field of real numbers (usual operations). Find a function from V into V wh… a linear transformation on the above vector space, but which is not a linear… formation on C^n, i.e., which is not complex linear.

11. Let V be the space of $n \times 1$ matrices over F and let W be the space of… matrices over F. Let A be a fixed $m \times n$ matrix over F and let T be the… transformation from V into W defined by $T(X) = AX$. Prove that T is the… transformation if and only if A is the zero matrix.

12. Let V be an n-dimensional vector space over the field F and let T be a li… transformation from V into V such that the range and null space of T are iden… Prove that n is even. (Can you give an example of such a linear transformatio…

13. Let V be a vector space and T a linear transformation from V into V. P… that the following two statements about T are equivalent.

(a) The intersection of the range of T and the null space of T is the … subspace of V.

(b) If $T(T\alpha) = 0$, then $T\alpha = 0$.

3.2. The Algebra of Linear Transformations

In the study of linear transformations from V into W, it is of fund… mental importance that the set of these transformations inherits a natu… vector space structure. The set of linear transformations from a space… into itself has even more algebraic structure, because ordinary composi… of functions provides a 'multiplication' of such transformations. We sh… explore these ideas in this section.

Theorem 4. *Let* V *and* W *be vector spaces over the field* F. *Let* T *an…* U *be linear transformations from* V *into* W. *The function* (T + U) *defined…*

$$(T + U)(\alpha) = T\alpha + U\alpha$$

is a linear transformation from V *into* W. *If* c *is any element of* F, *the functi…* (cT) *defined by*

$$(cT)(\alpha) = c(T\alpha)$$

is a linear transformation from V *into* W. *The set of all linear transformatio… from* V *into* W, *together with the addition and scalar multiplication defin… above, is a vector space over the field* F.

Proof. Suppose T and U are linear transformations from V int… W and that we define $(T + U)$ as above. Then

$$\begin{aligned}(T + U)(c\alpha + \beta) &= T(c\alpha + \beta) + U(c\alpha + \beta)\\ &= c(T\alpha) + T\beta + c(U\alpha) + U\beta\\ &= c(T\alpha + U\alpha) + (T\beta + U\beta)\\ &= c(T + U)(\alpha) + (T + U)(\beta)\end{aligned}$$

which shows that $(T + U)$ is a linear transformation. Similarly,

$$\begin{aligned}(cT)(d\alpha + \beta) &= c[T(d\alpha + \beta)]\\ &= c[d(T\alpha) + T\beta]\\ &= cd(T\alpha) + c(T\beta)\\ &= d[c(T\alpha)] + c(T\beta)\\ &= d[(cT)\alpha] + (cT)\beta\end{aligned}$$

which shows that (cT) is a linear transformation.

To verify that the set of linear transformations of V into W (together with these operations) is a vector space, one must directly check each of the conditions on the vector addition and scalar multiplication. We leave the bulk of this to the reader, and content ourselves with this comment: The zero vector in this space will be the zero transformation, which sends every vector of V into the zero vector in W; each of the properties of the two operations follows from the corresponding property of the operations in the space W. ▌

We should perhaps mention another way of looking at this theorem. If one defines sum and scalar multiple as we did above, then the set of *all* functions from V into W becomes a vector space over the field F. This has nothing to do with the fact that V is a vector space, only that V is a non-empty set. When V is a vector space we can define a linear transformation from V into W, and Theorem 4 says that the linear transformations are a subspace of the space of all functions from V into W.

We shall denote the space of linear transformations from V into W by $L(V, W)$. We remind the reader that $L(V, W)$ is defined only when V and W are vector spaces over the same field.

Theorem 5. *Let* V *be an* n*-dimensional vector space over the field* F*, and let* W *be an* m*-dimensional vector space over* F*. Then the space* L(V, W) *is finite-dimensional and has dimension* mn.

Proof. Let

$$\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\} \quad \text{and} \quad \mathcal{B}' = \{\beta_1, \ldots, \beta_m\}$$

be ordered bases for V and W, respectively. For each pair of integers (p, q) with $1 \leq p \leq m$ and $1 \leq q \leq n$, we define a linear transformation $E^{p,q}$ from V into W by

$$\begin{aligned}E^{p,q}(\alpha_i) &= \begin{cases}0, & \text{if } i \neq q\\ \beta_p, & \text{if } i = q\end{cases}\\ &= \delta_{iq}\beta_p.\end{aligned}$$

According to Theorem 1, there is a unique linear transformation from V into W satisfying these conditions. The claim is that the mn transformations $E^{p,q}$ form a basis for $L(V, W)$.

