Chapter 7

Scheduling by PERT and CPM

7.1 Introduction

A *project* is defined as a combination of interrelated activities all of which must be executed in a certain order to achieve a set goal. A large and complex project involves usually a number of interrelated activities requiring men, machines and materials. It is impossible for the management to make and execute an optimum schedule for such a project just by intuition, based on the organisational capabilities and work experience. A systematic scientific approach has become a necessity for such projects. So a number of methods applying network scheduling techniques has been developed. ***Programme Evaluation Review Technique*** (PERT) and ***Critical Path*** Method (CPM) are two of the many network techniques which are widely used for planning, scheduling and controlling large complex projects.

The three main managerial functions for any project are

1. Planning
2. Scheduling
3. Control

Planning

This phase involves a listing of tasks or jobs that must be performed to complete a project under consideration. In this phase, men, machines and materials required for the project in addition to the estimates of costs and durations of various activities of the project are also determined.

Scheduling

This phase involves the laying out of the actual activities of the projects in a ***logical sequence*** of time in which they have to be performed.

Men and material requirements as well as the ***expected completion time*** of each activity at each stage of the project are also determined.

Control

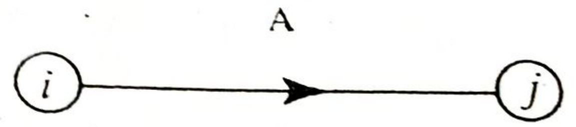
This phase consists of reviewing the progress of the project whether the actual performance is according to the planned schedule and finding the reasons for difference, if any, between the schedule and performance. The basic aspect of control is to analyse and correct this difference by taking remedial action wherever possible.

PERT and CPM are especially useful for scheduling and controlling.

7.2 Basic Terminologies

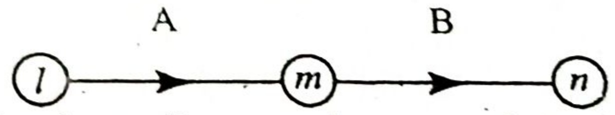
Activity is a task or an item of work to be done in a project. An activity consumes resources like time, money, labour etc.

An activity is represented by an arrow with a node (event) at the beginning and a node (event) at the end indicating the start and termination (finish) of the activity. Nodes are denoted by circles. Since this is a logical diagram length or shape of the arrow has no meaning. The direction indicates the progress of the activity. Initial node and the terminal node are numbered as $i - j$ $(j > i)$ respectively. For example If A is the activity whose initial node is i and the terminal node is j then it is denoted diagrammatically by



The name of the activity is written over the arrow, ***not inside the circle.*** The diagram in which arrow represents an activity is called ***arrow diagram.*** The initial and terminal nodes of activities are also called tail and head events.

If an activity B can start immediately after an activity A then it is denoted by



A is called the ***immediate predecessor*** of B and B is called the ***immediate successor*** of A. If C can start only after completing activities A and B then it is diagrammatically represented as follows :

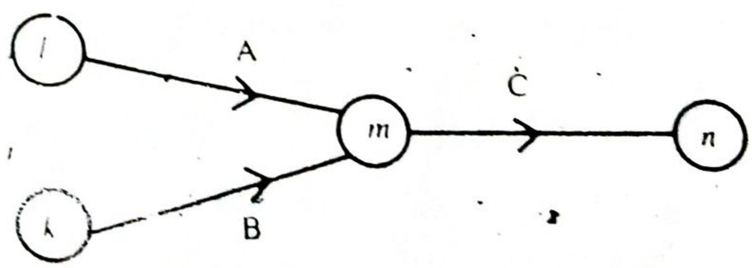


Fig. 1

Notation : "A is a predecessor of B" is denoted as "A < B" & "B is successor of A" is denoted by "B > A".

If the project contains two or more activities which have some of the immediate predecessors in common then there is a need for introducing what is called ***dummy activity***. Dummy activity is an imaginary activity which does not consume any resource and which serves the purpose of indicating the predecessor or successor relationship clearly in any activity on arrow diagram. The need for a dummy activity is illustrated by the following usual example.

Let P, Q be the predecessors of R and Q be the only predecessor of S.

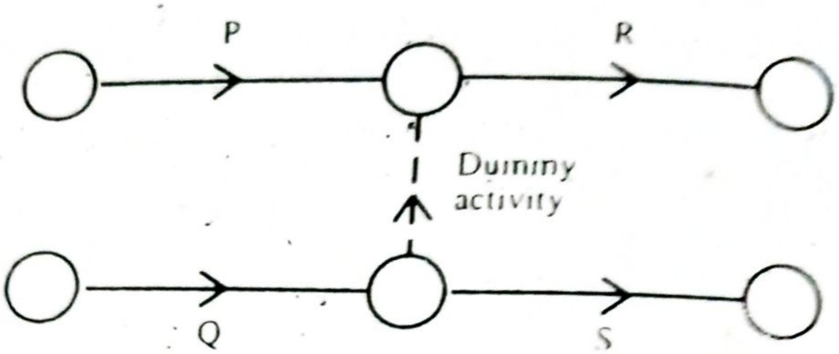


Fig. 2

Activities which have no ***predecessors*** are called ***start activities*** of the project. All the ***start activities*** can be made to have the ***same initial*** node. Activities which have ***no successors*** are called ***terminal activities*** of the project. These can be made to have ***the same*** terminal node (end node of the project.

A project consists of a number of activities to be performed in some technological sequence. For example while constructing a building the activity of laying the foundation should be done before the activity of erecting the walls for the building. The diagram denoting all the activities of a project by arrows taking into account the technological sequence of the activities is called the ***project network*** represented by ***activity on arrow diagram*** or simply ***arrow diagram.***

Note: There is another representation of a project network representing activities on nodes called AON diagram. To avoid confusion we use only activity on arrow diagram throughout the text.

