UNIT - I

System of Linear Equations :-

Definitions :-

If F be a field. Consider the n scalars $x_1, x_2, \ldots, x_n$ in F. Which satisfy the equation

$$A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n = y_1$$
$$A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n = y_2$$
$$A_{31}x_1 + A_{32}x_2 + \cdots + A_{3n}x_n = y_3$$
$$- - - - - - - - -$$
$$- - - - - - - - - -$$
$$A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n = y_m$$

where $y_1, y_2, \ldots, y_m$ and A_{ij}, $1 \le i \le m$, $1 \le j \le n$ are given elements in F. There equation are called a system of m linear equation in n unknowns.

Solution of System of equation :

Any n tuple $(x_1, x_2, \ldots, x_n)$ of elements of F. which satisfies each of the equation in,

$$A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n = y_1$$

$$A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n = y_2$$

$$\cdots \quad \cdots \quad \cdots$$

$$\cdots \quad \cdots \quad \cdots$$

$$A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n = y_m$$

is a solution of the system of linear equation.

Definition

The system is said to be homogeneous if all the number $y_1, y_2, \ldots, y_m$ are equal zero.

(ie) $y_1 = y_2 = \cdots = y_m = 0$.

Definition:-

An expression of the form

$$(c_1 A_{11} + c_2 A_{21} + \cdots + c_m A_{m1}) x_1 + \cdots +$$

$$(c_1 A_{1n} + c_2 A_{2n} + \cdots + c_m A_{mn}) x_n =$$

$$c_1 y_1 + \cdots + c_m y_m.$$

Where $c_1, c_2, \ldots, c_m$ are scalars is called a linear combination of the system is a linear combination of the equation in the other systems.

S.N. **Theorem:-**

Equivalent system of linear equations have exactly the same solutions.

Proof:-

Consider the system of m equations in n unknowns.

$$\left.\begin{aligned} &A_{11}x_1+A_{12}x_2+\cdots+A_{1n}x_n=y_1 \\ &A_{21}x_1+A_{22}x_2+\cdots+A_{2n}x_n=y_2 \\ &\text{- - - - - - - - - - - -} \\ &\text{- - - - - - - - - - - -} \\ &A_{m1}x_1+A_{m2}x_2+\cdots+A_{mn}x_n=y_n \end{aligned}\right\} \longrightarrow (1)$$

Let $C_1, C_2, \ldots, C_m$ be m Scalars and multiply the ith equation by C_j and adding we get,

$$C_1(A_{11}x_1+\cdots+A_{1n}x_n)+C_2(A_{21}x_1+\cdots+A_{2n}x_n)+\cdots+C_m(A_{m1}x_1+\cdots+A_{mn}x_n)=C_1y_1+C_2y_2+\cdots+C_ny_n$$

$$\left.\begin{aligned} &(C_1A_{11}+\cdots+C_mA_{m1})x_1+(C_1A_{12}+\cdots+C_mA_{m2})x_2 \\ &+\cdots+(C_1A_{1n}+\cdots+C_mA_{mn})x_n=C_1y_1+\cdots+C_my_n \end{aligned}\right\} (2)$$

This is a lineer Combination of m equation in n unknowns.

Evidently any solution of 1 will be a Solution of 2 also.

If we have another system of equations

$$\left.\begin{array}{l} B_{11} x_1 + \cdots + B_{1n} x_n = z_1 \\ - - - - - - - \\ - - - - - - - \\ B_{k1} x_1 + \cdots + B_{kn} x_n = z_n \end{array}\right\} \text{— } 3$$

in which each of the k equation is a linear Combination of equations in 1. Then every solution A (1) is a Solution of the new system 3.

Since 1 and 3 are equivalent system. If each equation in 1 is a linear Combinations of equations of 3 then any solution of 3 will be a solution of 1 also.

∴ Equivalent system of linear equations have exactly the Same solutions.

This complete the proof of the theorem.

S.N.

Remark:-

If the equation in ① cannot be expressed at a linear combination equation in then some solution of ③ may not be solutions of ①.

Matrices and elementary Row operations:-

Matrix representation of system of linear equation is $AX = Y$

where $A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ - & - & - \\ - & - & - \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}$

$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $Y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$

We call A the matrix of co-efficient of the system.

An $m \times n$ matrix over the field F is a function A from the set of pairs of integers (i, j)

$1 \le i \le m$, $1 \le j \le n$ into the field F

The entries of the matrix A are the scalars $A(i, j) = A_{ij}$

Definition:-

An elementary row operations on an $m \times n$ matrix A over the field F is an operations of one of the following three types.

(i). Multiplication of the one row of A by a non-zero scalar c.

(ii) Replacement of the r^{th} row of the A by row r plus c times row s, c any scalar and $r \neq s$.

(iii) interchange of two rows of A.

Theorem:-2

To each elementary row operations 'e' there corresponds on elementary row operation 'e_1' of the same type as e. such that $e_1(e(A)) = e(e_1(A)) = A$ for each A.

In other words, the inverse expression (or) function of an elementary row operation exists and is an elementary row operation of the same type.

Proof:

(i) Suppose 'e' is the operation which multiplies the rth row of a matrix by the non-zero scalar c.

