$$

$$\left(\frac{6-8}{3}\right)^2=\rho^2 \Rightarrow (-2/3)^2=\rho^2.$$

$$4/9=\rho^2 \Rightarrow \sqrt{\rho^2}=\sqrt{4/9}$$

$$\boxed{\rho=2/3}$$

The equation of the osculating sphere is,

$$(\bar{r}-\bar{c})^2=\rho^2.$$

$$\{(x\bar{i}+y\bar{j}+z\bar{k}) - (\bar{i}+\tfrac{8}{3}\bar{j}+3\bar{k})\}^2 = \rho^2$$

$$[(x-1)\bar{i} + (y-\tfrac{8}{3})\bar{j} + (z-3)\bar{k}]^2 = 4/9$$

$$(x-1)^2 + (y-8/3)^2 + (z-3)^2 = 4/9$$

$$x^2+1-2x+y^2+\frac{64}{9} - \frac{16}{3}y + z^2+9-6z = 4/9$$

$$x^2+y^2+z^2 - 2x - \frac{16}{3}y - 6z + \frac{154}{9} - 4/9 = 0$$

$$x^2+y^2+z^2 - 2x - \frac{16}{3}y - 6z + \frac{150}{9} = 0.$$

$$\frac{1}{9}\{9(x^2+y^2+z^2) - 18x - 48y - 54z + 150\} = 0.$$

$$9(x^2+y^2+z^2) - 18x - 48y - 54z + 150 = 0.$$

Case (ii):-

The osculating circle is the intersection of the osculating plane and the above sphere.

The eqn of the osculating plane at $u=0$ is,

W.K.T $[\bar{R}-\bar{r}(0), \bar{r}'(0), \bar{r}''(0)] = 0$

$$[\bar{R}-\bar{r}, \dot{\bar{r}}, \ddot{\bar{r}}] = 0.$$

W.K.T
$$\begin{vmatrix} X-x & Y-y & Z-z \\ \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix} = 0.$$

Given the point $u=(1,2,3)$ on the curve,

$x=1,\ y=2,\ z=3$

$$\begin{vmatrix} X-1 & Y-2 & Z-3 \\ 2 & 0 & 0 \\ 0 & 6 & 0 \end{vmatrix} = 0. \qquad [\dot{\bar{r}} = (2,0,0),\ \ddot{\bar{r}} = (0,6,0)]$$

$$(X-1)(0) - (Y-2)(0) + (Z-3)(12) = 0.$$

$$(Z-3)(12) = 0 \Rightarrow (Z-3) = 0$$

$$\therefore\ Z-3 = 0.$$

Hence the eqn of the osculating circle is,

$$9(x^2+y^2+z^2) - 18x - 48y - 54z + 150 = 0\ \ \&\ \ z-3=0$$

Ex: 6-5 P.T $\frac{d(\sigma\rho')}{ds} + \rho/\sigma = 0$. Prove also the converse

Soln:-

If the curve lies on a sphere, then the sphere will be the osculating sphere for every point on the curve so that the radius R of the osculating sphere is constant

$$R^2 = \rho^2 + (\sigma\rho')^2 \quad \text{——} ①$$

Diff w.r.t 's',

$$0 = 2\rho\rho' + 2(\sigma\rho')\frac{d}{ds}(\sigma\rho')$$

$$2\rho'\left[\rho + \sigma\frac{d}{ds}(\sigma\rho')\right] = 0$$

$$\because \rho' \neq 0.$$

$$\rho + \sigma\frac{d}{ds}(\sigma\rho') = 0 \quad (\text{or}) \quad \frac{\rho}{\sigma} + \frac{d}{ds}(\sigma\rho') = 0 \quad \text{——} ②$$