7.3 Rules for constructing a project network

1. There must be no loops. For example, the activities F, D, E.

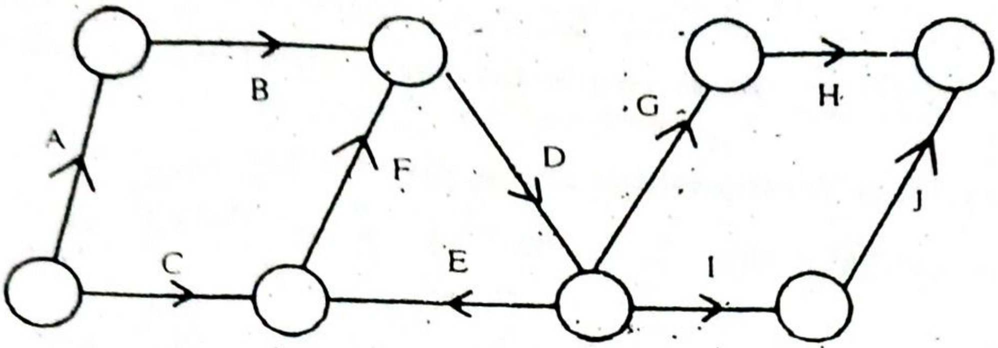


Fig. 3

Obviously form a loop which is obviously not possible in any real
ject network.

2. Only one activity should connect any two nodes.

3. No dangling should appear in a project network *i.e.* no node of any
ivity except the terminal node of the project should be left without any
ivity emanating from it. Such a node can be joined to the terminal
e of the project to avoid.

es may be numbered using the rule given below :

(Ford and Fulkerson's Rule)

1. Number the start node which has no predecessor activity, as 1.

2. Delete all the activities emanating from this node 1.

3. Number all the resulting start nodes without any predecessor as

4. Delete all the activities originating from the start nodes 2, 3, ... in

3

Number all the resulting new start nodes without any predecessor
o the last number used in step (3).

Repeat the process until the terminal node without any successor
y is reached and number this terminal node suitably.

***mediate predecessor (successor) will be simply called as
essor (successor) unless otherwise stated.***

**xample 1 : If there are five activities P, Q, R, S and T such that
R have no immediate predecessors but S and T have immediate
essors P, Q and Q, R respectively. Represent this situation by a
rk.**

Solution :

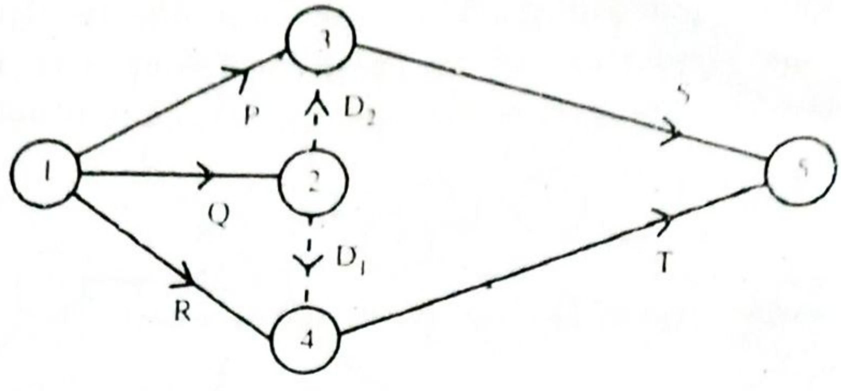


Fig 4

D_1 and D_2 are dummy activities

Example 2 : **Draw the network for the project whose activities and their precedence relationships are given below:**

Activity	:	**P**	**Q**	**R**	**S**	**T**	**U**
Predecessor	:	–	–	–	**P, Q**	**P, R**	**Q, R**

Solution :

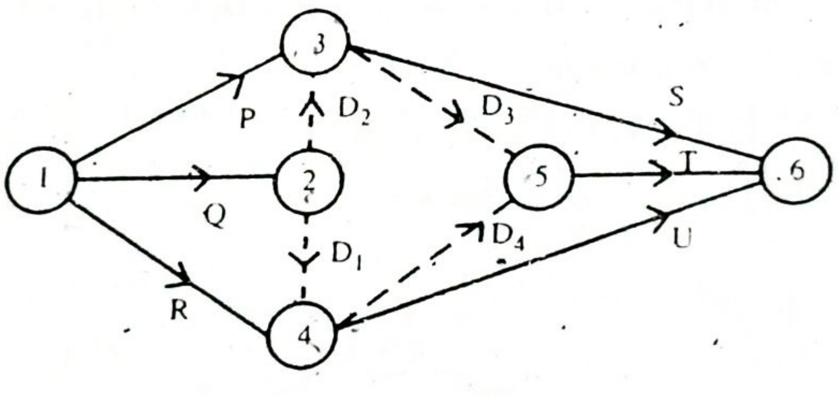


Fig. 5

D_1, D_2, D_3, D_4 are dummy activities

Example 3 : **Draw the network for the project whose activities with their predecessor relationships are given below:**

A,C,D can start simultaneously ; E > B, C ; F, G > D ; H, I > E, F ; J > I, G ; K > H ; B > A.

