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UNIT - IV CHAPTER - 6

ELEMENTARY CANONICAL FORMS

6.1 INTRODUCTION :-

Defn :-

Let V and W be vector space over the field F. A linear transformation from V into W is a function T from V into W such that $T(c\alpha+\beta) = c(T\alpha)+T\beta$ are α, β in the all scalar c.

$T : V_1 \to V_2$

Defn :-

If V is a vector space over a field F. A linear transformation from V into W is a function T from V into W such that $T(c\alpha+\beta) = c(T\alpha)+T\beta$ are α, β in all scalar c.

Defn :-

If V is a vector space over a field F, a linear operator on V is a linear transformation from V into V.

Defn :-

$T : V \to V$

The function T from V into V is called invertible if there exist a function U from V

(2) into V such that UT is identity ~~function~~ on V and TU is identity function on V.

if T is invertible on V then there exist $U(T^{-1})$ such that $TT^{-1} = T^{-1}T = I$

Defn:-

A linear transformation is non-singular if $T\alpha = 0 \Rightarrow \alpha = 0$.

A linear transformation is said to be singular then $\exists$ linear transformation U on V such that $TU = UT = 0$.

Defn:-

The set of all linear transformation from V into W is denoted by $L(V, W)$.

The set of all linear transformation from V into itself is denoted by $L(V, V)$.

Let T be a linear operator on an n dimensional space V. $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ for V we could find an ordered basis. In which where represented by a diagonal matrix D.

(3) Defn:-

Let V be a vector space F and let T be a linear operator of a characteristic value of T is a scalar such that there is a non-zero vector with $T\alpha = c\alpha$. If c is a characteristic of T, then

(a) any α such that $T\alpha = c\alpha$ is characteristic vector of T associated characteristic value c.

(b) the collection of all α such $T\alpha = c\alpha$ is called the characteristic associated with c.

Remark:-

1. If T is any linear operator and c scalar, the set of vectors α such $T\alpha = c\alpha$ is a subspace of V.

Proof:- Let there exist a non-zero $\alpha \in V$ such that $T\alpha = c\alpha$, for some c

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Let $W = \{\alpha \in V : T\alpha = c\alpha\}$.

Consider α, β any two vectors in W, then

$T\alpha = c\alpha$, $T\beta = c\beta$.

For $a, b \in F$ we have

$$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta) \quad (\because \text{linearity of } T)$$

$$= ac\alpha + bc\beta$$

$$= c(a\alpha + b\beta)$$

$\Rightarrow a\alpha + b\beta \in W$

$\therefore W$ is a subspace of V.

Theorem :- (1)

Let T be a linear operator on a finite-dimensional space V and let c be a scalar. The following are equivalent.

(i) c is a characteristic value of T.

(ii) The operation $(T - cI)$ is singular (not invertible)

(iii) $\det(T - cI) = 0$.

Proof :- (i) $\Rightarrow$ (ii) :-

Assume that c is a characteristic value of T. Then by defn there is a non-zero vector $\alpha \neq 0$ such that $T\alpha = \alpha c$.

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ie $T\alpha - c\alpha = 0$

$\alpha(T - cI) = 0$

$(T - cI) = 0 \quad (\because \alpha \neq 0)$

$\Rightarrow (T - cI)$ is singular.

(ii) $\Rightarrow$ (iii) :-

Since $(T - cI)$ is singular. is not invertible

Since $(T - cI)$ is singular, $\exists$ a non-zero vector $\alpha \in V$, $\det(T - cI) = 0$.

(iii) $\Rightarrow$ (i) :-

Assume that $\det(T - cI) = 0$.

$\Rightarrow (T - cI)$ is not invertible

$\Rightarrow (T - cI)$ is singular.

$\Rightarrow (T - cI)(\alpha) = 0$ where α is non-zero vector

$\Rightarrow T\alpha = cI(\alpha) = c\alpha$

$\Rightarrow T\alpha = c\alpha$

$\Rightarrow c$ is a characteristic value of T.

This completes the proof of the theorem.

Defn :- (characteristic value of a matrix)

If A is an $n \times n$ matrix over the field F, a characteristic value of A in F is a

Similar matrix :- Let A and B be $n \times n$ (square) matrices over the field F. We say that B is similar to A over F if there is an invertible $n \times n$ matrix P over F such that $B = P^{-1}AP$.

scalar $c \in F$ such that the matrix $(A - cI)$ is singular. (not invertible) is $\det(A - cI) = 0$.

NOTE :-

(i) c is the characteristic value of A iff $\det(A - cI) = 0$.

(ii) The polynomial $F = \det(xI - A)$. clearly the characteristic value of A in F are just scalar c in F such that $f(c) = 0$. Hence f is called the characteristic polynomial of A.

Lemma :-

Similar matrices have the same characteristic polynomial.