Let T be a linear transformation from V into W. For each j, $1 \leq j \leq n$, let $A_{1j}, \ldots, A_{mj}$ be the coordinates of the vector $T\alpha_j$ in the ordered basis $\mathcal{B}'$, i.e.,

$$T\alpha_j = \sum_{p=1}^{m} A_{pj}\beta_p. \tag{3-1}$$

We wish to show that

$$T = \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq}E^{p,q}. \tag{3-2}$$

Let U be the linear transformation in the right-hand member of (3-2). Then for each j

$$\begin{aligned} U\alpha_j &= \sum_p \sum_q A_{pq}E^{p,q}(\alpha_j) \\ &= \sum_p \sum_q A_{pq}\delta_{jq}\beta_p \\ &= \sum_{p=1}^{m} A_{pj}\beta_p \\ &= T\alpha_j \end{aligned}$$

and consequently $U = T$. Now (3-2) shows that the $E^{p,q}$ span $L(V, W)$; we must prove that they are independent. But this is clear from what we did above; for, if the transformation

$$U = \sum_p \sum_q A_{pq}E^{p,q}$$

is the zero transformation, then $U\alpha_j = 0$ for each j, so

$$\sum_{p=1}^{m} A_{pj}\beta_p = 0$$

and the independence of the β_p implies that $A_{pj} = 0$ for every p and j. ∎

Theorem 6. *Let* V, W, *and* Z *be vector spaces over the field* F. *Let* T *be a linear transformation from* V *into* W *and* U *a linear transformation from* W *into* Z. *Then the composed function* UT *defined by* (UT)(α) = U(T(α)) *is a linear transformation from* V *into* Z.

Proof.

$$\begin{aligned} (UT)(c\alpha + \beta) &= U[T(c\alpha + \beta)] \\ &= U(cT\alpha + T\beta) \\ &= c[U(T\alpha)] + U(T\beta) \\ &= c(UT)(\alpha) + (UT)(\beta). \end{aligned}$$ ∎

In what follows, we shall be primarily concerned with linear transformation of a vector space into itself. Since we would so often have to write 'T is a linear transformation from V into V,' we shall replace this with 'T is a linear operator on V.'

Definition. *If* V *is a vector space over the field* F, *a* **linear operator on** V *is a linear transformation from* V *into* V.

In the case of Theorem 6 when $V = W = Z$, so that U and T are linear operators on the space V, we see that the composition UT is again a linear operator on V. Thus the space $L(V, V)$ has a 'multiplication' defined on it by composition. In this case the operator TU is also defined, and one should note that in general $UT \neq TU$, i.e., $UT - TU \neq 0$. We should take special note of the fact that if T is a linear operator on V then we can compose T with T. We shall use the notation $T^2 = TT$, and in general $T^n = T \cdots T$ (n times) for $n = 1, 2, 3, \ldots$. We define $T^0 = I$ if $T \neq 0$.

Lemma. *Let* V *be a vector space over the field* F; *let* U, T_1 *and* T_2 *be linear operators on* V; *let* c *be an element of* F.

(a) $IU = UI = U$;
(b) $U(T_1 + T_2) = UT_1 + UT_2$; $(T_1 + T_2)U = T_1U + T_2U$;
(c) $c(UT_1) = (cU)T_1 = U(cT_1)$.

Proof. (a) This property of the identity function is obvious. We have stated it here merely for emphasis.

(b)
$$\begin{aligned}[U(T_1 + T_2)](\alpha) &= U[(T_1 + T_2)(\alpha)]\\ &= U(T_1\alpha + T_2\alpha)\\ &= U(T_1\alpha) + U(T_2\alpha)\\ &= (UT_1)(\alpha) + (UT_2)(\alpha)\end{aligned}$$

so that $U(T_1 + T_2) = UT_1 + UT_2$. Also

$$\begin{aligned}[(T_1 + T_2)U](\alpha) &= (T_1 + T_2)(U\alpha)\\ &= T_1(U\alpha) + T_2(U\alpha)\\ &= (T_1U)(\alpha) + (T_2U)(\alpha)\end{aligned}$$

so that $(T_1 + T_2)U = T_1U + T_2U$. (The reader may note that the proofs of these two distributive laws do not use the fact that T_1 and T_2 are linear, and the proof of the second one does not use the fact that U is linear either.)