Solution : Identify the start activities *i.e.*, activities which have no predecessors. They are A, C and D as given. These three activities should start with the same start node. Also identify the terminal activities –

activities which have no successors. They are J and K. These two activities should end with the same end node, the last terminal node indicating the completion of the project. Taking into account the predecessor relationships given, the required network is as follows

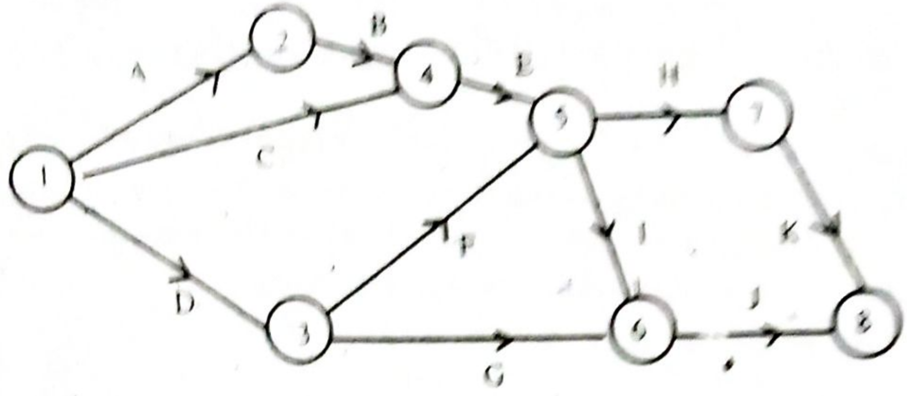


Fig. 6

Example 4 : **Construct the network for the project whose activities and their relationships are as given below:**

Activities : A, D, E can start simultaneously.

Activities : B, C > A ; G, F > D, C ; H > E, F.

Solution : Start activities are A, D, E

End activities are H, G, B

∴ The required network is

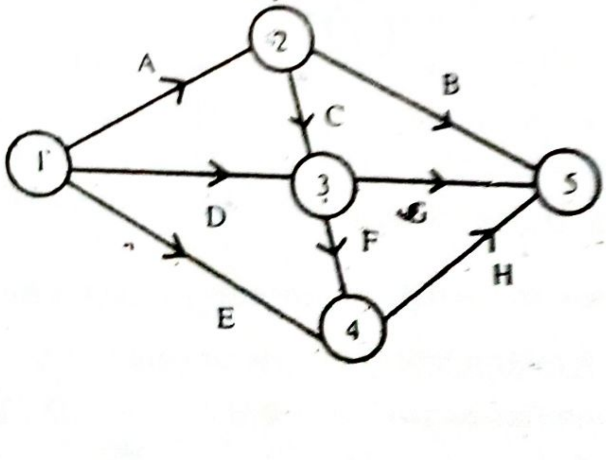


Fig. 7

Note : See how the nodes of the activity F are numbered. Can we number C as 2 – 4 and F as 4 – 3 ?

7.

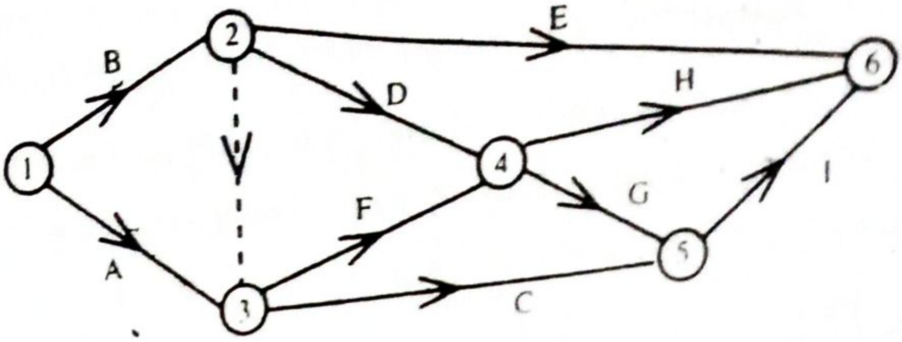


Fig. 17

8.

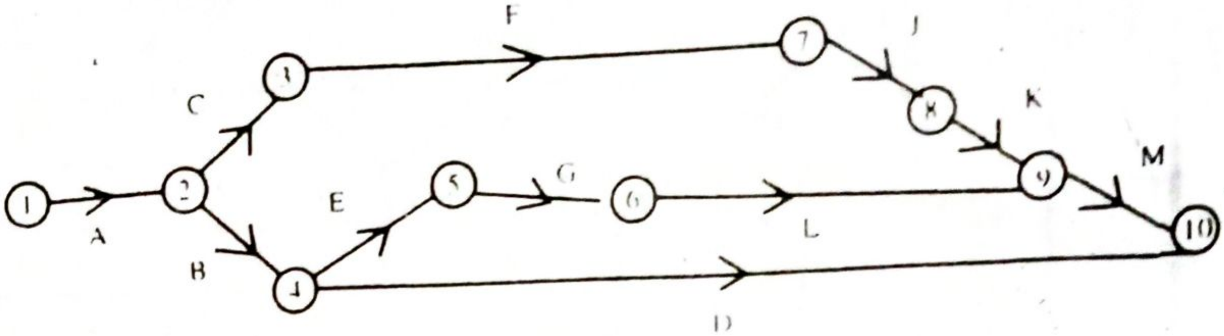
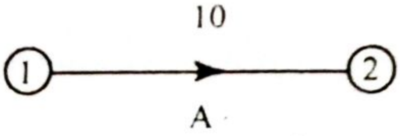


Fig. 18

7.4 Network Computations
(Earliest Completion time of a Project and Critical path)

It is obvious that the completion time of the project is one of the very important things to be calculated knowing the durations of each activity. In real world situation the duration of any activity has an element of uncertainty because of sudden unexpected shortage of labour, machines, materials etc. Hence the completion time of the project also has an element of uncertainty. We first consider the situation where the duration of each activity is deterministic without taking the uncertainty into account.



The above diagram represents an activity whose direction is 10 time units (hours or days or weeks or months etc)

The first net work calculation one does is the computation of earliest start and earliest finish (completion) time of each activity given the duration of each activity. The method used is called **forward pass calculation** and it is best illustrated by means of the following example.

Example 1 : Compute the earliest start, earliest finish latest start and latest finish of each activity of the project given below :

Activity	1–2	1–3	2–4	2–5	3–4	4–5
Duration (in days)	8	4	10	2	5	3

First draw the network.