Proof :-

Let A and B be any two similar matrices then for a invertible matrix P, we have $B = P^{-1}AP$.

$$\det(xI - B) = \det(xI - P^{-1}AP)$$

$$= \det\left(P^{-1}(xI - A)P\right)$$

$$= \det P^{-1} \cdot \det(xI - A) \cdot \det P$$

$$= \det(xI - A).$$

This completes the proof of the lemma.

Ex :-

1. Let T be the linear operator on R^2 which is represented in the standard order matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The characteristic polynomial for T (or) f

$$= \det(xI - A)$$

$$= x\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = \begin{vmatrix} x & 1 \\ -1 & x \end{vmatrix}$$

$$= x^2 + 1$$

The characteristic values are i and $-i$.

The matrix A has no characteristic value in R. But has two characteristic values i and $-i$ in C.

$\therefore$ It discussing the characteristic value we must careful the field involve.

2. Find the characteristic vectors T of a real matrix

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}.$$

Solution :-

$$\text{Let } A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$$

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$\therefore$ Thus space of characteristic vectors associated the characteristic value 1 is $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

$\therefore$ vector $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ spans the nullspace of $T - I$.

Thus $T\alpha = 1\alpha \Leftrightarrow \alpha$ is a scalar multiple of α_1.

Given $c_2 = 2$.

$$A - c_2 I = A - 2I$$

$$= \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \\ 2 & 2 & -2 \end{pmatrix}$$

Let $\alpha_2 = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ be the characteristic vector for c_2 then $(A - c_2 I)\alpha_2 = (A - c_2 I)\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$$(A - c_2 I)\alpha_2 = 0$$

$$(A - c_2 I)\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \\ 2 & 2 & -2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 0$$

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$$y_1 + y_2 - y_3 = 0 \quad \text{--- (1)}$$

$$2y_1 + 0y_2 - y_3 = 0 \quad \text{--- (2)}$$

$$2y_1 + 2y_2 - 2y_3 = 0 \quad \text{--- (3)}$$

From this system of linear eqns (1) & (2) are linearly independent.

$$\begin{array}{ccccc} y_1 & y_2 & y_3 & & \\ 1 & 1 & -1 & 1 & 1 \\ 2 & 0 & -1 & 2 & 0 \end{array}$$

$$\frac{y_1}{-1+0} = \frac{y_2}{-2+1} = \frac{y_3}{0-2}$$

$$\Rightarrow \frac{y_1}{-1} = \frac{y_2}{-1} = \frac{y_3}{-2} \Rightarrow \frac{y_1}{1} = \frac{y_2}{1} = \frac{y_3}{2}$$

$\therefore$ Thus space of characteristic vectors associated the characteristic value 2 is $\alpha_2 = ($

$\therefore$ vector $\alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ spans the nullspace of $T - I$.

Thus $T\alpha = 2\alpha \Leftrightarrow \alpha$ is a scalar multiple of α_2.

(12) NOTE :

If there is an ordered basis $\mathcal{B} = \{\alpha_1, \ldots, \alpha_n\}$ for V in which each α_i is a characteristic vector of T, then the matrix of T in the ordered basis $\mathcal{B}$ is diagonal. If $T\alpha_i = c_i\alpha_i$, then

$$[T]_{\mathcal{B}} = \begin{bmatrix} c_1 & 0 & \ldots & 0 \\ 0 & c_2 & \ldots & 0 \\ \vdots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & c_n \end{bmatrix}.$$

(13) Let T be a linear operator on the finite dimensional space V. We say that T is diagonalizable if there is a basis for V each vector of which is a characteristic vector of T.

Lemma :-

Suppose that $T\alpha = c\alpha$. If f is any polynomial, then $f(T)\alpha = f(c)\alpha$.

Proof :-

Given (i) $T\alpha = c\alpha$

(ii) f is any polynomial

To P.T $f(T)\alpha = f(c)\alpha$.

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k$$

Now $T^2\alpha = (TT)\alpha$

$$= T(T\alpha) = T(c\alpha)$$

$$= c(T\alpha) = c(c\alpha)$$

$$= c^2\alpha$$

$T^k\alpha = c^k\alpha$ for every +ve integer k.

$$f(T) = a_0 + a_1 T + a_2 T^2 + \cdots + a_k T^k$$

$$f(T)\alpha = \left(a_0 + a_1 T + a_2 T^2 + \cdots + a_k T^k\right)\alpha$$

$$= (a_0 T^0)\alpha + (a_1 T)\alpha + (a_2 T^2)\alpha + \cdots + (a_k T^k)\alpha$$

$$= a_0(T^0\alpha) + a_1(T^1\alpha) + a_2(T^2\alpha) + \cdots + a_k(T^k\alpha)$$

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$= a_0(c\alpha) + a_1(c^1\alpha) + \cdots + a_k(c^k\alpha)$

$= (a_0 + a_1 c + a_2 c^2 + \cdots + a_k c^k)\alpha$

$f(T)\alpha = f(c)\alpha$

This completes the proof of the lemma.