Conversely,

Let the condition ② satisfied at every point on the curve. multiplying by $2\rho'\sigma$ we get,

$$2\rho'\sigma\left(\frac{\rho}{\sigma}\right) + 2\rho'\sigma\frac{d}{ds}(\sigma\rho') = 0.$$

$$2\rho'\rho + 2\sigma\rho'\frac{d}{ds}(\sigma\rho') = 0.$$

$$2\int \rho\frac{d\rho}{ds}\,ds + 2\int(\sigma\rho')\frac{d}{ds}(\sigma\rho')\,ds = 0$$

$$2\left(\rho^2/2\right) + 2\,\frac{[\sigma\rho']^2}{2} = R^2.$$

$$\rho^2 + (\sigma\rho')^2 = R^2. \quad [\text{Const}]$$

(i.e) R the radius of the osculating sphere is constant at every point of the curve. Also the centre of the osculating sphere is,

$$\bar{c} = \bar{r} + \rho\bar{n} + \sigma\rho'\bar{b}$$

Differentiating w.r.t 's',

$$\frac{d\bar{c}}{ds} = \bar{\gamma}' + \rho'\bar{n} + \rho\bar{n}' + \frac{d}{ds}(\sigma\rho')\bar{b} + \sigma\rho'(\bar{b}')$$

$$= \bar{t} + \rho'\bar{n} + \rho(-k\bar{t} + \tau\bar{b}) + \frac{d}{ds}(\sigma\rho')\bar{b} + \sigma\rho'(-\tau\bar{n})$$

$$= \bar{t} + \rho'\bar{n} - \rho k\bar{t} + \rho\tau\bar{b} + \frac{d}{ds}(\sigma\rho')\bar{b} - (\sigma\tau)\rho'\bar{n}$$

$$= \cancel{\bar{t}} + \cancel{\rho'\bar{n}} - \cancel{\bar{t}} + \rho\left(\frac{1}{\sigma}\right)\bar{b} + \frac{d}{ds}(\sigma\rho')\bar{b} - \cancel{\rho'\bar{n}} \quad [\because k\rho = 1, \tau\sigma = 1]$$

$$= \left[\frac{\rho}{\sigma} + \frac{d}{ds}(\sigma\rho')\right]\bar{b}$$

$$\frac{d\bar{c}}{ds} = 0.$$

$\bar{c}$ is a constant vector. Hence the curve must lie on a sphere.

Ex: 3 Find the involutes and evolutes of the circular helix.

$\bar{\gamma} = a(\cos\theta, \sin\theta, \theta\tan\alpha)$.

Soln:

$$\dot{\bar{\gamma}} = a(-\sin\theta, \cos\theta, \tan\alpha) \quad \text{—— (1)}$$

$$\dot{s} = |\dot{\bar{\gamma}}| = \sqrt{a^2(\sin^2\theta + \cos^2\theta + \tan^2\alpha)}$$

$$= a\sqrt{\sin^2\theta + \cos^2\theta + \tan^2\alpha} = a\sqrt{1+\tan^2\alpha}$$

$$= a\sqrt{\sec^2\alpha} \quad [\because 1 + \tan^2\alpha = \sec^2\alpha]$$

$$\dot{s} = a\sec\alpha. \quad \text{—— (2)}$$

$$\bar{t} = \bar{\gamma}' = \frac{\dot{\bar{\gamma}}}{\dot{s}}$$

$$= \frac{a(-\sin\theta, \cos\theta, \tan\alpha)}{a\sec\alpha} = (-\sin\theta, \cos\theta, \tan\alpha)\cos\alpha.$$

Integrating eqn (2) we get,

$$s = \int_0^\theta a\sec\alpha\, d\theta = a\int_0^\theta \sec\alpha\, d\theta.$$

$$= a\sec\alpha \int_0^\theta d\theta$$

$= a \sec\alpha\,[\theta - 0]$

$s = a\sec\alpha\,\theta$ —— (3).

Case (i): To Find Involute:

Involutes are given by $\bar{R} = \bar{r} + (c-s)\bar{t}$.

$\bar{R} = a(\cos\theta, \sin\theta, \theta\tan\alpha) + [c - \theta . a\sec\alpha]$

$(-\sin\theta, \cos\theta, \tan\alpha)\cos\alpha$.