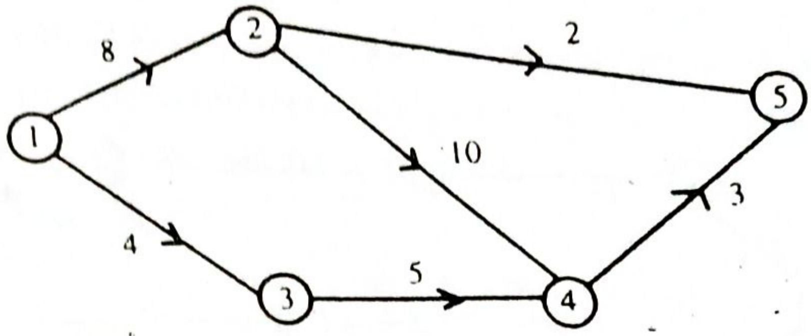


Fig 19

We take the earliest time of all the start activities as zero.

So earliest starts of 1 – 2 and 1 – 3 are zero.

To find earliest start of 2–4.

The activity 2 – 4 can start only after finishing the only preceding activity 1 – 2 i.e., after 8 days.

∴ Earliest start of 2 – 4 is 8 days. Similarly earliest start of 2 – 5 is also 8 days.

Similarly earliest start of 3 – 4 is 4 days

To find the earliest start of 4–5 we first notice that the activity 4 – 5 ***has more than one predecessor*** and also the activity 4 – 5 can start only after finishing all its preceding activities.

There are two paths leading to the activity 4 – 5 : namely 1 – 2 – 4 which takes 18 days and 1 – 3 – 4 which takes 9 days. Obviously after 18 days all the activities. 1 – 2, 1 – 3, 2 – 4, 3 – 4 can be finished but earlier than that.

Earliest start of 4 – 5 is 18 days.

Note: Earliest start of an activity $i - j$ can be denoted as ES_i or ES_i can also be called the ***earliest occurrence of the event i***.

Earliest finish of any activity $i - j$ is got by adding the duration of activity denoted by t_{ij} to the earliest start of $i - j$.

Hence the earliest finish of 1 – 2, 1 – 3, 2 – 4, 2 – 5, 3 – 4, 4 – 5 8,4, 18, 10, 9, 21 respectively

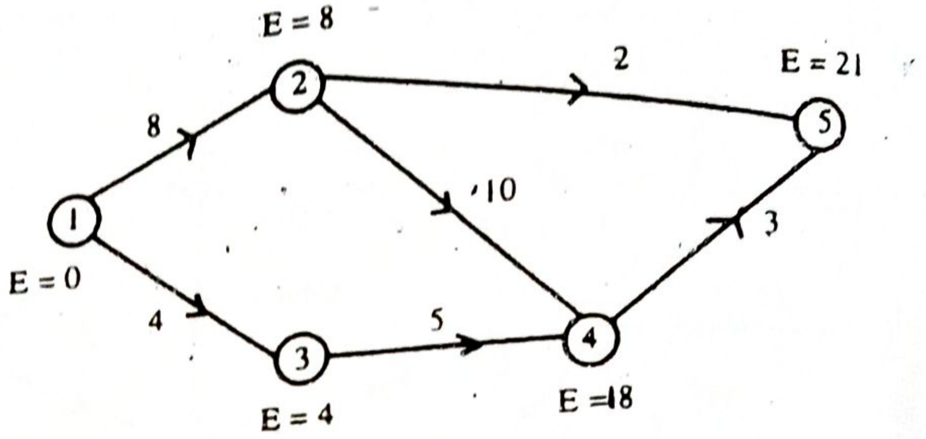


Fig. 20

Obviously ***earliest completion time of the project is 21 days, the greatest number among all these*** since all the activities can be finished only after 21 days.

Formula for Earliest Start of an activity $i - j$ in a project network is given by

$$ES_j = \text{Max}\,[ES_i + t_{ij}] \text{ where}$$

ES_i denotes the earliest start time of all the activities emanating from node i and t_{ij} is the estimated duration of the activity $i - j$.

To compute the latest finish and latest start of each activity

The method used here is called ***backward pass calculation*** since we start with the terminal activity and go back to the very first node.

We first calculate the latest finish of each activity as follows :

Latest finish of all the terminating (end) activities is taken as the earliest completion time of the project. Similarly latest finish of all the vities is obviously taken as the same as the earliest start of these ivities.

the latest finish of the terminal activities 2 –5 and 4 – 5 are 21 ch is the earliest completion time of the project.

st finish of the activity 2 – 4 and 3 – 4 are 21 – 3 = 18 days.

st finish of 1 – 3 is 18 – 5 = 13 days

ind the latest finish of the activity 1 – 2, we observe that the 1 – 2 has ***more than one successor activity***. Therefore the latest f the activity 1 – 2 is the smaller of the two numbers 19 and 18 – 10 = 8, *i.e.*, 8 days.

: Latest finish of an activity can be denoted by LF_j or LF_{ij}. It can called the ***latest occurrence of the event j.***

st start of each activity is the latest finish of that activity minus the of that activity.

latest start of the activities 4 – 5, 2 – 5, 2 – 4, 3 – 4, 1 – 3, e 21, 21, 18, 18, 13, 8 respectively.