Let e_1 be the operation which multiplies row r by the c^{-1}.

(ii) Suppose 'e' is the operation which replace row r by row r plus c times row s, $r \neq s$

Let e_1 be the operation which replace row r by r plus $(-c)$ times row s.

(iii) If e interchanges row r and s.

Let $e_1 = e$ in each of these three cases. We clearly have $e_1(e(A)) = e(e_1(A)) = A$ for each A

Hence the proof.

Definition

If A and B are $m \times n$ matrices over the field F we say that B is row equivalent to A. If B can be obtained from A by a finite square of elementary row operation.

Theorem:- 3

If A and B are row equivalent $m \times n$ matrixes. The homogeneous system of linear equations $AX = 0$ and $BX = 0$ have exactly the same solutions.

Proof:-

Suppose we pass from A to B by a finite sequence of elementary row equations

$$A = A_0 \rightarrow A_1 \rightarrow \cdots \rightarrow A_k = B$$

It is enough to prove that the systems $A_j x = 0$ and $A_{j+1} x = 0$ have the same solutions.

(4), one elementary row operation does not disturb the set of solutions.

So suppose that B is obtained from A by a single elementary row equation.

No whether which of the three types the operation is (i), (ii), (iii) each equation in the system $BX=0$ will be a linear combination of the equations in the system $AX=0$.

Since the inverse of an elementary row operation is an elementary row operation each eqn ① in $AX=0$ will also be a linear combination of the eqn in $BX=0$

Hence these two systems are equivalent By know theorem.

Equivalent system of linear equations have exactly the same solutions.

$\therefore$ They have the same solutions

Hence the theorem.

Examples:-

1. Suppose F is the field of rational numbers and $A = \begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & 5 \end{bmatrix}$ we

Shall perform a finite sequence of elementary row operations. A indicating by numbers perenthesis the type of operation performed.

Solutions:-

$$A = \begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & 5 \end{bmatrix} \xrightarrow[(2)]{R_1 \to -2R_2+R_1} \begin{bmatrix} 0 & -9 & 3 & 4 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & 5 \end{bmatrix}$$

$$\xrightarrow{R_b = -2R_2 + R_3} \begin{bmatrix} 0 & -9 & 3 & 4 \\ 1 & 4 & 0 & -1 \\ 0 & -2 & -1 & -1 \end{bmatrix} \xrightarrow{R_3 \to -\frac{R_3}{2}}$$

$$\begin{bmatrix} 0 & -9 & 3 & 4 \\ 1 & 4 & 0 & -1 \\ 0 & 1 & 1/2 & -1/2 \end{bmatrix} \xrightarrow{R_2 = R_2 - 4R_3} \begin{bmatrix} 0 & -9 & 3 & 4 \\ 1 & 0 & -2 & 13 \\ 0 & 1 & 1/2 & -1/2 \end{bmatrix}$$

$$\xrightarrow[(2)]{R_1 = 9R_3 + R_1} \begin{bmatrix} 0 & 0 & 15/2 & -55/2 \\ 1 & 0 & -2 & 13 \\ 0 & 1 & 1/2 & -1/2 \end{bmatrix}$$

$$\xrightarrow[(2)]{R_3 = \frac{-R_1}{3} + R_3} \begin{bmatrix} 0 & 0 & 1 & -11/3 \\ 1 & 0 & 0 & 17/3 \\ 0 & 1 & 0 & -5/3 \end{bmatrix}$$

The row equivalance of A with the final matrix is the above sequence tells us in particular that the solution of

$$2x_1 - x_2 + 3x_2 + 2x_4 = 0$$

$$x_1 + 4x_2 + 0x_2 - x_4 = 0$$

$$2x_1 + 6x_2 - x_3 + 5x_4 = 0$$

and $\quad 0x_1 + 0x_2 + x_3 - 11/3\, x_4 = 0$

$$x_1 + 0x_2 + 0x_3 + \frac{17}{3} x_4 = 0$$

$$0x_1 + x_2 + 0x_3 - 5/3\, x_4 = 0$$

are exactly the same in the second system. It is apparent that if we assign any rational value c to x_4, we obtain a solution, $(-17/3\,c, 5/3\,c, 11/3\,c, c)$ and also that every solution is of this form.

2) Suppose F is the field of complex numbers and $A = \begin{bmatrix} -1 & i \\ -i & 3 \\ 1 & 2 \end{bmatrix}$ in performing two operations if after coverient to combine several operations of type (2)

Solution: -

$$\begin{bmatrix} -1 & i \\ -i & 3 \\ 1 & 2 \end{bmatrix} \xrightarrow[(2)]{} \begin{bmatrix} 0 & 2+i \\ 0 & 3+2i \\ 1 & 2 \end{bmatrix} \xrightarrow[(1)]{}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 3+2i \\ 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Thus the system of equations

$$-x_1 + i x_2 = 0$$

$$-i x_1 + 3 x_2 = 0$$

$$x_1 + 2 x_2 = 0$$ has only the trivial

solution $x_1 = x_2 = 0$.

Definition:-

An $m \times n$ matrix R is called row reduced if,

(i) The first non-zero entry in each non-zero row of R is equal to 1.

(ii) Each column of R which contains the leading non-zero entry of some row all its other entries zero.