BASES AND DIMENSION :

Definition :

Let V be a vector space over F. A subset S of V is said to be linearly dependent (or) simply dependent if there exists distinct vectors $\alpha_1, \alpha_2 \dots \alpha_n$ in S and scalar $c_1, c_2 \dots c_n$ in F not all of which are zero. such that $c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n = 0$.

Definition :

A set which is not linearly dependent is called linearly independent.

Remark :

If the set S contains only finite many vectors $\alpha_1, \alpha_2 \dots \alpha_n$ we some times say $\alpha_1, \alpha_2 \dots \alpha_n$ are dependent (or) independent instead of saying S is dependent (or) independent.

Remark : 1

Any subset of a linearly independent set is linearly independent.

Proof :

Let V be a vector space over a field F.

Let $S = \{\alpha_1, \alpha_2 \dots \alpha_n\}$ be a linearly independent set.

Let S' be a subset of S without loss of generality we take $S' = \{\alpha_1, \alpha_2 \dots \alpha_k\}$ when $k < n$.

Suppose S' is a linearly dependent set. Then there exist $\beta_1, \beta_2 \dots \beta_k$ in F not all zero. such that $\beta_1\alpha_1 + \beta_2\alpha_2 + \dots \beta_k\alpha_k = 0$.

Hence $(\beta_1\alpha_1 + \beta_2\alpha_2 + \dots + \beta_k\alpha_k) + 0 \cdot \alpha_{k+1} + \dots + 0\alpha_n = 0$

is a non trivial linear combination given tha zero vector.

Hence S' is a linearly dependent set which is a contradiction.

Hence S' is linearly independent.

2. Any set which contains a linearly dependent set is linearly dependent.

proof:

Let V be a vector space

Let S be a linearly dependent set.

Let $S' \supset S$ in S' is linearly independent S is also linearly independent by known "Remark ①" which is a contradiction

Hence S' is linearly dependent.

3. Any set which contains the zero vector is linearly dependent.

$\alpha_1 V_1 + \alpha_2 V_2 +$

proof:

$\alpha \cdot 0 \quad \alpha \cdot \alpha_1 + 0 \cdot \alpha_2$

Let $S = \{0, \alpha_1, \dots \alpha_n\}$ clearly $\alpha \cdot 0 + 0 \cdot \alpha_1 + \dots + 0\alpha_n = 0$

where α is any element of F. Hence for any $\alpha \neq 0$

we get a non-trivial linear combination of vector in S given the zero vector.

Hence S is linearly dependent.

A set S of vectors is linearly independent if each finite subsets of S is linearly independent.

i.e for any distrinct vectors $\alpha_1, \alpha_2 \dots \alpha_n$ of S

$c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n = 0 \Rightarrow$ each $c_i = 0$.

Definition:

Let V be a vector space. A basis for V is a linearly independent set of vectors in V. which spans the space V. the space V is finite dimensional if it has a finite basis.

Let V be a vector space and S be a subset of vectors we say that S spans V if any $v \in V$ can be expressed as a linear combination of finite number of element of S.

Example: 1

In F^3 the vectors $\alpha_1 = (1,3,2)$, $\alpha_2 = (1,-7,-8)$ $\alpha_3 = (2,-1,-1)$ are linearly dependent.

Solutions:

$\alpha_1 = (1,3,2)$, $\alpha_2 = (1,-7,-8)$, $\alpha_3 = (2,1,-1)$ using known scalars a_1, a_2, a_3 from a linear combination of the given vectors equal to the zero vectors.

i.e; $a_1(1,3,2) + a_2(1,-7,-8) + a_3(2,1,-1) = (0,0,0)$

$(a_1+a_2+2a_3,\ 3a_1-7a_2+a_3,\ 2a_1-8a_2-a_3) = (0,0,0)$

equating corresponding components equal to each other.

we got the equivalent homogeneous system.

$a_1 + a_2 + 2a_3 = 0 \longrightarrow (1)$

$3a_1 - 7a_2 + a_3 = 0 \longrightarrow (2)$

$2a_1 - 8a_2 - a_3 = 0 \longrightarrow (3)$

$(2) + (3) \Rightarrow 5a_1 - 15a_2 = 0 \longrightarrow (4)$

$① + 2 \times ③ \Rightarrow 5a_1 - 15a_2 = 0 \rightarrow ⑤$

If we get $a_2 = 1$ then $a_1 = 3$ and put in these values in eqn ① we get $a_3 = -2$.

Hence the given system of vectors has a non-zero solution showing that the given system of vectors in linearly dependent.

$2(A+B)=0$.
$A+B=0$
$A=-B$

Example : 1

Let F be a subfield of the complex number in F^3 the vectors $\alpha_1 = (3,0,-3)$ $\alpha_2 = (-1,1,2)$ $\alpha_3 = (4,2,-2)$, $\alpha_4 = (0,1,1)$ are linearly dependent.

Example : 2.

In F^3 the vectors $E_1 = (1,0,0)$ $E_2 = (0,1,0)$ $E_3 = (0,0,1)$ are linearly independent.

Example : 3.