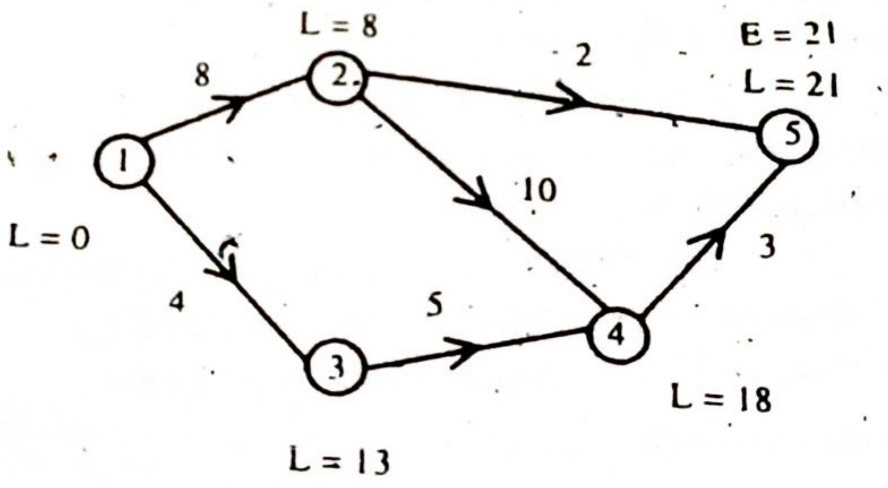


Fig. 21

a for the latest start time of all the activities emanating from of the activity $i - j$, $LS_i = \text{Min}\ [LS_j - t_{ij}]$ for all defined $i - j$ ere t_{ij} is the estimated duration of the activity $i - j$.

tabulate the results and represent these earliest and latest of the events in the network diagram as follows :

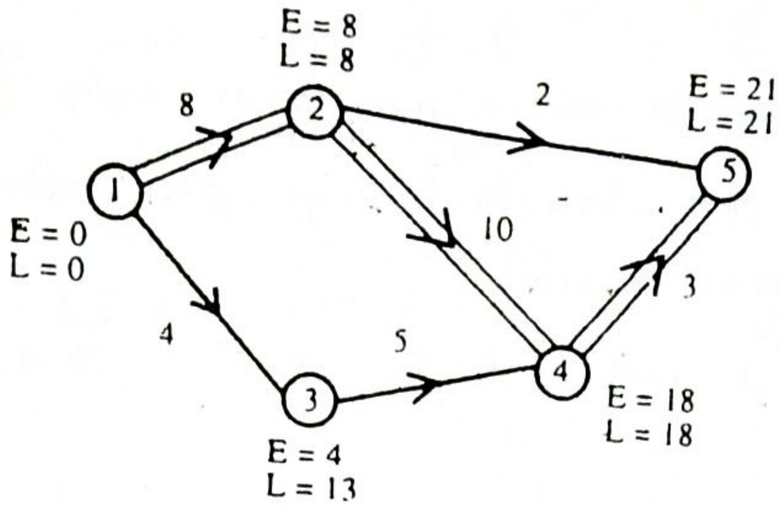


Fig. 22

		Earliest		Latest	
Activity	Duration days	Start ES	Finish EF EF= ES + t_{ij}	Start LS LF − t_{ij}	Finish LF
1 – 2	8	0	8	0	8
1 – 3	4	0	4	9	13
2 – 4	10	8	18	8	18
2 – 5	2	8	10	19	21
3 – 4	5	4	9	13	18
4 – 5	3	**18**	**21**	18	**21**

Note : For small networks, it is not difficult to draw the network with E and L values calculated directly by looking at the diagram itself and constructing the table given above.

* ***Critical path :*** **Path,** connecting the first initial node to the very last terminal node, ***of longest duration*** in any project network is called the ***Critical path.***

All the activities in any critical path are called ***Critical activities.*** Critical path is 1 – 2 – 4 – 5. usually denoted by double lines. (Ref fig. 22)

Critical path plays a very important role in project scheduling problems.

7.5 Floats

Total float of an activity (T.F) is defined as the ***difference*** between the ***latest finish*** and the ***earliest finish of the activity*** or the difference between the ***latest start*** and the ***earliest start*** of the activity.

$$\text{Total float of an activity } i-j = (LF)_{ij} - (EF)_{ij}$$

$$\text{or} = (LS)_{ij} - (ES)_{ij}$$

Total float of an activity is the amount of time by which that particular activity may be delayed without affecting the duration of the project. If the total float is positive then it may indicate that the resources for the activity are more than adequate. If the total float of an activity is zero it may indicate that the resources are just adequate for that activity. If the total float is negative, it may indicate that the resources for that activity are inadequate.

Note : (L – E) of an event is called the ***slack*** of the event j

Slack is defined for an event whereas float is defined for an activity. Some authors use these as the same.

There are three other types of floats for an activity, namely, Free float, Independent float and interference (interfering) float.

Free Float of an activity (F.F.) is that ***portion of the total float*** which can be used for rescheduling that activity without affecting the succeeding activity. It can be calculated as follows :

Free float of an activity $i-j$ = Total float of $i-j$ – (L – E) of the event j

= Total float of $i-j$ – Slack of the head event j

= Total float of I - J – Slack of the head event j

where L = Latest occurrence

E = Earliest occurrence

Obviously Free Float ≤ Total float for any activity.

Independent float (I.F) of an activity is the amount of time by which the activity can be rescheduled without affecting the preceding or succeeding activities of that activity.

Independent float of an activity $i-j$ = Free float of $i-j$ – (L –E) of event i.

= Free float of $i-j$ – Slack of the tail event i.

Clearly,

Independent float ≤ Free float for any activity

Thus I.F ≤ F.F ≤ T.F.

Interfering Float or Interference Float of an activity $i-j$ is nothing the slack of the head event j.

Obviously,

Interfering Float of $i-j$ = Total Float of $i-j$ − Free Float of $i-j$.

Example 3 : **Calculate the total float, free float and independent float for the project whose activities are given below :**

Activity	1 – 2	1 – 3	1 – 5	2 – 3	2 – 4
Duration (weeks)	8	7	12	4	10
Activity	3 – 4	3 – 5	3 – 6	4 – 6	5 – 6
Duration (weeks)	3	5	10	7	4

The data is the same as given in example 2 above.