If $\bar{R} = x\bar{i} + y\bar{j} + z\bar{k}$ then the Cartesian equations of the involutes are,

$x = a\cos\theta - \cos\alpha\sin\theta\,[c - a\theta\sec\alpha]$

$y = a\sin\theta + \cos\alpha\cos\theta\,[c - a\theta\sec\alpha]$

$z = a\theta\tan\alpha + \sin\alpha .\,[c - a\theta\sec\alpha]$

Case (ii): To Find Evolute:

The Evolutes are given by,

$\bar{R} = \bar{r} + \rho\bar{n} + \rho\cot\left(\int\tau ds + c\right)\bar{b}$

$\bar{R} = \bar{r} + \rho\bar{n} + \rho\cot(\phi + c)\bar{b}$, where $\phi = \int\tau ds$.

Find ρ & $\bar{n}$

$$\bar{t} = \frac{\dot{\bar{r}}}{\dot{s}} = \frac{a(-\sin\theta, \cos\theta, \tan\alpha)}{a\sec\alpha}.$$

$$\bar{t} = \cos\alpha(-\sin\theta, \cos\theta, \tan\alpha)$$

$$\bar{t}' = k\bar{n} = \frac{\dot{\bar{t}}}{\dot{s}} = \frac{\cos\alpha(-\cos\theta, -\sin\theta, 0)}{a\sec\alpha}.$$

$$\bar{t}' = \frac{\cos^2\alpha}{a}(-\cos\theta, -\sin\theta, 0)$$

where, $k = \dfrac{\cos^2\alpha}{a}$ (or) $\rho = \dfrac{1}{k} = \dfrac{a}{\cos^2\alpha} = a\sec^2\alpha$.

$$\bar{n} = (-\cos\theta, -\sin\theta, 0)$$

Find $\bar{b}$:

$$\bar{b} = \bar{t}\times\bar{n} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \cancel{-\cos\alpha\cos\theta} - \cos\alpha\sin\theta & \cos\alpha\cos\theta & \cos\alpha\tan\alpha \\ -\cos\theta & -\sin\theta & 0 \end{vmatrix}$$

$$= \bar{i}\,[0+\sin\theta\cos\alpha\tan\alpha] - \bar{j}\,[0+\cos\alpha\tan\alpha\cos\theta] + \bar{k}\,[\sin^2\theta\cos\alpha + \cos^2\theta\cos\alpha]$$

$$= \bar{i}\,[\sin\theta\cos\alpha\tan\alpha] - \bar{j}\,[\cos\alpha\tan\alpha\cos\theta] + \bar{k}\,[\cos\alpha]$$

$$= (\sin\theta\cos\alpha\tan\alpha,\ \cos\alpha\tan\alpha\cos\theta,\ \cos\alpha]$$

$$= \cos\alpha\,(\sin\theta\tan\alpha,\ \tan\alpha\cos\theta,\ 1)$$

Find τ:

$$\bar{b} = \cos\alpha\,(\sin\theta\tan\alpha,\ \tan\alpha\cos\theta,\ 1)$$

$$\bar{b}' = -\tau\bar{n} = \frac{\dot{\bar{b}}}{\dot{s}} = \frac{\cos\alpha\,(\cos\theta\tan\alpha,\ -\sin\theta\tan\alpha,\ 0)}{a\sec\alpha}$$

$$= \frac{\cos^2\alpha}{a}\,(\cos\theta\tan\alpha,\ -\sin\theta\tan\alpha,\ 0)$$

$$= \frac{\cos^2\alpha}{a}\times\tan\alpha\,(\cos\theta,\ -\sin\theta,\ 0)$$

$$\bar{b}' = \frac{\cos^2\alpha}{a}\times\frac{\sin\alpha}{\cos\alpha}\,(\cos\theta,\ -\sin\theta,\ 0)$$

$$\tau = \frac{1}{a}\cos\alpha\sin\alpha.$$

$$\phi = \int\tau\,ds = \int\frac{1}{a}\cos\alpha\sin\alpha\,ds = \frac{1}{a}\cos\alpha\sin\alpha\int ds.$$

