10-7-18 Unit - II

Finite Differences:

(1) Let $y = f(x)$ be a given function of x and let $y_0, y_1, y_2 \cdots$ be the values of y corresponding to $x_0, x_0+h, x_0+2h \cdots$ of the values of x ie, $y_0 = f(x_0)$, $y_1 = f(x_0+h)$, $y_2 = f(x_0+2h) \cdots\cdots y_n = f(x_0+nh)$.

Here the independent variable (or) argument x proceeds at equally spaced intervals and 'h' (constants) the difference between two consecutive value of x is called the **interval of differencing.**

Now $y_1 - y_0, y_2 - y_1, \cdots y_n - y_{n-1}$ are called the first differences of the function y and the differences of the given values are denoted by

$$\Delta y_n = y_{n+1} - y_n \quad [n = 0, 1, 2 \cdots].$$

Here 'Δ' acts as an operator called forward differences operator.

Thus,

$$\Delta y_0 = y_1 - y_0$$
$$\Delta y_1 = y_2 - y_1$$
$$\Delta y_n = y_{n+1} - y_n$$

The differences of these first differences are called second differences

Thus,

$$\Delta^2(y_0) = \Delta(y_0) = \Delta(y_1 - y_0)$$

$$\Delta^2(y_0) = \Delta(y_1) - \Delta(y_0)$$

$$\Delta^2(y_0) = y_2 - y_1 - [y_1 - y_0]$$

$$\Delta^2(y_0) = y^2 - y_1 - y_1 + y_0$$

$$\Delta^2(y_0) = y^2 - 2y_1 + y_0$$

$$\Delta^2(y_1) = \Delta(\cancel{x}y_1) = \Delta(y_2 - y_1) = \Delta y_2 - \Delta y_1$$

$$\Delta^2(y_1) = y_3 - y_2 - [y_2 - y_1]$$

$$\Delta^2(y_1) = y_3 - y_2 - y_2 + y_1$$

$$\Delta^2(y_1) = y_3 - 2y_2 + y_1 \text{ and so on.}$$

In general $\Delta^n \cdot y_K = \Delta^{n-1} y_{K+1} - \Delta^{n-1} y_K$ defines n^{th} differences where K and n are integers.

The difference table is a standard format for displaying finite differences and is explained in the following table called **forward difference table.**

x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		
		Δy_3			
x_4	y_4				

Here each differences proves to be a combination of y values.

for example,

(3)

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$$

$$\Delta^3 y_0 = (\Delta^2 y_0 - \Delta y_1) - (\Delta y_1 - \Delta y_0)$$

$$\Delta^3 y_0 = \{(y_3 - y_2) - (y_2 - y_1)\} - \{(y_2 - y_1) - (y_1 - y_0)$$

$$\Delta^3 y_0 = y_2 - y_2 - y_2 + y_1 - y_2 - y_1 - y_1 + y_0$$

$$\Delta^3 y_0 = y_3 - 2y_2 + y_1 - y_2 - 2y_1 + y_0$$

$$\Delta^3 y_0 = y_3 - 3y_3 + 3y_1 - y_0$$

Newton's forward:

- Interpolation formula

we know that

$\Delta y_0 = y_1 - y_0$ ie ... $y_1 = y_0 + \Delta y_0 = (1+\Delta) y_0$

$\Delta y_1 = y_2 - y_1$ ie ... $y_2 = y_1 + \Delta y_1 = (1+\Delta) y_1$

$$= (1+\Delta)(1+\Delta) y_0$$

$$= (1+\Delta)^2 y_0$$

$\Delta y_2 = y_3 - y_2$ ie ... $y_3 = y_2 + \Delta y_2 = (1+\Delta)^3 y_0$

In general

$$y_n = (1+\Delta)^n y_0$$

Expanding $(1+\Delta)^n$ by using Binomial theoram we have,

$$y_n = \left\{1 + \Delta n + \frac{n(n-1)\Delta^2}{2!} + \frac{n(n-1)(n-2)\Delta^3}{3!} + \dots\right\} y_0$$

$$y_n = f(x_0 + nh) = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \dots$$

The result is known as Gregory newton forward interpolation (or) newton's formula for equal intervals.