The network with L and E of every event is given by

Solution :

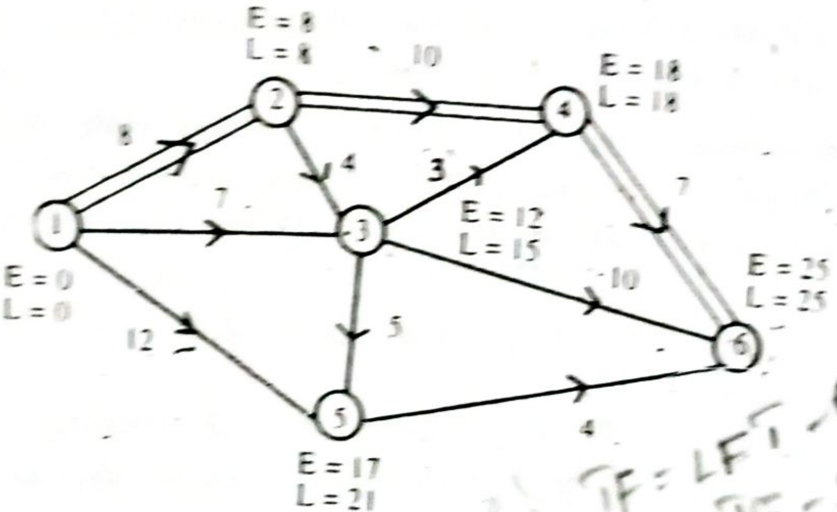


Fig 24

Activity	Duration (in weeks)	Earliest Start	Earliest Finish	Latest Start	Latest Finish	Floats TF	Floats FF	Floats IF
1 – 2	8	0	8	0	8	0	0	0
1 – 3	7	0	7	8	15	8	5	5
1 – 5	12	0	12	9	21	9	5	5
2 – 3	4	8	12	11	15	3	0	0
2 – 4	10	8	18	8	18	0	0	0
3 – 4	3	12	15	15	18	3	3	0
3 – 5	5	12	17	16	21	4	0	3
3 – 6	10	12	22	15	25	3	3	0
4 – 6	7	18	25	18	25	0	0	0
5 – 6	4	17	21	21	25	4	4	0

Explanation : To find the total float of 2 – 3

Total float of (2 – 3) = (LF – EF) of (2 – 3) = 15 – 12 = 3 from the table against the activity 2 – 3.

Free Float of (2 – 3) = Total float of (2 – 3) – (L – E) of event 3

= 3 – (15 – 12) from the figure for event 3 = 0

Free Float of (1 – 5) = Total float of (1 – 5) – (L – E) of event 5

= (21 – 12) – (21 – 17) from the figure for event 5

= 9 – 4 = 5

Independent float of (1 – 5) = Free float of (1 – 5) – (L–E) of event 1

= 5 – (0 – 0) = 5

Important Note : Note that all the critical activities have their total float as zero. ***In fact the critical path can also be defined as the path of least (zero) total float.*** As we have noticed total float is 3 for the activity 2 – 3. This means that the activity 2 – 3 can be delayed by 3 weeks without delaying the duration (completion date) of the project.

Free float of 3 – 4 is 3. This means that the activity 3 – 4 can be delayed by 3 weeks without affecting its succeeding activity 4 – 6.

Independent float of 1 – 5 is 5 means that the activity 1 – 5 can be delayed by 5 weeks without affecting its preceding or succeeding activity. Of course 1 – 5 has no preceding activity.

Uses of floats : Floats are useful in resource levelling and resource allocation problems which will be discussed in the last section of this chapter. Floats give some flexibility in rescheduling some activities so as to smoothen the level of resources or allocate the limited resources as best as possible.

Example 4 : Construct the network for the project whose activities are given below and compute the total, free and independent float of each activity and hence determine the critical path and the project duration.

Activity	**0 – 1**	**1 – 2**	**1 – 3**	**2 – 4**	**2 – 5**
Duration (in weeks)	**3**	**8**	**12**	**6**	**3**
Activity	**3 – 4**	**3 – 6**	**4 – 7**	**5 – 7**	**6 – 7**
Duration (in weeks)	**3**	**8**	**5**	**3**	**8**

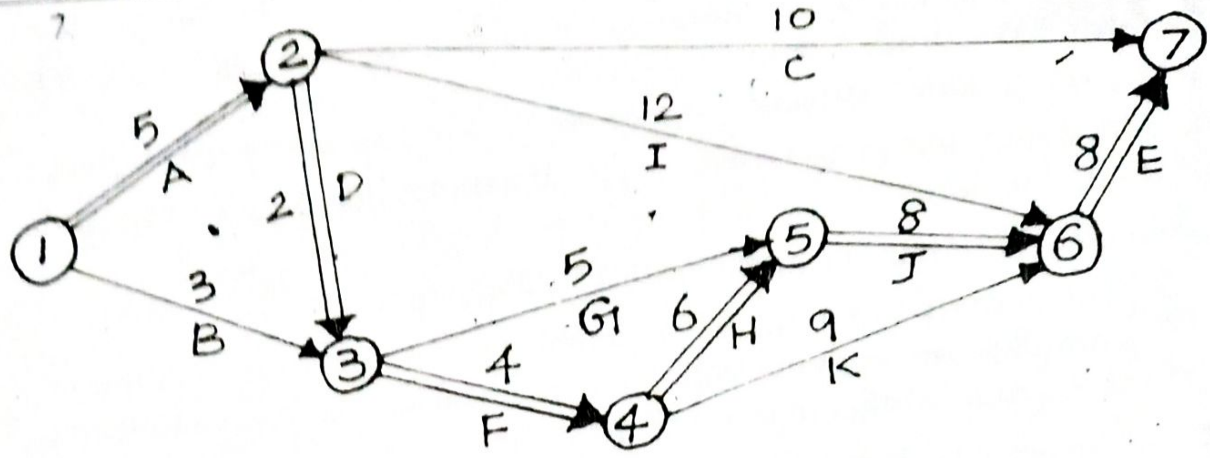


Fig. 33 (a)