$$= \frac{s}{a}\cos\alpha\sin\alpha. \qquad (\because s = a\theta\sec\alpha)$$

$$= \frac{a\theta\sec\alpha}{a}\,\frac{1}{\sec\alpha}\,\sin\alpha.$$

$$\phi = \theta\sin\alpha.$$

The evolutes are given by,

$$\bar{R} = a(\cos\theta, \sin\theta, \theta\tan\alpha) + a\sec^2\alpha\,(-\cos\theta, -\sin\theta, 0) +$$

$$a\sec^2\alpha\cot(\theta\sin\alpha + c)\cos\alpha\,(\sin\theta\tan\alpha, \tan\alpha\cos\theta, 1)$$

Ex: 1 (1.70) S.T the intrinsic eqn of the curve given by $x = ae^u \cos u$, $y = ae^u \sin u$, $z = be^u$ are $k = \frac{a\sqrt{2}}{s\sqrt{2a^2+b^2}}$, $\tau = \frac{b}{\sqrt{2a^2+b^2}} \cdot \frac{1}{s}$.

Soln:-

Given: $x = ae^u \cos u$, $y = ae^u \sin u$, $z = be^u$.

$\bar{r} = (ae^u \cos u, ae^u \sin u, be^u)$

$\dot{\bar{r}} = \{a[e^u \cos u + e^u(-\sin u)], a[e^u \cos u + e^u \sin u], be^u\}$

$\dot{\bar{r}} = \{ae^u[-\sin u + \cos u], ae^u[\cos u + \sin u], be^u\}$ — ①

$|\dot{\bar{r}}| = \dot{s} = \sqrt{\{ae^u[-\sin u + \cos u]\}^2 + \{ae^u[\cos u + \sin u]\}^2 + (be^u)^2}$

$= e^u \sqrt{a^2(-\sin u + \cos u)^2 + a^2(\cos u + \sin u)^2 + b^2}$

$= e^u \sqrt{a^2[\sin^2 u + \cos^2 u - 2\sin u \cos u] + a^2[\cos^2 u + \sin^2 u + 2\cos u \sin u] + b^2}$

$= e^u \sqrt{a^2[2\cos^2 u + 2\sin^2 u] - 2a^2 \sin u \cos u + 2a^2 \sin u \cos u + b^2}$

$= e^u \sqrt{2a^2[\cos^2 u + \sin^2 u] + b^2} = e^u \sqrt{2a^2(1) + b^2}$

$|\dot{\bar{r}}| = e^u \sqrt{2a^2 + b^2} = \dot{s}$

$\int$ing w.r.t 'u' we get,

$\int \frac{ds}{du} \cdot du = \int_{-\infty}^{u} e^u \sqrt{2a^2+b^2}\, du$ $\quad [\dot{s} = \frac{ds}{du}]$

$\int ds = \sqrt{2a^2+b^2} \int_{-\infty}^{u} e^u du$

$s = \sqrt{2a^2+b^2}\,[e^u]_{-\infty}^{u} = \sqrt{2a^2+b^2}\,[e^u - e^{-\infty}]$

$[e^{-\infty} = 0]$

$s = e^u \sqrt{2a^2+b^2} = \dot{s}$

$\boxed{s = \dot{s}}$

$\bar{r}' = \frac{d\bar{r}}{du} \cdot \frac{du}{ds} = \frac{d\bar{r}}{du} \cdot \frac{1}{ds/du} = \frac{d\bar{r}}{du} \cdot \frac{1}{\dot{s}} = \frac{\dot{\bar{r}}}{\dot{s}}$