(c) We leave the proof of part (c) to the reader. ∎

The contents of this lemma and a portion of Theorem 5 tell us that the vector space $L(V, V)$, together with the composition operation, is what is known as a linear algebra with identity. We shall discuss this in Chapter 4.

Example 8. If A is an $m \times n$ matrix with entries in F, we have the linear transformation T defined by $T(X) = AX$, from $F^{n\times 1}$ into $F^{m\times 1}$. If B is a $p \times m$ matrix, we have the linear transformation U from $F^{m\times 1}$ into $F^{p\times 1}$ defined by $U(Y) = BY$. The composition UT is easily described:

$$\begin{aligned}(UT)(X) &= U(T(X))\\ &= U(AX)\\ &= B(AX)\\ &= (BA)X.\end{aligned}$$

Thus UT is 'left multiplication by the product matrix BA.'

$\gamma = 0$, i.e., if the null space of T is $\{0\}$. Evidently, T is 1:1 if and only if T is non-singular. The extension of this remark is that non-singular linear transformations are those which preserve linear independence.

Theorem 8. *Let T be a linear transformation from V into W. Then T is non-singular if and only if T carries each linearly independent subset of V onto a linearly independent subset of W.*

Proof. First suppose that T is non-singular. Let S be a linearly independent subset of V. If $\alpha_1, \ldots, \alpha_k$ are vectors in S, then the vectors $T\alpha_1, \ldots, T\alpha_k$ are linearly independent; for if

$$c_1(T\alpha_1) + \cdots + c_k(T\alpha_k) = 0$$

then

$$T(c_1\alpha_1 + \cdots + c_k\alpha_k) = 0$$

and since T is non-singular

$$c_1\alpha_1 + \cdots + c_k\alpha_k = 0$$

from which it follows that each $c_i = 0$ because S is an independent set. This argument shows that the image of S under T is independent.

Suppose that T carries independent subsets onto independent subsets. Let α be a non-zero vector in V. Then the set S consisting of the one vector α is independent. The image of S is the set consisting of the one vector $T\alpha$, and this set is independent. Therefore $T\alpha \neq 0$, because the set consisting of the zero vector alone is dependent. This shows that the null space of T is the zero subspace, i.e., T is non-singular. ∎

EXAMPLE 11. Let F be a subfield of the complex numbers (or a field of characteristic zero) and let V be the space of polynomial functions over F. Consider the differentiation operator D and the 'multiplication by x' operator T, from Example 9. Since D sends all constants into 0, D is singular; however, V is not finite dimensional, the range of D is all of V, and it is possible to define a right inverse for D. For example, if E is the indefinite integral operator:

$$E(c_0 + c_1x + \cdots + c_nx^n) = c_0x + \frac{1}{2}c_1x^2 + \cdots + \frac{1}{n+1}c_nx^{n+1}$$

then E is a linear operator on V and $DE = I$. On the other hand, $ED \neq I$ because ED sends the constants into 0. The operator T is in what we might call the reverse situation. If $xf(x) = 0$ for all x, then $f = 0$. Thus T is non-singular and it is possible to find a left inverse for T. For example if U is the operation 'remove the constant term and divide by x':

$$U(c_0 + c_1x + \cdots + c_nx^n) = c_1 + c_2x + \cdots + c_nx^{n-1}$$

then U is a linear operator on V and $UT = I$. But $TU \neq I$ since every function in the range of TU is in the range of T, which is the space of polynomial functions f such that $f(0) = 0$.

Example 12. Let F be a field and let T be the linear operator on F^2 defined by

$$T(x_1, x_2) = (x_1 + x_2, x_1).$$

Then T is non-singular, because if $T(x_1, x_2) = 0$ we have

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_1 &= 0 \end{aligned}$$

so that $x_1 = x_2 = 0$. We also see that T is onto; for, let (z_1, z_2) be any vector in F^2. To show that (z_1, z_2) is in the range of T we must find scalars x_1 and x_2 such that

$$\begin{aligned} x_1 + x_2 &= z_1 \\ x_1 &= z_2 \end{aligned}$$

and the obvious solution is $x_1 = z_2$, $x_2 = z_1 - z_2$. This last computation gives us an explicit formula for T^{-1}, namely,

$$T^{-1}(z_1, z_2) = (z_2, z_1 - z_2).$$

We have seen in Example 11 that a linear transformation may be non-singular without being onto and may be onto without being non-singular. The present example illustrates an important case in which that cannot happen.