Activity		Duration	Earliest Start	Earliest Finish	Latest Start	Latest Finish	TF	FF
A	1 - 2	5	0	5	0	5	0	0
B	1 - 3	3	0	3	4	7	4	4
D	2 - 3	2	5	7	5	7	0	0
I	2 - 6	12	5	17	13	25	8	8
C	2 - 7	10	5	15	23	33	18	18
G	3 - 5	5	7	12	12	17	5	5
F	3 - 4	4	7	11	7	11	0	0
H	4 - 5	6	11	17	11	17	0	0
K	4 - 6	9	11	20	16	25	5	5
J	5 - 6	8	17	25	17	25	0	0
E	6 - 7	8	25	33	25	33	0	0

Critical path : 1 - 2 - 3 - 4 - 5 - 6 - 7

7.6 Programme Evaluation Review Technique : (PERT)

This technique, unlike CPM, takes into account the uncertainty of project durations into account.

PERT calculations depend upon the following three time estimates.

Optimistic (least) time estimate : (t_0 or a) is the duration of any activity when everything goes on very well during the project. i.e., labourers are available and come in time, machines are working properly, money is available whenever needed, there is no scarcity of raw material needed etc.

***Pessimistic (greatest) time estimate* : (t_p or b)** is the duration of any activity when almost every thing goes against our will and a lot of difficulties is faced while doing a project.

***Most likely time estimate* : (t_m or m)** is the duration of any activity when sometimes things go on very well, sometimes things go on very bad while doing the project.

Two main assumptions made in PERT calculations are

(i) The activity durations are independent, i.e., the time required to complete an activity will have no bearing on the completion times of any other activity of the project.

(ii) The activity durations follow β – distribution

β distribution is a probability distribution with density function $k(t - a)^\alpha \ (b - t)^\beta$ with mean $t_e = \frac{1}{3}\left[2t_m + \frac{1}{2}(t_o - t_p)\right]$ and the standard deviation $\sigma_t = \frac{t_p - t_o}{6}$

PERT Procedure

(1) Draw the project net work

(2) Compute the expected duration of each activity $t_e = \frac{t_o + 4t_m + t_p}{6}$

(3) Compute the expected variance $\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$ of each activity.

(4) Compute the earliest start, earliest finish, latest start, latest finish and total float of each activity.

(5) Determine the critical path and identify critical activities.

(6) Compute the expected variance of the Project length (also called the variance of the critical path) σ_c^2 which is the sum of the variances of all the critical activities.

(7) Compute the expected standard deviation of the project length σ_c and calculate the standard normal deviate $\frac{T_S - T_E}{\sigma_c}$ where

T_S = Specified or Scheduled time to complete the project

T_E = Normal expected project duration

σ_c = Expected standard deviation of the project length.

(8) Using (7) one can estimate the probability of com[...] project within a specified time using the normal cu[...] tables.

Note : (2), (3) are valid because of assumption (ii). (because of assumption *(i)*.

7.7 Basic differences between PERT and CPM

PERT

1. PERT was developed in a brand new R and D Project consider and deal with the uncertainties associated w projects. Thus the project duration is regarded as a variable and therefore probabilities are calculated s characterise it.
2. Emphasis is given to important stages of completion rather than the activities required to be performed to particular event or task in the analysis of network. i.e. network is essentially an event – oriented network.
3. PERT is usually used for projects in which time estima uncertain. Example : R & D activities which are usual! repetitive.
4. PERT helps in identifying critical areas in a project s suitable necessary adjustments may be made to me scheduled completion date of the project.

CPM

1. CPM was developed for conventional projects like constr project which consists of well known routine tasks w resource requirement and duration were known with certainty
2. CPM is suited to establish a trade off for optimum balar between schedule time and cost of the project.
3. CPM is used for projects involving well known activitie repetitive in nature.

However the distinction between PERT and CPM is m historical.

Example 1 : Construct the network for the project whose ctivities and the three time estimates of these activities (in weeks) are ven below: Compute

(a) Expected duration of each activity

(b) Expected variance of each activity

(c) Expected variance of the project length

Activity	t_o	t_m	t_p
1 – 2	3	4	5
2 – 3	1	2	3
2 – 4	2	3	4
3 – 5	3	4	5
4 – 5	1	3	5
4 – 6	3	5	7
5 – 7	4	5	6
6 – 7	6	7	8
7 – 8	2	4	6
7 – 9	1	2	3
8 – 10	4	6	8
9 – 10	3	5	7