$\bar{r}' = \frac{\{ae^u[-\sin u + \cos u], ae^u[\cos u + \sin u], be^u\}}{e^u \sqrt{2a^2+b^2}}$

$$= \frac{1}{e^{u}\sqrt{2a^2+b^2}}\, e^{u} \{a[-\sin u+\cos u], a[\cos u+\sin u], b\}$$

$$\bar{\gamma}' = \frac{1}{\sqrt{2a^2+b^2}} \{a[-\sin u+\cos u], a[\cos u+\sin u], b\} \quad \text{— (2)}$$

$$\bar{\gamma}'' = \frac{d\bar{\gamma}'}{du}\cdot\frac{du}{ds}$$

$$\bar{\gamma}'' = \frac{d}{du}\left\{\frac{1}{\sqrt{2a^2+b^2}}\{a[-\sin u+\cos u], a[\cos u+\sin u], b\}\right\}\cdot\frac{du}{ds}$$

$$= \frac{1}{\sqrt{2a^2+b^2}}\{a[-\cos u-\sin u], a[-\sin u+\cos u], 0\}\cdot\frac{1}{ds/du}$$

$$\bar{t}' = k\bar{n} = \frac{1}{\sqrt{2a^2+b^2}}\{a[-\cos u-\sin u], a[-\sin u+\cos u]\}\,\frac{1}{\dot{s}} \quad \text{— (3)}$$

$$k\bar{n} = \frac{1}{\sqrt{2a^2+b^2}}\{a[-\cos u-\sin u], a[-\sin u+\cos u]\}\,\frac{1}{s} \quad [\dot{s}=s]$$

$$|k\bar{n}| = \sqrt{\left(\frac{1}{\sqrt{2a^2+b^2}}\right)^2}\sqrt{a^2[-\cos u-\sin u]^2+a^2[-\sin u+\cos u]^2}\;\frac{1}{\sqrt{s^2}}$$

$$= \frac{1}{\sqrt{2a^2+b^2}}\sqrt{a^2[\cos^2u+\sin^2u+\cancel{2\cos u\sin u}]+a^2[\sin^2u+\cos^2u - \cancel{2\sin u\cos u}]}\cdot\frac{1}{s}$$

$$= \frac{1}{\sqrt{2a^2+b^2}}\sqrt{a^2 2[\cos^2u+\sin^2u]}\cdot\frac{1}{s}$$

$$k = \frac{a\sqrt{2}}{\sqrt{2a^2+b^2}}\cdot\frac{1}{s} \quad \text{— (A)}$$

From (3) using $\dot{s}=s$

$$\bar{\gamma}'' = \frac{1}{\sqrt{2a^2+b^2}}\{a[-\cos u-\sin u], a[-\sin u+\cos u], 0\}\,\frac{1}{s}$$

$$s\bar{\gamma}'' = \frac{1}{\sqrt{2a^2+b^2}}\{a[-\cos u-\sin u], a[-\sin u+\cos u], 0\} \quad \text{— (4)}$$

Diff w.r.t 's';

$$s\bar{\gamma}''' + \bar{\gamma}''(1) = \frac{1}{\sqrt{2a^2+b^2}}\{a[\sin u-\cos u], a[-\cos u-\sin u], 0\}\cdot\frac{1}{\dot{s}}$$

$$s[s\bar{\gamma}'''+\bar{\gamma}''] = \frac{1}{\sqrt{2a^2+b^2}}\{a[\sin u-\cos u], a[-\cos u-\sin u], 0\}$$

$$s^2\bar{\gamma}''' + s\bar{\gamma}'' = \frac{1}{\sqrt{2a^2+b^2}}\{a[\sin u-\cos u], a[-\cos u-\sin u], 0\}$$

$[\dot{s}=s]$ — (5)

(5) - (4) gives,

$$s^2\bar{r}''' + s\bar{r}'' - s\bar{r}'' = \frac{1}{\sqrt{2a^2+b^2}}\{a[\sin u - \cos u],\ a[-\cos u - \sin u],\ 0\} - \frac{1}{\sqrt{2a^2+b^2}}\{a[-\cos u - \sin u],\ a[-\sin u + \cos u],\ 0\}$$

$$s^2\bar{r}''' = \frac{1}{\sqrt{2a^2+b^2}}\{[a\sin u - a\cos u + a\cos u + a\sin u],\ [-a\cos u - a\sin u + a\sin u - a\cos u],\ 0\}$$

