Theorem :- 14

Let V be a finite-dimensional vector space over the field F, and let $\mathcal{B} = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ and $\mathcal{B}' = \{\alpha'_1, \alpha'_2, \dots, \alpha'_n\}$ be ordered bases for V. Suppose T is a linear operator on V. If $P = [P_1, \dots, P_n]$ is the $n \times n$ matrix with columns $P_j = [\alpha'_j]_{\mathcal{B}}$, then $[T]_{\mathcal{B}'} = P^{-1}[T]_{\mathcal{B}} P$. Alternatively, if U is the invertible operator on V defined by $U\alpha_j = \alpha'_j$, $j = 1, 2, \dots, n$, then $[T]_{\mathcal{B}'} = [U]_{\mathcal{B}}^{-1}[T]_{\mathcal{B}}[U]_{\mathcal{B}}$.

Proof :-

Let $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$, $\mathcal{B}' = \{\alpha'_1, \dots, \alpha'_n\}$ be the ordered bases for V. Then $[\alpha]_{\mathcal{B}} = P[\alpha]_{\mathcal{B}'}$.

$$\alpha'_j = \sum_{i=1}^{n} P_{ij}\alpha_i \longrightarrow (1)$$

$x'_1, x'_2, \dots, x'_n$ are the co-ordinates of α with the standard basis $\mathcal{B}'$.

Then
$$\alpha = x'_1\alpha'_1 + x'_2\alpha'_2 + \dots + x'_n\alpha'_n$$
$$= \sum_{j=1}^{n} x'_j \alpha'_j$$
$$= \sum_{j=1}^{n} x'_j \sum_{j=1}^{n} P_{ij}\alpha_i \quad [\because \text{ by } (1)]$$
$$= \sum^{n} \sum^{n} (P_{ij} x'_j)\alpha_i$$

$$[\alpha]_{B} = PX'$$

Then $[\alpha]_{B} = P[\alpha]_{B'} \rightarrow (2)$

$P = [P_1, P_2, \dots, P_n]$ is the matrix where

$P_j = [\alpha_j']_{B}$.

$$[T\alpha]_{B} = [T]_{B}[\alpha]_{B} \rightarrow (3)$$

Applying (2) to the vector $T\alpha$

$$[T\alpha]_{B} = P[T\alpha]_{B'} \longrightarrow (4)$$

From (3) and (4), we get $[\alpha]_{B} = \begin{bmatrix} P_{11}x_1 & P_{12}x_2 & \dots \\ P_{21}x_1 & \dots \end{bmatrix}$

$$[T]_{B}[\alpha]_{B'} = P[T\alpha]_{B'}$$

$$P^{-1}[T]_{B}[\alpha]_{B'} = P^{-1}P[T\alpha]_{B'}$$

$$P^{-1}[T]_{B}P[\alpha]_{B'} = [T\alpha]_{B'} \quad [\text{using } (2)]$$

$$= [T]_{B'}[\alpha]_{B'} \quad [\text{using } (3)]$$

$$[T]_{B'} = P^{-1}[T]_{B}P.$$

Let U be a unique linear operator which carries B into B'.

$$U\alpha_j = \alpha'_j, \quad j = 1, 2, \dots, n.$$

This operator U is invertible since it carries the basis for V onto a basis of V.

$$P \text{ is defined by } \alpha'_j = \sum_{i=1}^{n} P_{ij}\alpha_i.$$

$$\text{But } \alpha'_j = U\alpha_j = \sum_{i=1}^{n} P_{ij}\alpha_i, \quad j = 1, 2, \dots, n.$$

Matrix of U in the ordered basis $\mathcal{B} = [U]_{\mathcal{B}} = P$.

$$\therefore \; [T]_{\mathcal{B}'} = P^{-1}[T]_{\mathcal{B}}P \text{ becomes}$$

$$[T]_{\mathcal{B}'} = [U]^{-1}_{\mathcal{B}}[T]_{\mathcal{B}}[U]_{\mathcal{B}}$$

This completes the proof of the theorem.