Let F be a field, let $F(n) = \{x_1, x_2 \ldots x_n / x_i \in F\}$

Than $f(n)$ is an n-dimensional vector space over F.

$E_1 = (1,0,0,\ldots 0)$, $E_2 = (0,1,0,\ldots 0)\ldots$

$E_n = (0,0,0 \ldots 1)$ is a standard basis. If $\alpha = (x_1, x_2 \ldots x_n) \in F^n$. The we can write

$\alpha = x_1E_1 + x_2E_2 + \ldots + x_nE_n$.

Theorem : 4

Let V be a vector space which is spanned by a finite set of vectors $\beta_1, \beta_2 \ldots \beta_m$. then every (any) independent set of vectors in V is finite and contains no more than m elements.

Proof:

Let V be a vector space which is spanned by a finite set of vectors $\beta_1, \beta_2 \dots \beta_m$

Let $S = \{\beta_1, \beta_2 \dots \beta_m\}$

To prove that every subset S of V which contains no more than m vectors is linearly independent.

It is enough to prove that every subset $S(V)$ which contain more than m elements linearly dependent.

Let S be such a set. In S there are distinct vectors $\alpha_1, \alpha_2 \dots \alpha_n$ where $n > m$.

$\because \beta_1, \beta_2 \dots \beta_m$ span V. there exist scalar $A_{ij} \in F$ such that $\alpha_j = \sum_{i=1}^{m} A_{ij} \beta_i$. For any n scalars $x_1, x_2 \dots x_n$ we have $x_1\alpha_1 + \dots + x_n\alpha_n = \sum_{j=1}^{n} x_j \alpha_j$

$$= \sum_{j=1}^{n} x_j \sum_{i=1}^{m} A_{ij} \beta_i$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{m} (A_{ij} x_j) \beta_i$$

$$= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} (A_{ij} x_j) \right) \beta_i$$

$\therefore n > m$ by known theorem.

"If A is a $m \times n$ matrix and $m < n$ then there homogeneous system of linear equation $Ax = 0$ has a non-trivial solution.

There exists scalars $x_1, x_2 \dots x_n$ not all zero such that $\sum_{j=1}^{n} (A_{ij} x_j) = 0 \quad 1 \le i \le m$.

Hence $x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n = 0$

$\therefore$ S is a linearly dependent set.

This complete the proof of the theorem.

Corollary: 1

If V is a finite dimensional vectors space then any two basis of V. have the same (finite) numbers of elements.

Proof:

Since V is finite dimensional it has a finite basis $\{\beta_1, \beta_2 \cdots \beta_m\}$

taking $S_1 = \{\beta_1, \beta_2 \cdots \beta_m\}$ as a basis for V and $S_2 = \{\alpha_1, \alpha_2 \cdots \alpha_n\}$ a set of linearly independent elements.

By known theorem "$n \le m$" $\longrightarrow$ ①.

Taking $S_2 = \{\alpha_1, \alpha_2 \cdots \alpha_n\}$ as a basis for V and $S_1 = \{\beta_1, \beta_2 \cdots \beta_m\}$ a set of linearly independent elements. Then by known theorem.

"$m \le n$" $\longrightarrow$ ②

From ① and ② $\Rightarrow m = n$

This complete the proof of the theorem.

Corollary: 2

Let V be a finite dimensional vector space and let $n = \dim(V)$. Then.

a) Any subset of V which ~~contino~~ contains more than n vectors is linearly dependent.

b) No subset of V which contains less than n vectors ~~cannot space~~ can span V.

Proof (a):

Let $S_1 = \{\alpha_1, \alpha_2 \dots \alpha_n\}$ be a basis of V.

Let S_2 be any consists of m vector where $m > n$.

Suppose Now S_2 is linearly independent. By known theorem "$m \leq n$" which is a contradiction.

Hence S_2 is linearly dependent.

Proof (b):

Let S_2 be a set consisting of m vectors where $m < n$

Suppose $S_1 = \{\alpha_1, \alpha_2 \dots \alpha_n\}$ is a basis for V and hence linearly independent.

Hence by known theorem "$n \leq m$"

which is a contradiction

Hence S_2 cannot space V.

This is complete the proof of the corollary.

$\boxed{S_2 \text{ Cannot span } V}$

Lemma:

Let S be a linearly independent subset of a vector space V. Suppose β is a vector in V which is not in the subspace spanned by S. Then the set obtained by adjoining β to S is linearly independent.

Proof: $\alpha_1, \dots \alpha_m$ are distinct vectors in S and that

suppose $c_1\alpha_1 + c_2\alpha_2 + \dots + c_m\alpha_m + b\beta = 0$

$b=0$.

Then b must be zero. otherwise we have

$\beta = (-c_1/b)\alpha_1 + (-c_2/b)\alpha_2 + \cdots (-c_m/b)\alpha_m$

i.e; β is in the subspace spanned by S. which is a contradiction.

Hence put in $b=0$ in equation (1) we get,

$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_m\alpha_m = 0 \rightarrow$ (2).