$$= \frac{1}{\sqrt{2a^2+b^2}}\{2a\sin u,\ -2a\cos u,\ 0\} \quad \text{— (6).}$$

Now,

$$[\bar{r}'\ \ s\bar{r}''\ \ s^2\bar{r}'''] = \begin{vmatrix} \frac{a(-\sin u + \cos u)}{\sqrt{(2a^2+b^2)}} & \frac{a(\cos u + \sin u)}{\sqrt{2a^2+b^2}} & \frac{b}{\sqrt{2a^2+b^2}} \\ \frac{a(-\cos u - \sin u)}{\sqrt{2a^2+b^2}} & \frac{a(-\sin u + \cos u)}{\sqrt{2a^2+b^2}} & 0 \\ \frac{2a\sin u}{\sqrt{2a^2+b^2}} & \frac{-2a\cos u}{\sqrt{2a^2+b^2}} & 0 \end{vmatrix}$$

$$= \frac{b}{\sqrt{2a^2+b^2}}\left[\frac{-2a^2\cos u\,[\cos u - \sin u]}{(2a^2+b^2)} - \frac{2a^2\sin u\,[-\sin u + \cos u]}{(2a^2+b^2)}\right]$$

$$= \frac{b}{(2a^2+b^2)^{1/2}(2a^2+b^2)}\{2a^2\cos^2 u + 2a^2\cos u\sin u + 2a^2\sin^2 u - 2a^2\sin u\cos u\}$$

$$= \frac{b}{(2a^2+b^2)^{3/2}}\, 2a^2[\cos^2 u + \sin^2 u]$$

$$= \frac{2a^2 b}{(2a^2+b^2)^{3/2}}.$$

(i.e) $s^3[\bar{r}'\ \ \bar{r}''\ \ \bar{r}'''] = \dfrac{2a^2 b}{(2a^2+b^2)^{3/2}}.$

$$s^3\kappa^2\tau = \frac{2a^2 b}{(2a^2+b^2)^{3/2}} \qquad [\because [\bar{r}'\ \bar{r}''\ \bar{r}'''] = \kappa^2\tau]$$

$$\tau = \frac{2a^2 b}{(2a^2+b^2)^{3/2}}\cdot\frac{1}{s^3\kappa^2} = \frac{2a^2 b}{(2a^2+b^2)^{3/2}}\cdot\frac{1}{s^3(\kappa\cdot\kappa)}$$

$$= \frac{2a^2 b}{(2a^2+b^2)^{3/2}} \times \frac{(2a^2+b^2)\,s^2}{2a^2} \times \frac{1}{s^3} \qquad [\text{By (1)}]$$

$$\tau = \frac{b}{\sqrt{2a^2+b^2}}\cdot\frac{1}{s} \qquad \left[\kappa = \frac{\sqrt{2}\,a}{\sqrt{2a^2+b^2}}\cdot\frac{1}{s}\right]$$

Ex : 2 S.T the angle b/w the principal normals at two neighbouring points O & P is $s\left[\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right]^{1/2}$ where s is the arcual distance b/w O & P.

Soln:

Let $\bar{n}$ & $\bar{n}+d\bar{n}$ be the principal normals at O & P on a Curve $\bar{r}=\bar{r}(s)$. If θ is the angle b/w these normals

$$\cos\theta = \frac{\bar{n}\cdot(\bar{n}+d\bar{n})}{|\bar{n}|\cdot|\bar{n}+d\bar{n}|} = \frac{\bar{n}\cdot\bar{n}+\bar{n}\cdot d\bar{n}}{|\bar{n}|\cdot|\bar{n}+d\bar{n}|} = \frac{1+\bar{n}\cdot d\bar{n}}{|\bar{n}|\cdot|\bar{n}+d\bar{n}|}$$