(4)

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\dots$$

$$(1+x)^n = 1-nx+\frac{n(n-1)}{2!}x^2-\dots$$

$$(1-x)^n = 1-nx+\frac{n(n-1)}{2!}x^2-\dots$$

$$(1-x)^{-n} = 1+xn+\frac{n(n-1)}{2!}x^2+\dots$$

Backward differences:

We use another operator called the Backward differences operator ∇ and is defined by

$$\Delta y_n = y_n - y_{n-1}$$

For

$[n = 0, 1, 2 \dots]$ we get.

$$\nabla y_0 = y_0 - y_1$$

$$\nabla y_1 = y_1 - y_0$$

$\nabla y_2 = y_2 - y_1$ and so on.

The second Backward difference is

$$\begin{aligned}\nabla^2 y_n &= \nabla(\nabla y_n)\\ &= \nabla(y_n - y_{n-1})\\ &= \nabla y_n - \Delta y_{n-1}\\ &= (y_n - y_{n-1}) - (y_{n-1} - y_{n-2})\\ &= y_n - 2y_{n-1} + y_{n-2}.\end{aligned}$$

Similarly the third Backward difference is.

$$\nabla^2 y_n = \nabla^2 y_n - \nabla^2 y_{n-1}.$$

$= (y_n - 2y_{n-1} + y_{n-2}) - (y_{n-1} - 2y_{n-2} + y_{n-3})$

$= y_n - 3y_{n-1} + 3y_{n-2} - y_{n-3}$ and so on

Derive newton's Backward table.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
$x_{-4} = x_0 - 4h$	y_{-4}				
		∇y_{-3}			
$x_{-3} = x_0 - 3h$	y_{-3}		$\nabla^2 y_{-2}$		
		∇y_{-2}		$\nabla^3 y_{-1}$	
$x_{-2} = x_0 - 2h$	y_{-2}		$\nabla^2 y_{-1}$		$\nabla^4 y_0$
		∇y_{-1}		$\nabla^3 y_0$	
$x_{-1} = x_0 - 1h$	y_{-1}		$\nabla^2 y_0$		
		∇y_0			
$x_0 = y_0$	y_0				

Newton's Backward Interpolation formula:

We know that $\nabla y = y_1 - y_0$ (or)

$(1-\nabla) y_1 - y_0$ (or) $y_1 = (1-\nabla)^{-1} y_0 \longrightarrow (1)$

Also we know that $y_1 = (1+\nabla) y_0 \longrightarrow (2)$

[By definition forward difference operator].

from (1) and (2) we get

$(1-\nabla)^{-1} = (1+\nabla)$

Hence $y_n = (1+\nabla)^n y_0 = (1-\nabla)^{-n} y_0$

$= 1 + n\nabla + \frac{n(n+1)}{2!}\nabla^2 + \frac{n(n+1)(n+2)}{3!}\nabla^3 \ldots$

ie,

$y = (x_0 + nh)$

$= y_0 + n\nabla y_0 + \frac{n(n+1)}{2!}\nabla^2 y_0 + \frac{n(n+1)(n+2)}{3!}\nabla^3 y_0 \ldots$

This is Gregory - Newton Backward interpolation formula.

Central Differences:

The central difference operator δ is defined by the relation.

$$y_1 - y_0 = \delta y_{1/2},\ y_2 - y_1 = \delta y_{3/2} \ldots y_n - y_{n-1} = \delta y_{n-1/2}.$$

Similarly high order central differences can be defined with the value of x and y in the proceding two tables, a central differences table can be formed, thus,

x	y	δ	δ_2	δ_3	δ_4	δ_5	δ_6
x_0	y_0						
		$\delta y_{1/2}$					
x_1	y_1		$\delta^2 y_1$				
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$			
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$		
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$	
x_3	y_3		$\delta^2 y_3$		$\delta^4 y_3$		$\delta^6 y_3$.
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$		$\delta^5 y_{7/2}$	
x_4	y_4		$\delta^2 y_4$		$\delta^4 y_4$		
		$\delta y_{9/2}$		$\delta^3 y_{9/2}$			
x_5	y_5		$\delta^2 y_5$				
		$\delta y_{11/2}$					
x_6	y_6						

It is clear from the tables that in a define numerical case, the same numbers occurs in the same position whether we use forward, backward or central differences,

Thus, we obtain,

$$\Delta y_0 = \nabla y_1 \, \delta y^{1/2}$$

$$\Delta^3 y_2 = \nabla^3 y_n = \nabla^3 y^{7/2} \text{ etc}$$

Newton's formula for Interpolation:

- **Newton forward difference Interpolation formula:**

Give the set of $(n+1)$ values viz ...
$(x_0, y_0), (x_1, y_1), (x_2, y_2) \ldots (x_n, y_n)$ of x and y it is required to find $y_n(x)$ a polynomial of the n^{th} degree such that y and $y_n(x)$ agree at the tabulated points. Let the value of x be equidistant

ie,

$$\text{let } x_i = x + ih \; ; i = 0, 1, 2 \ldots n.$$

Since $y_n(x)$ is polynomial of the n^{th} degree it may be written as

$$y_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) + \cdots (x - x_{n-1})$$

Imposing now the condition that y and y_n should agree at the set of tabulated points we obtain

$$a_0 = y_0 \; ; \; a_1 = \frac{y - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h} \; ; \; a_2 = \frac{\Delta^2 y_0}{h^2 2!}$$

$$a_3 = \frac{\Delta^3 y_0}{h^3 3!} , \ldots \quad a_n = \frac{\Delta^n y_0}{h^n n!}$$

Setting $x = x_0 + Ph$ and Substituting for $a_0 \cdots$ an equation ① gives.

$$y_n(x) = y_0 + P\nabla y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0$$
$$+ \cdots \frac{P(P-1)(P-2)\cdots(P-n-1)}{n!}\Delta^n y_0$$

which is newton's forward differences interpolation formula.

- **Newton Backward Interpolation formula:**

Give the set of $(n+1)$ values viz... $(x_0, y_0), (x_1, y_1), (x_2, y_2) \cdots (x_n, y_n)$ of x, and y. It is required to find $y_n(x)$ a polynomial of the n^{th} degree such that y and $y_n(x)$ degree at the tabulated points. Let the values of x be equidistant ie...

let $x_i = x_0 + ih$; $i = 0, 1, 2 \cdots n$

Since $y_n(x)$ is a polynomial of the n^{th} degree it may be written as.

$$y_n(x) = a_0 + a(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) + \cdots + a_n(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{n-1})$$

Instead of assuming $y_n(x)$ as in ① If we choose it in the form.

$$\left.\begin{aligned} y_n(x) = a_0 + a_1(x-x_n) + a_2(x-x_n)(x-x_{n-1}) + a_3(x-x_n)(x-x_{n-1}) \\ (x-x_{n-2}) + \cdots + a_n(x-x_n)(x-x_{n-3}) + \cdots + (x-x_1) \end{aligned}\right\} - ①$$

and then impose the condition on

that y and $y_n(x)$ should agree at the tabulated points

$x_n, x_{n-1}, \dots, x_2, x_1, x_0$. we obtain (after some simplification)

Backward difference formula :-

$$y_n(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \dots$$

$$\frac{p(p+1)\dots(p+n-1)}{n!}\nabla^2 y_n \rightarrow (6)$$

The Newton's interpolation formula

Problem: 1

The population of the town in the decimal census was as given below. estimate the population for the year 1895 (in thousands)

Ans 54.85

Year x :	1891	1901	1911	1921	1931
Population y :	46	66	81	93	101

1895

Soln:

The forward difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1891	46				
		(66-46) 20			
1901	66		(15-20) -5		
		(81-66) 15		(-3+5) 2	
1911	81		(12-15) -3		(-1-2) -3
		(93-81) 12		(-4+3) -1	
1921	93		(+8-12) -4		
		(101-93) 8			
1931	101				

To find P

$x_0 \rightarrow$ initial value

$x \rightarrow$ gn value

$h \rightarrow$ difference b/w x value

$$\boxed{x = x_0 + Ph}$$

$x_0 = 1891$; $x = 1895$; $h = 10$.

$$1895 = 1891 + P(10)$$

$$10P = 1895 - 1891$$

$$P = \frac{1895 - 1891}{10}$$

$$\boxed{P = 0.4}$$

Newton's forward formula is.

$$y_n(x) = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \frac{P(P-1)(P-2)(P-3)}{4!}\Delta^4 y_0 + \cdots$$

$y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0$

$$= 46 + (0.4)(20) + \frac{(0.4)(0.4-1)}{2!}(-5) + \frac{(0.4)(0.4-1)(0.4-2)}{3!}(2) + \cancel{(0.4)(0.4-1)} \frac{(0.4)(0.4-1)(0.4-2)(0.4-3)}{4!}(-3).$$

$$= 46 + 8 + \frac{(-0.24)}{2!}(-5) + \frac{(-0.896)}{3!}(2) + \frac{(1.5584)}{4!}(-3).$$

$$= 54 + (-0.12)(-5) + (-0.1493)(2) + (-0.0649)(-3)$$

$$= 54 + 0.6 - 0.2986 + 0.1947$$

$$= 54.85.$$