Example:-

One example of a row-reduced matrix is the $n \times n$ identity matrix I this is the $n \times n$ matrix defined by

$$I_{ij} = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Theorem:- 4

5 mark

Every $m \times n$ matrix over field F is row equivalent to a row reduced matrix.

Proof:-

Let A be an $m \times n$ matrix over F.

if every entry in the first row of A is zero, then condition (i) is satisfied in so far as row 1 is concerned.

If row 1 has a non-zero entry. Let k be the smallest the integer j for which $A_{ij} \neq 0$.

Multiply row 1 by A_{1k}^{-1}, and the condition (i) is satisfied with regard to row 1.

Now for each $i \geq 2$, add $(-A_{ik})$ times row 1 to row i. Now the leading non zero entry is row 1 and every other entry in column k is 0.

Now consider the matrix which has resulted from above.

If ~~some~~ every entry in row 2 is 0 we do nothing to row 2. If some entries in row 2 different from zero by a scalar so that the zero leading non-zero entry is, 1.

In the event that row 1 and a leading non-zero entry in column k. This leading non-zero entry of row 2 cannot occur in column k say it occurs in column $k_r \neq k$.

By adding suitable multiples of row 2 to the various rows, we an arrange that all entry in Columns k^{-1} are 0, except that in row 2.

In carrying out these least ~~Bx~~ operation we will not change the entries of row 1 in Columns $1 \ldots k$ for will we change any entry of Column.

If row 1 was Identically 0, the operation with row 2 will not affect row 1.

working with one row at a time in the above manner it is Clear that in a finite number of steps we will arrive at a row reduced matrix.

Every $m \times n$ matrix over the field F is row equivalent to a row-reduced matrix.

Hence the theorem.

MATRIX MULTIPLICATION

Definition:-

Let A be an $m\times n$ matrix over the field F and let B be an $n\times p$ matrix over the field F the product AB is the $m\times p$ matrix.

i.e C whose C_{ij} entry is

$$C_{ij} = \sum_{r=1}^{n} A_{ir} B_{rj}.$$

Example:-

(i) If I is $m\times m$ identity matrix and A is an $m\times n$ matrix $IA = A_{m\times n}$.

(ii) If I is the $m\times m$ identity matrix and A is an $m\times n$ matrix $AI = A$.

(iii) If $0^{k,m}$ is the $k\times m$ zero matrix

$0^{k,n} = 0^{k,m}$, lly $A0^{n,p} = 0^{m,p}$.

Theorem:-

If A, B, C are matrices over the field F such that the products BC and $A(BC)$ are defined then so are the product AB, $AB(C)$ and $A(BC) = (AB)C$

Proof :-

Let $A = (a_{ij})$ be an $m \times n$ matrix.

Let $B = (b_{jk})$ be an $n \times p$ matrix and

$C = C_{kl}$ be an $p \times q$ matrix

$\therefore B_{n\times p} = C_{p\times q} = BC_{n\times q}$

Since BC is defined C is a matrix with p rows and BC has n rows because $A(BC)$ is defined where A is an $m \times n$ matrix.

Thus the product AB exist and is $m \times p$ matrix from which it follows then the product $(AB)C = m \times q$ exists.

To prove that $A(BC) = AB(C)$

It is enough to prove that,

$[A(BC)]_{ij} = [(AB)C]_{ij}$ for each i, j

$$[A(BC)]_{ij} = \sum_{r=1}^{n} A_{ir} (BC)_{rj}$$

$$= \sum_{r} A_{ir} \sum_{s} B_{rs} C_{sj}$$

$$= \sum_{r} \sum_{s} A_{ir} B_{rs} C_{sj}$$

$$= \sum_{s} \sum_{r} A_{ir} B_{rs} C_{sj}$$

$$= \sum_{s} \left(\sum_{r} A_{ir} B_{rs} \right) C_{sj}$$

$$= \sum_{s} (AB)_{is} \, C_{sj}$$

$$[A(BC)]_{ij} = [(AB)C]_{ij}$$

Hence the proof.

Definition:

An $m \times n$ matrix is said to be an elementary matrix if it can be obtained from the $m \times m$ identity matrix by means of a single elementary row operations.

Example:-

A 2×2 elementary matrix is necessarily one of the following

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix} \; c \neq 0$$

$$\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix} c \neq 0$$

Corollary:-

Let A and B be $m \times n$ matrices over the field F. iff then B is row-equivalent in A iff $B = PA$. where P is a product of $m \times n$ elementary matrices.

Proof:-

Let A and B be $m \times n$ matrices over the field F. Assume that $B = PA$ whose $P = E_s \cdots E_2 E_1$ and the E_i are $m \times m$ elementary matrix.

Then $E_1 A$ is row - equivalent to A and $E_2(E_1 A)$ is row equivalent to $E_1 A$.

$E_2 E_1 A \sim$ is

Continuing is this way we get $(E_s \cdots E_1)$

A is row - equivalent to A

PA is row - equivalent to A

(i) B is row - equivalent to A. Conversely assume that B is row - equivalent to A.

To prove that $B = PA$.

Row - Reduced Echelon matrices:-

Definition:-

An $m \times n$ matrix R is called a row reduced echelon matrix if

$$\begin{bmatrix} 0 & 1 & -3 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(i) R is row reduced

(ii) every row of R which has all it is entries zero.

Theorem :- 5

Every $m \times n$ matrix A is row - equivalent to a row reduced echelon matrix.

proof :-

W.K.T every matrix A is row equivalent to a row reduced. matrix by theorem (4).

Hence by performing finite numbers of row interchanges on this row reduced matrix being it to row reduced echolon.

Theorem :- 6

If A is an $m \times n$ matrix and $m < n$ there the homogeneous system of this linear equation $AX = 0$ has a non - trivial solution.

proof :-

Let R be a row - reduced echolon matrix which is row equivalent to A.

Then the system $AX = 0$ and $RX = 0$ have the same solution.

If r is the numbers of non - zero

rows in $1 \leq r \leq m$ certainly $r \leq m$ and $r \leq m$, $m < n$, $r < n$, since $m < n$ and from $r < n$.

Let us leading non-zero entry of in the column k_i

The system $Rx = 0$ then consists of r non-trivial equation.

Also the unknown x_{k_i} will occur only in the ith equation.

If we let $u_1 \dots u_{n-r}$ denote the $(n-r)$ unknowns which are different from $x_{k_1}, \dots x_{k_r}$ then the r non-trivial equation in $Rx = 0$ are of the form.

$$\left.\begin{aligned} x_{k_1} + \sum_{j=1}^{n-r} c_{1j} u_j &= 0 \\ x_{k_r} + \sum_{j=1}^{n-r} c_{rj} u_j &= 0 \end{aligned}\right\} \quad (1)$$

All the solution to the system of equations $Rx = 0$ are obtained by assigning any values what so ever to $v_1, \ldots, v_{n-r}$ and then computing the corresponding values of $x_{k_1}, \ldots, x_{k_r}$ for ①.

(ii) $Rx = 0$ has a non-trivial solution

$Ax = 0$ has a non-trivial solution

Hence the theorem.