DEFINITION :-

Let A and B be $n \times n$ (square) matrices over the field F. We say that B is similar to A over F, if there is an invertible $n \times n$ matrix P over F, such that $B = P^{-1}AP$.

LINEAR FUNCTIONALS

$f(c(\alpha+\beta)) = cf(\alpha) + f(\beta)$

$f = V \to V'$

DEFINITION :-

V is a vector space over F, A linear transformation f from V into the scalar field F is called a linear functional on V.

is If f is a function from V into F such that $f(c\alpha+\beta) = c f(\alpha) + f(\beta)$, $\forall \alpha, \beta \in V$, $c \in F$.

Ex :-

1. Let F be a field and let $a_1, \dots, a_n$ be scalars in F. Define a function f on F^n by $f(x_1, \dots, x_n) = a_1 x_1 + \dots + a_n x_n$. Then f is a linear functional on F^n.

Proof :- Let $\alpha, \beta \in F^n$.

$$\alpha = (x_1, x_2, \dots, x_n) \; ; \; \beta = (y_1, y_2, \dots, y_n)$$

$$c\alpha = (cx_1, cx_2, \dots, cx_n)$$

$$c\alpha+\beta = (cx_1+y_1, cx_2+y_2, \dots, cx_n+y_n)$$

$$\begin{aligned} f(c\alpha+\beta) &= f(cx_1+y_1, cx_2+y_2, \dots, cx_n+y_n) \\ &= a_1(cx_1+y_1) + a_2(cx_2+y_2) + \dots + a_n(cx_n+y_n) \\ &= c(a_1x_1 + \dots + a_nx_n) + (a_1y_1 + \dots + a_ny_n) \\ &= c\, f(x_1, \dots, x_n) + f(y_1, \dots, y_n) \\ &= c f(\alpha) + f(\beta) \end{aligned}$$

$\therefore f$ is a linear functional on F^n.

2. Let n be a +ve integer and F be a field. If A is an $n \times n$ matrix with entries in F, the trace of A is the scalar $\operatorname{tr} A = A_{11} + A_{22} + \dots + A_{nn}$.

$$tr(cA+B) = \sum_{i=1}^{n}(cA_{ii} + B_{ii})$$

$$= c\sum_{i=1}^{n} A_{ii} + \sum_{i=1}^{n} B_{ii}$$

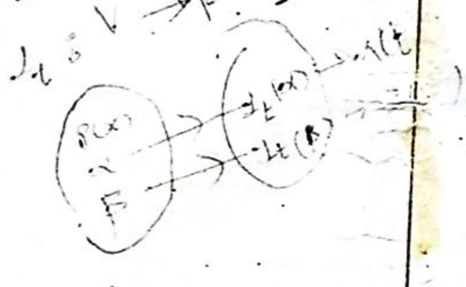
$$= c\,tr A + tr B.$$

$\therefore$ Trace function is a linear functional on $F^{n\times n}$.

3. Let V be the space of all polynomial functions from the field F into itself. Let t be an element of F. Define $L_t(p) = p(t)$, then L_t is a linear functional on V.

Proof :-

Let $\alpha, \beta \in V$, $c \in F$, then $L_t(\alpha) = \alpha(t)$, $L_t(\beta) = \beta(t)$.



$$\therefore L_t(c\alpha+\beta) = (c\alpha+\beta)(t)$$

$$= c\alpha(t) + \beta(t)$$

$$= cL_t(\alpha) + L_t(\beta)$$

$\therefore L_t$ is a linear functional on V.

4. Let $[a, b]$ be a closed interval on the real line and let $C([a,b])$ be the space of continuous real-valued functions on $[a,b]$. Then $L(g) = \int_a^b g(t)dt$ defines a linear functional on $C([a,b])$.

Proof :- Let $f, g \in C([a,b])$, $c' \in F$.

Then $L(c'f+g) = \int_a^b (c'f+g)(t)\,dt$

$= c'\int_a^b f(t)\,dt + \int_a^b g(t)\,dt$

$= c'L(f) + L(g)$

$\therefore L$ is a linear functional on $C([a,b])$.