$\frac{\bar{a}\cdot\bar{b}}{|\bar{a}|\cdot|\bar{b}|}$

$$= \frac{1+0}{\sqrt{(\bar{n})^2}\cdot|\bar{n}+d\bar{n}|} = \frac{1}{|\bar{n}+d\bar{n}|} \quad \text{--- (1)} \quad \left[\begin{matrix} n\cdot n=1 \\ n\cdot dn=0\end{matrix}\right]$$

$$|\bar{n}+d\bar{n}| = |\bar{n}+\bar{n}'ds| \qquad \left[\because \bar{n}' = \frac{d\bar{n}}{ds}\right]$$

$$= |\bar{n}+(-k\bar{t}+\tau\bar{b})s| \qquad [\because ds = \overset{\circ}{s} = s]$$

$$= \sqrt{(\bar{n})^2+(-k\bar{t}+\tau\bar{b})^2 s^2}$$

$$= \sqrt{1+(k^2+\tau^2)s^2} \quad \text{--- (2)}$$

$$\cos\theta = \frac{1}{\sqrt{1+(k^2+\tau^2)s^2}} = [1+s^2(k^2+\tau^2)]^{-1/2} \quad \text{--- (3)}$$

$$\left[\ \cos\theta = 1-\frac{\theta^2}{2!}+\frac{\theta^3}{3!}\cdots\right.$$

$$(1+x)^{-1} = 1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots$$

$$\left.(1+x)^{-2} = 1-2x+\frac{2x^2}{2!}-\frac{2x^3}{3!}+\cdots\ \right]$$

therefore we get,

$$(3) \Rightarrow \quad 1-\frac{\theta^2}{2!}+\frac{\theta^3}{3!}\cdots = 1-\frac{1(k^2+\tau^2)s^2}{2}+\frac{1}{2}\,\frac{(k^2+\tau^2)^2 s^4}{2!}\cdots$$

$$1-\frac{\theta^2}{2!} = 1-\frac{1}{2}(k^2+\tau^2)s^2 \quad \text{(approximately)}$$

$$\theta^2 = (k^2+\tau^2)s^2$$

$$\theta^2 = \left(\frac{1}{\rho^2}+\frac{1}{\sigma^2}\right)s^2 \qquad \left[\frac{1}{\rho}=k,\ \frac{1}{\sigma}=\tau\right]$$

Square root on both sides,

$$\sqrt{\theta^2} = \sqrt{\left(\frac{1}{\sigma^2}+\frac{1}{\rho^2}\right)}\cdot\sqrt{s^2}$$

$$\theta = s\left[\frac{1}{\sigma^2}+\frac{1}{\rho^2}\right]^{1/2}$$

P. No: 1.37
10

Find the Curvature and Torsion of the Curve $\bar{r} = \{a(1+\cos u), a\sin u, 2a\sin u/2\}$.

Soln:-

Given: $\bar{r} = \{a(1+\cos u), a\sin u, 2a\sin u/2\}$

$\dot{\bar{r}} = a\{-\sin u, \cos u, 2\cos u/2 \cdot 1/2\}$.

$\ddot{\bar{r}} = a\{-\cos u, -\sin u, -\sin u/2 \cdot 1/2\}$

$\dddot{\bar{r}} = a\{\sin u, -\cos u, -\cos u/2 \cdot 1/4\}$.

$|\dot{\bar{r}}| = a\sqrt{\sin^2 u + \cos^2 u + \cos^2 u/2} = a\sqrt{1+\cos^2 u/2}$

$= a\sqrt{1 + \left(\frac{1+\cos 2(u/2)}{2}\right)} = a\sqrt{\frac{2+1+\cos u}{2}}$.

$|\dot{\bar{r}}| = a\sqrt{\frac{3+\cos u}{2}}$

$|\dot{\bar{r}}|^3 = a^3\left[\frac{3+\cos u}{2}\right]^{3/2}$ —— (1)

$$\dot{\bar{r}} \times \ddot{\bar{r}} = a\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -\sin u & \cos u & \cos u/2 \\ -\cos u & -\sin u & -\sin u/2 \cdot 1/2 \end{vmatrix}$$

$= a[\bar{i}\{-\cos u \sin u/2 \cdot 1/2 + \sin u \cos u/2\} - \bar{j}\{\sin u \sin u/2 \cdot 1/2 + \cos u \cos u/2\} + \bar{k}\{\sin^2 u + \cos^2 u\}]$

$= a\{\bar{i}[-\cos u \sin u/2 \cdot 1/2 + \sin u \cos u/2] - \bar{j}[1/2 \sin u \sin u/2 + \cos u \cos u/2] + \bar{k}\}$.