From eqn (2) we get,

$c_1 = 0, c_2 = 0 \ldots c_m = 0$. because the set, $\{\alpha_1, \alpha_2 \ldots \alpha_m\}$ is linearly independent as it is a subset of a linearly independent set S.

Hence from eqn (1) we have.

$c_1 = 0; c_2 = 0, \ldots c_m = 0, b = 0$.

$\Rightarrow$ The set $\{\alpha_1, \alpha_2 \ldots \alpha_m, \beta\}$ is linearly independent

if S_1 is set obtained by adjoined S to S', then we have prove that every finite subset of S_1 is linearly independent.

Hence it follows that S_1 is linearly independent.

10M 2012

Theorem: 5

If W is a subspace of a finite dimensional vector space V. P.T every linearly independent subset of W is finite and is part of a (finite) basis for W.

Proof:

Suppose S_0 is a linearly independent subset of W.

If S is a linearly independent subset of W containing S_0. then S is also a linearly independent subset of V.

$\therefore$ V is finite dimensional S contains no more than dim V elements.

If S_0 spanns W then S_0 is a basis for W.

S_0 does not span W then by known lemma to find a vector β_2 in W such that $S_1 = S_0 \cup \{\beta_1\}$, $S_2 = S_1 \cup \{\beta_2\}$ is independent.

Containing in this way we get

$S_m = S_0 \cup \{\beta_1 \ldots \beta_m\}$ which is a basis for W.

Thus every linearly independent subset of W is finite and is part of a (finite) basis for W. This complete the proof of the theorem.

Corollary : 1

5M 2010

If W is a proper subset if a finite dimensional vector space V. Then W is finite dimensional and $\dim W \leq \dim V$

Proof:

Given that W is a proper subspace of a finite dimensional vector space V. then

To prove that W is finite dimensional and $\dim W < \dim V$

Suppose W contains a vector $\alpha \neq 0$. By known theorem, "If W is a subspace of a finite dimensional vector space V. every linearly independent subset of W is finite and is part of a (finite) basis for W."

There is a basis of W containing α which contains no more than dimension V. element.

Hence W is finite dimensional.

$\dim W \leq \dim V$.

$\therefore$ W is a proper subspace, There is a vector β in V is not in W. Atjoining β to any basis of W, we get a linearly independent subset of V

Thus $\dim W < \dim V$.

This complete the proof of the corollory.

Corollary: 2

In a finite dimensional vector space V every non-empty ... linearly independent set of vector is part of a basis.

5M-2012 Corollary: 3.

10M 2011.

Let A be an $n \times n$ matrix over a field F and suppose the row vector space V. Then $W_1 + W_2$ is finite dimensional and $\dim W_1 + \dim W_2 = \dim(W_1 \cap W_2) + \dim(W_1 + W_2)$

proof:

Given that W_1 and W_2 are subspace of V. Hence $W_1 \cap W_2$ is subspace of V. By known theorem "if W is a subspace of finite dimensional vector space V, every linearly independent subset of W is finite and is part of a (finite) basis for W"

$W_1 \cap W_2$ has a finite basis $\alpha_1, \alpha_2 \dots \alpha_k$.

which is part of a basis $\{\alpha_1, \alpha_2 \dots \alpha_k, \beta_1 \dots \beta_m\}$ for W_1

And part of a basis $\{\alpha_1, \alpha_2 \dots \alpha_k, \gamma_1, \dots \gamma_n\}$ for W_2.

The subspace $W_1 + W_2$ is spanned by the vectors $\{\alpha_1, \dots \alpha_k \dots, \beta_1 \dots \beta_m, \gamma_1, \dots \gamma_n\}$ is a basis for $W_1 + W_2$.

Suppose $\sum x_i \alpha_i + \sum y_j \beta_j + \sum z_r \gamma_r = 0$.

Then $-\sum z_r \gamma_r = \sum x_i \alpha_i + \sum y_i \beta_i$

Which shows that $\sum z_r \gamma_r \in W_1$ as $\sum z_r \gamma_r \in W_2$.

It follows that $\sum z_r \gamma_r = \sum c_i \alpha_i$

For certain scalars $c_1, c_2 \dots c_k$ because the set $\{\alpha_1, \alpha \dots \alpha_k, \gamma_1 \dots \gamma_n\}$ is independent each of the scalars $z_r = 0$. Thus $\sum x_i \alpha_i + \sum y_j \beta_j = 0$

Since $\{\alpha_1 \dots \alpha_k, \beta_1 \dots \beta_m\}$ is also an independent set, each $x_i = 0$ and each $y_i = 0$.

Thus $\{\alpha_1 \dots \alpha_k, \beta_1 \dots \beta_m, \gamma_1 \dots \gamma_n\}$ is a basis for $W_1 + W_2$.

Hence $\dim(W_1 + W_2) = K + m + n$

Also $\dim W_1 = K + m$, $\dim W_2 = K + n$ and $\dim(W_1 \cap W_2) = K$

$$\dim W_1 + \dim W_2 = (K+m) + (K+n)$$
$$= K + (m + K + n)$$
$$= \dim(W_1 \cap W_2) + \dim(W_1 + W_2).$$