DEFINITION :-

Let V be a vector space. The collection of all linear functional on V is also a vector space. It is the space $L(V,F)$ and we denote this space by V^* and call it as the dual space of V.

$f : V \to F$

$L(V,F) \Rightarrow V^*$

REMARK :-

Let $\dim V = n$. $\dim F = 1$. $V^* = L(V,F)$.

$\dim V^* = \dim V . \dim F$

$f : V^* \to F$

$= n . 1 = n = \dim V$. $L(V^*, F) \to V^{**}$

$\therefore \dim V^* = \dim V = \dim V^{**}$

DEFINITION :-

Let $\mathcal{B} = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis for V.

According to theorem 1, there is for each i, a unique functional f_i on V such that $f_i(\alpha_j) = \delta_{ij}$.

In this way we obtain from $\mathcal{B}$, a set of n distinct linear function $f_1, f_2, \ldots, f_n$ on V.

$$f_j(\alpha) = \sum_{i=1}^{n} x_i f_j(\alpha_i)$$

$$= \sum_{i=1}^{n} x_i \delta_{ij} = x_j \rightarrow (4)$$

$\therefore$ From (3) and (4),

$$\alpha = \sum_{i=1}^{n} f_i(\alpha)\, \alpha_i$$

$$= \text{l.c of } \alpha_i \quad (i = 1, 2, \ldots, n)$$

Also from (1) and (2),

$$f = \sum_{i=1}^{n} c_i f_i = \sum_{i=1}^{n} f(\alpha_i) f_i .$$

This completes the proof of the theorem.

DEFINITION :-

If V is a vector space over the field F and S is a subset of V, the annihilator of S is the set S^0 of linear functionals f on V such that $f(\alpha) = 0$ for every α in S.

Theorem :- 16

Let V be a finite-dimensional vector space over the field F, and let W be a subspace of V. Then $\dim W + \dim W^0 = \dim V$.

Proof :- Let k be the dimension of W and $\{\alpha_1, \ldots, \alpha_k\}$

Choose vectors $\alpha_{k+1}, \ldots, \alpha_n$ in V such that $\{\alpha_1, \ldots, \alpha_n\}$ is a basis for V.

Let $\{f_1, \ldots, f_n\}$ be the basis for V^* which is dual to this basis for V.

We claim that $\{f_{k+1}, \ldots, f_n\}$ is a basis for the annihilator W^0. Certainly f_i belongs to W^0 for $i \geq k+1$.

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These functions are l.i.

$$\text{Let } f = \sum_{i=1}^{n} c_i f_i \quad \text{--- (1)}$$

$$f(\alpha_j) = \sum_{i=1}^{n} c_i f_i(\alpha_j)$$

$$= \sum_{i=1}^{n} c_i \delta_{ij} = c_j .$$

If $f = 0$ then $f(\alpha_j) = 0$ for each j.

$$\Rightarrow c_j = 0 \text{ for } j = 1, 2, \ldots, n \quad \text{--- (2)}$$

$\therefore$ From (1) and (2), $\{f_1, f_2, \ldots, f_n\} \in V^*$

l.i. since $\dim V^* = n$, the set $\{f_1, f_2, \ldots, f_n\}$ is a spanning set. so $\mathcal{B}^* = \{f_1, f_2, \ldots, f_n\}$ is a basis for V^*. This basis is called the dual basis of $\mathcal{B}$.

Theorem:-15

Let V be a finite-dimensional vector space over the field F, and let $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ be a basis for V. Then there is a unique dual basis $\mathcal{B}^* = \{f_1, \ldots, f_n\}$ for V^* such that $f_i(\alpha_j) = \delta_{ij}$.

For each linear functional f on V we have

$f = \sum_{i=1}^{n} f(\alpha_i) f_i$ and for each vector α in V we

$\alpha = \sum f_i(\alpha)\alpha_i$

$\sum^{n} f_i(\alpha)\alpha$

Proof :- Let $B = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a basis for V.

By theorem ①, for each i there is a unique linear functional f_i on V such that $f_i(\alpha_j) = \delta_{ij}$.

In this way, we obtain from B, a set of n distinct linear functionals $f_1, f_2, \dots, f_n$ on V. These functions are l.i.