Theorem 9. *Let V and W be finite-dimensional vector spaces over the field F such that dim V = dim W. If T is a linear transformation from V into W, the following are equivalent:*

(i) T *is invertible.*
(ii) T *is non-singular.*
(iii) T *is onto, that is, the range of* T *is* W.

Proof. Let $n = \dim V = \dim W$. From Theorem 2 we know that

$$\text{rank } (T) + \text{nullity } (T) = n.$$

Now T is non-singular if and only if nullity $(T) = 0$, and (since $n = \dim W$) the range of T is W if and only if rank $(T) = n$. Since the rank plus the nullity is n, the nullity is 0 precisely when the rank is n. Therefore T is non-singular if and only if $T(V) = W$. So, if either condition (ii) or (iii) holds, the other is satisfied as well and T is invertible. ∎

We caution the reader not to apply Theorem 9 except in the presence of finite-dimensionality and with $\dim V = \dim W$. Under the hypotheses of Theorem 9, the conditions (i), (ii), and (iii) are also equivalent to these.

(iv) *If* $\{\alpha_1, \ldots, \alpha_n\}$ *is basis for* V, *then* $\{T\alpha_1, \ldots, T\alpha_n\}$ *is a basis for* W.

(v) *There is some basis* $\{\alpha_1, \ldots, \alpha_n\}$ *for* V *such that* $\{T\alpha_1, \ldots, T\alpha_n\}$ *is a basis for* W.

We shall give a proof of the equivalence of the five conditions which contains a different proof that (i), (ii), and (iii) are equivalent.

(i) → (ii). If T is invertible, T is non-singular. (ii) → (iii). Suppose T is non-singular. Let $\{\alpha_1, \ldots, \alpha_n\}$ be a basis for V. By Theorem 8, $\{T\alpha_1, \ldots, T\alpha_n\}$ is a linearly independent set of vectors in W, and since the dimension of W is also n, this set of vectors is a basis for W. Now let β be any vector in W. There are scalars $c_1, \ldots, c_n$ such that

$$\begin{aligned}\beta &= c_1(T\alpha_1) + \cdots + c_n(T\alpha_n)\\ &= T(c_1\alpha_1 + \cdots + c_n\alpha_n)\end{aligned}$$

which shows that β is in the range of T. (iii) → (iv). We now assume that T is onto. If $\{\alpha_1, \ldots, \alpha_n\}$ is any basis for V, the vectors $T\alpha_1, \ldots, T\alpha_n$ span the range of T, which is all of W by assumption. Since the dimension of W is n, these n vectors must be linearly independent, that is, must comprise a basis for W. (iv) → (v). This requires no comment. (v) → (i). Suppose there is some basis $\{\alpha_1, \ldots, \alpha_n\}$ for V such that $\{T\alpha_1, \ldots, T\alpha_n\}$ is a basis for W. Since the $T\alpha_i$ span W, it is clear that the range of T is all of W. If $\alpha = c_1\alpha_1 + \cdots + c_n\alpha_n$ is in the null space of T, then

$$T(c_1\alpha_1 + \cdots + c_n\alpha_n) = 0$$

or

$$c_1(T\alpha_1) + \cdots + c_n(T\alpha_n) = 0$$

and since the $T\alpha_i$ are independent each $c_i = 0$, and thus $\alpha = 0$. We have shown that the range of T is W, and that T is non-singular, hence T is invertible.

The set of invertible linear operators on a space V, with the operation of composition, provides a nice example of what is known in algebra as a 'group.' Although we shall not have time to discuss groups in any detail, we shall at least give the definition.

Definition. *A* **group** *consists of the following.*

1. *A set* G;
2. *A rule (or operation) which associates with each pair of elements* x, y *in* G *an element* xy *in* G *in such a way that*
   (a) x(yz) = (xy)z, *for all* x, y, *and* z *in* G *(associativity)*;
   (b) *there is an element* e *in* G *such that* ex = xe = x, *for every* x *in* G;
   (c) *to each element* x *in* G *there corresponds an element* x^{-1} *in* G *such that* $xx^{-1} = x^{-1}x = e$.