Solution : *(a) & (b)*

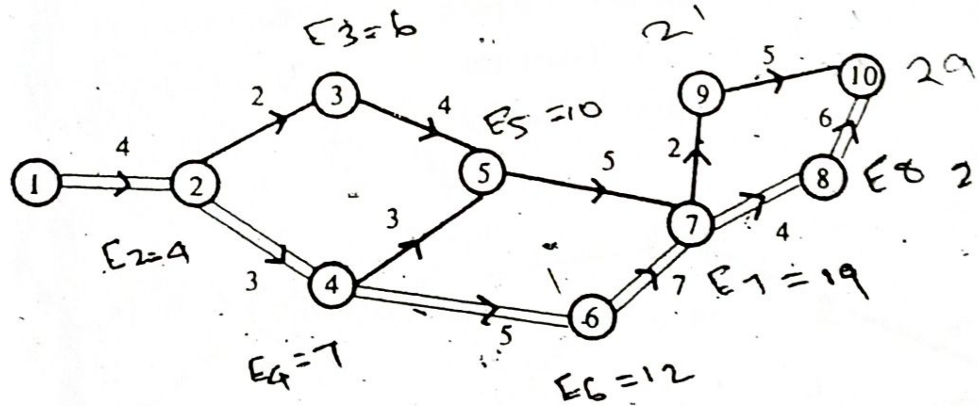


Fig. 34

Activity	t_o	t_m	t_p	Expected duration $t_e = \frac{t_o + 4t_m + t_p}{6}$	Expected Variance $\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$
1 – 2	3	4	5	4	**1/9 = 0.11** nearly
2 – 3	1	2	3	2	1/9 = 0.11
2 – 4	2	3	4	3	**1/9 = 0.11**
3 – 5	3	4	5	4	1/9 = 0.11
4 – 5	1	3	5	3	4/9 = 0.44
4 – 6	3	5	7	5	**4/9 = 0.44**
5 – 7	4	5	6	5	1/9 = 0.11
6 – 7	6	7	8	7	**1/9 = 0.11**
7 – 8	2	4	6	4	**4/9 = 0.44**
7 – 9	1	2	3	2	1/9 = 0.11
8 – 10	4	6	8	6	**4/9 = 0.44**
9 – 10	3	5	7	5	4/9 = 0.44

Critical path 1–2–4–6–7–8–10. Expected Project duration = 29 weeks.

(c) Expected variance of the project length = Sum of the expected variances of all the critical activities

$$= \frac{1}{9} + \frac{1}{9} + \frac{4}{9} + \frac{1}{9} + \frac{4}{9} + \frac{4}{9} = \frac{15}{9} \qquad \frac{15}{9} = \frac{5}{3} = 1.67$$

or $(0.11 + 0.11 + 0.44 + 0.11 + 0.44 + 0.44 = 1.32 + 0.33 = 1.65)$

Example 2 : **The following table indicates the details of a project. The durations are in days. '*a*' refers to optimistic time, '*m*' refers to most likely time and '*b*' refers to pessimistic time duration.**

Activity	1–2	1–3	1–4	2–4	2–5	3–5	4–5
a	2	3	4	8	6	2	2
m	4	4	5	9	8	3	5
b	5	6	6	11	12	4	7

(a) Draw the network

(b) Find the critical path

(c) Determine the expected standard deviation of the completion time.

Solution :

Activity	a	m	b	Expected duration t_e	Expected variance σ^2
1 – 2	2	4	5	3.83	**1/4**
1 – 3	3	4	6	4.17	$\frac{1}{4}$
1 – 4	4	5	6	5	$\frac{1}{9}$
2 – 4	8	9	11	9.17	$\mathbf{\frac{1}{4}}$
2 – 5	6	8	12	8.33	1
3 – 4	2	3	4	3	$\frac{1}{9}$
4 – 5	2	5	7	4.83	$\mathbf{\frac{25}{36}}$

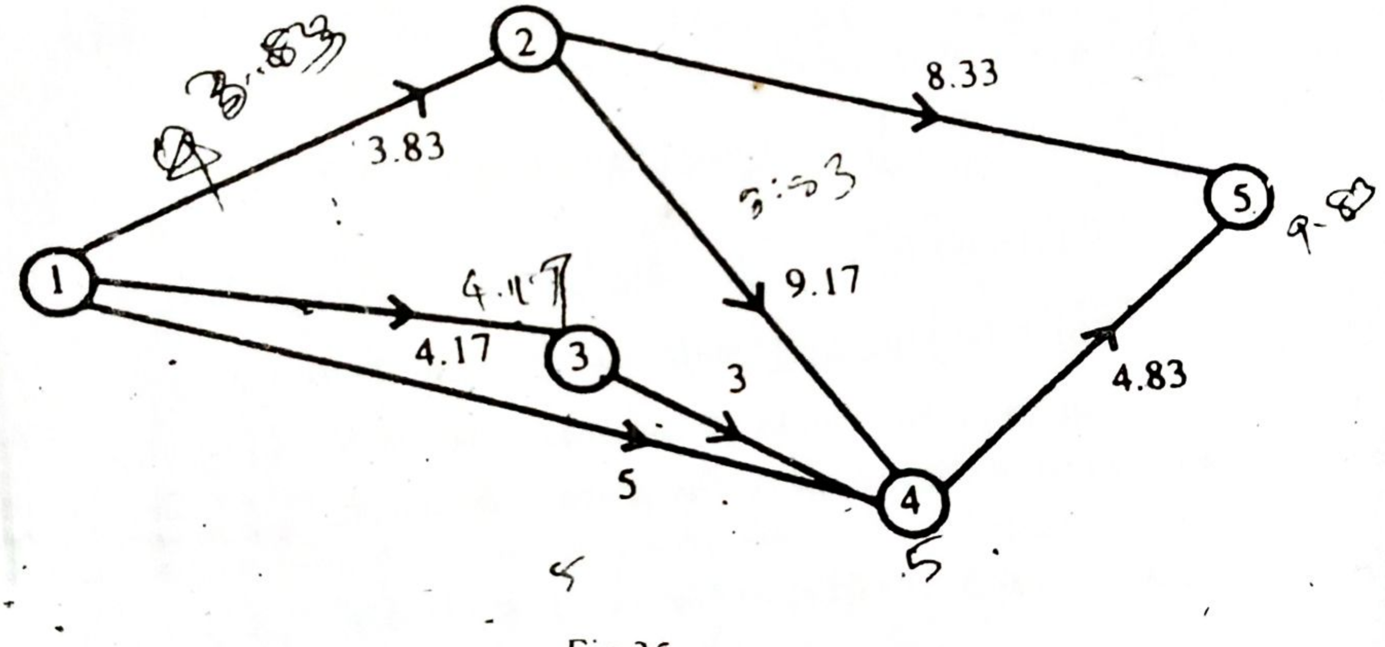


Fig 35

Critical path 1 – 2 – 4 – 5

Expected Project duration = 17.83 days

Expected variance of the completion time $= \frac{1}{4} + \frac{1}{4} + \frac{25}{36} = \frac{43}{36}$

Expected standard deviation of the completion time $= \sqrt{\frac{43}{36}} = 1.09$ nearly