Suppose $f = \sum_{i=1}^{n} c_i f_i$ —— ①

$$f(\alpha_j) = \sum_{i=1}^{n} c_i f_i(\alpha_j)$$

$$= \sum_{i=1}^{n} c_i \delta_{ij} = c_j \text{ —— ②}$$

If $f = 0 \Rightarrow f(\alpha_j) = 0$

$\Rightarrow c_j = 0$ for $j = 1, 2, \dots, n$.

$\{f_1, f_2, \dots, f_n\} \in V^*$ are l.i and also $\{f_1, \dots, f_n\}$ is a spanning set of V^*.

dim $V^* = n$. [$\because$ dim V^* = dim $V = n$]

By known theorem, $B^* = \{f_1, f_2, \dots, f_n\}$ form a basis for V^*. Let $\alpha \in V$. Then

(∵ finite dimensional vector space)

$$\alpha = \sum_{i=1}^{n} x_i \alpha_i \text{ —— ③}$$

$(f_i(\alpha) = x_i)$

These functions are l.i.

Let $f = \sum_{i=1}^{n} c_i f_i$ — ①

$$f(\alpha_j) = \sum_{i=1}^{n} c_i f_i(\alpha_j)$$

$$= \sum_{i=1}^{n} c_i \delta_{ij} = c_j.$$

If $f = 0$ then $f(\alpha_j) = 0$ for each j.

$\Rightarrow c_j = 0$ for $j = 1, 2, \ldots, n$ — ②

$\therefore$ From ① and ②, $\{f_1, f_2, \ldots, f_n\} \in V^*$

l.i. since $\dim V^* = n$, the set $\{f_1, f_2, \ldots, f_n\}$ is a spanning set. so $B^* = \{f_1, f_2, \ldots, f_n\}$ is a basis for V^*. This basis is called the dual basis of B.

Theorem:-15

Let V be a finite-dimensional vector space, over the field F, and let $B = \{\alpha_1, \ldots, \alpha_n\}$ be a basis for V. Then there is a unique dual basis $B^* = \{f_1, \ldots, f_n\}$ for V^* such that $f_i(\alpha_j) = \delta_{ij}$. For each linear functional f on V we have $f = \sum_{i=1}^{n} f(\alpha_i) f_i$ and for each vector α in V we have $\alpha = \sum_{i=1}^{n} f_i(\alpha)\, \alpha_i$.

Let $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ ...

By theorem ①, for each i there ... linear functional f_i on V such that $f_i(\alpha_j)$ ...

In this way, we obtain from B, a set of n distinct linear functionals $f_1, f_2, \ldots, f_n$ on V. These functions are l.i.

Suppose $f = \sum_{i=1}^{n} c_i f_i$ —— ①

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$\dim V^* = n$. $[\because \dim V^* = \dim V = n]$

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$$\alpha = \sum_{i=1}^{n} x_i \alpha_i \quad \text{—— ③}$$

$$\therefore f_j(\alpha) = \sum_{i=1} x_i f_j(\alpha_i)$$

$$= \sum_{i=1}^{n} x_i \delta_{ij} = x_j \rightarrow (4)$$

$\therefore$ From (3) and (4),

$$\alpha = \sum_{i=1}^{n} f_i(\alpha)\alpha_i$$

$= l.c$ of α_i $(i = 1, 2, \ldots, n)$

Also from (1) and (2),

$$f = \sum_{i=1}^{n} c_i f_i = \sum_{i=1}^{n} f(\alpha_i) f_i .$$

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Proof :- Let k be the dimension of W and $\{\alpha_1, \ldots, \alpha_k\}$ a basis for W.

choose vectors $\alpha_{k+1}, \ldots, \alpha_n$ in V ...

$\{\alpha_1, \ldots, \alpha_n\}$ is a basis for V.

Let $\{f_1, \ldots, f_n\}$ be the basis for

dual to this basis for V.

We claim that $\{f_{k+1}, \ldots, f_n\}$

for the annihilator W^0. Certainly f_i

for $i \geq k+1$.