$|\dot{\bar{r}} \times \ddot{\bar{r}}| = \sqrt{a^2} \cdot \sqrt{\frac{1}{4}\cos^2 u \sin^2 u/2 + \sin^2 u \cos^2 u/2 - 2 \cdot 1/2 \sin u \sin u/2 \cos u \cos u/2 + 1/4 \sin^2 u \sin^2 u/2 + \cos^2 u \cos^2 u/2 + 2 \cdot 1/2 \sin u \sin u/2 \cos u \cos u/2 + 1}$

$= a\sqrt{1/4 \sin^2 u/2 (1) + \cos^2 u/2 (1) + 1}$

$= a\sqrt{1/4\left(\frac{1-\cos u}{2}\right) + \left(\frac{1+\cos u}{2}\right) + 1}$

$= a\sqrt{\frac{1}{8} - \frac{\cos u}{8} + \frac{1}{2} + \frac{\cos u}{2} + 1}$

$$= a\sqrt{\frac{5}{8} + \frac{3\cos u}{8} + 1}$$

$$|\dot{r} \times \ddot{r}| = \frac{a}{4}\sqrt{\frac{13+3\cos u}{2}} \quad \text{--- (2)}$$

Curvature : $\bar{k} = \frac{|\dot{r} \times \ddot{r}|}{|\dot{r}|^3}$

$$\bar{k} = \frac{a}{4}\sqrt{\frac{13+3\cos u}{2}} \times \frac{2\sqrt{2}}{a^3[3+\cos u]^{3/2}}$$

$$= \frac{\sqrt{13+3\cos u}}{a^2(3+\cos u)^{3/2}} \quad \text{--- (3)}$$

$$[\dot{r}\ \ddot{r}\ \dddot{r}] = a^3 \begin{vmatrix} -\sin u & \cos u & \cos u/2 \\ -\cos u & -\sin u & -\sin u/2 \cdot 1/2 \\ \sin u & -\cos u & -\cos u/2 \cdot 1/4 \end{vmatrix}$$

$$= a^3 \{ -\sin u [\sin u \cos u/2 \cdot 1/4 - \cos u \cdot \sin u/2 \cdot 1/2] - \cos u$$
$$[\cos u \cos u/2 \cdot 1/4 + \sin u \sin u/2 \cdot 1/2] + \cos u/2 [\cos^2 u + \sin^2 u] \}$$

$$= a^3 \{ -\sin^2 u \cos u/2 \cdot 1/4 + \sin u \cos u \sin u/2 \cdot 1/2 - \cos^2 u \cos u/2 \cdot 1/4$$
$$- \cos u \sin u \sin u/2 \cdot 1/2] + \cos u/2 (1) \}$$

$$= a^3 \{ -\cos u/2\ 1/4 (1) + \cos u/2 \}$$

$$= a^3 \left[\frac{3}{4} \cos u/2 \right] \quad \text{--- (4)}$$

Torsion: $\tau = \frac{[\dot{r}\ \ddot{r}\ \dddot{r}]}{|\dot{r} \times \ddot{r}|^2}$

$$\tau = a^3 [3/4 \cos u/2] \times \frac{1}{(a/4)^2 \cdot \left(\sqrt{\frac{13+3\cos u}{2}}\right)^{2}}$$

$$= a^3 \left[\frac{3}{4} \cos u/2\right] \times \frac{1}{\frac{a^2}{4}\left(\frac{13+3\cos u}{2}\right)}$$

$$= \frac{a^3\ 3\cos u/2}{4} \times \frac{8}{a^2(13+3\cos u)}$$

$$\tau = \frac{6a\cos u/2}{(13+3\cos u)}$$