We have seen that composition $(U, T) \rightarrow UT$ associates with each pair of invertible linear operators on a space V another invertible operator on V. Composition is an associative operation. The identity operator I

satisfies $IT = TI$ for each T, and for an invertible T there is (by Theorem 7) an invertible linear operator T^{-1} such that $TT^{-1} = T^{-1}T = I$. Thus the set of invertible linear operators on V, together with this operation, is a group. The set of invertible $n \times n$ matrices with matrix multiplication as the operation is another example of a group. A group is called **commutative** if it satisfies the condition $xy = yx$ for each x and y. The two examples we gave above are not commutative groups, in general. One often writes the operation in a commutative group as $(x, y) \to x + y$, rather than $(x, y) \to xy$, and then uses the symbol 0 for the 'identity' element e. The set of vectors in a vector space, together with the operation of vector addition, is a commutative group. A field can be described as a set with two operations, called addition and multiplication, which is a commutative group under addition, and in which the non-zero elements form a commutative group under multiplication, with the distributive law $x(y + z) = xy + xz$ holding.

Exercises

1. Let T and U be the linear operators on R^2 defined by

$$T(x_1, x_2) = (x_2, x_1) \quad \text{and} \quad U(x_1, x_2) = (x_1, 0).$$

(a) How would you describe T and U geometrically?

(b) Give rules like the ones defining T and U for each of the transformations $(U + T)$, UT, TU, T^2, U^2.

2. Let T be the (unique) linear operator on C^3 for which

$$T\epsilon_1 = (1, 0, i), \qquad T\epsilon_2 = (0, 1, 1), \qquad T\epsilon_3 = (i, 1, 0).$$

Is T invertible?

3. Let T be the linear operator on R^3 defined by

$$T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3).$$

Is T invertible? If so, find a rule for T^{-1} like the one which defines T.

4. For the linear operator T of Exercise 3, prove that

$$(T^2 - I)(T - 3I) = 0.$$

5. Let $C^{2\times 2}$ be the complex vector space of 2×2 matrices with complex entries. Let

$$B = \begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix}$$

and let T be the linear operator on $C^{2\times 2}$ defined by $T(A) = BA$. What is the rank of T? Can you describe T^2?

6. Let T be a linear transformation from R^3 into R^2, and let U be a linear transformation from R^2 into R^3. Prove that the transformation UT is not invertible. Generalize the theorem.

7. Find two linear operators T and U on R^2 such that $TU = 0$ but $UT \neq 0$.

8. Let V be a vector space over the field F and T a linear operator on V. If $T^2 = 0$, what can you say about the relation of the range of T to the null space of T? Give an example of a linear operator T on R^2 such that $T^2 = 0$ but $T \neq 0$.

9. Let T be a linear operator on the finite-dimensional space V. Suppose there is a linear operator U on V such that $TU = I$. Prove that T is invertible and $U = T^{-1}$. Give an example which shows that this is false when V is not finite-dimensional. (*Hint:* Let $T = D$, the differentiation operator on the space of polynomial functions.)

10. Let A be an $m \times n$ matrix with entries in F and let T be the linear transformation from $F^{n\times 1}$ into $F^{m\times 1}$ defined by $T(X) = AX$. Show that if $m < n$ it may happen that T is onto without being non-singular. Similarly, show that if $m > n$ we may have T non-singular but not onto.

11. Let V be a finite-dimensional vector space and let T be a linear operator on V. Suppose that rank (T^2) = rank (T). Prove that the range and null space of T are disjoint, i.e., have only the zero vector in common.

12. Let p, m, and n be positive integers and F a field. Let V be the space of $m \times n$ matrices over F and W the space of $p \times n$ matrices over F. Let B be a fixed $p \times m$ matrix and let T be the linear transformation from V into W defined by $T(A) = BA$. Prove that T is invertible if and only if $p = m$ and B is an invertible $m \times m$ matrix.

3.3. Isomorphism

If V and W are vector spaces over the field F, any one-one linear transformation T of V onto W is called an **isomorphism of V onto W**. If there exists an isomorphism of V onto W, we say that V is **isomorphic** to W.