Lemma :-

Let T be a linear operator on a finite dimensional vector space V. Let $c_1, \ldots, c_k$ be the distinct characteristic values of T. Let W_i be the space of characteristic vectors associated with the characteristic values c_i. If $W = W_1 + \cdots + W_k$ then

$\dim W = \dim W_1 + \cdots + \dim W_k$.

In fact, if $\mathcal{B}_i$ is an ordered basis for W_i then $\mathcal{B} = (\mathcal{B}_1, \ldots, \mathcal{B}_k)$ is an ordered basis for W.

Proof :-

Given

(i) T be a linear operator on a finite dimensional vector space V.

(ii) $c_1, c_2, \ldots, c_k$ distinct characteristic value T.

(iii) W_i space characteristic vectors associated with characteristic value c_i.

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(iv) $W = W_1 + \cdots + W_k$

(v) $\mathcal{B}_i$ is an ordered basis for W_i

To P.T (i) $\dim W = \dim W_1 + \cdots + \dim W_k$

(ii) $\mathcal{B} = (\mathcal{B}_1, \ldots, \mathcal{B}_k)$.

The space $W = W_1 + \cdots + W_k$ is the subspace spanned by all the characteristic vector of

Then $\dim W = \dim W_1 + \cdots + \dim W_k$

Now to prove the characteristic spaces associated with different characteristic values are independence of one another.

Suppose that for each i we have a β_i in W_i and assume that $\beta_1 + \beta_2 + \cdots + \beta_k$

We shall S.T $\beta_i = 0$ for each i.

Let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k$ be polynomial.

$0 = f(T) \cdot 0$

$= f(T)(\beta_1 + \beta_2 + \cdots + \beta_k)$

$= f(T)\beta_1 + f(T)\beta_2 + \cdots + f(T)\beta_k$

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$= f(c_1)\beta_1 + f(c_2)\beta_2 + \quad + f(c_k)\beta_k$ [$\because$ by known lemma]

$\sum_i f(c_i)\beta_i = 0$

choose polynomials $f_1, f_2, \ldots, f_k$ such that $f_i(c_j) = \delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i\neq j. \end{cases}$

Then $0 = f_i(T)\,0$

$= \sum_j \delta_{ij}\beta_j = \delta_{i1}\beta_1 + \delta_{i2}\beta_2 + \cdots + \delta_{i\,i-1}\beta_{i-1} + \delta_{ii}\beta_i + \cdots + \delta_{ik}\beta_k$

$= \cancel{\delta_{i1}\beta_1 + \delta_{i2}\beta_2 + \cdots + \delta_{i\,i-1}\beta_{i-1}} + \delta_{ii}\beta_i + \cancel{\cdots + \delta_{ik}\beta_k}$

$0 = \beta_i$

$\beta_i = 0$, for each i.

Now let $\mathcal{B}_i$ be an ordered basis for W_i and let $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2 + \cdots + \mathcal{B}_k$ then $\mathcal{B}$ spans the subspace $W = W_1 + \cdots + W_k$.

Also $\mathcal{B}$ is a linearly independent sequence of vectors.

Any linear relation between the vectors in $\mathcal{B}$ will have the form $\beta_1 + \cdots + \beta_k = 0$ where β_i is the linear combination of vectors in $\mathcal{B}_i$. We proved $\beta_i = 0$ for each i.

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Since each $\mathcal{B}_i$ is linearly independent that $\mathcal{B}$ is linearly independent.

From the lemma observed that the characteristic vectors are linearly independent.

This completes the proof of the theorem.

Theorem :- (2)

Let T be a linear operator on a finite dimensional space V. Let $c_1, \ldots, c_k$ be the distinct characteristic values of T and let W_i be the null space of $(T - c_iI)$. The following are equivalent

(i) T is diagonalizable.

(ii) The characteristic polynomial for T is

$$f = (x-c_1)^{d_1} \cdots (x-c_k)^{d_k}$$

and $\dim W_i = d_i$, $i = 1, 2, \ldots, k$.

(iii) $\dim W_1 + \cdots + \dim W_k = \dim V$.

Proof :-

Given

(i) T linear operator on finite-dimensional space V.

(ii) $c_1, c_2, \ldots, c_k$ characteristic value of

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$$P^{-1} = \begin{pmatrix} \frac{1}{-1} & \frac{-2}{-1} & \frac{-2}{-1} \\ \frac{1}{-1} & -3/_{-1} & -2/_{-1} \\ -3/_{-1} & 6/_{-1} & 5/_{-1} \end{pmatrix} = \begin{pmatrix} -1 & 2 & 2 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} -1 & 2 & 2 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{pmatrix}\begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}\begin{pmatrix} 3 & 2 & 2 \\ -1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -5-2+6 & 6+8-12 & 6+4-8 \\ -5-3+6 & 6+12-12 & 6+6-8 \\ 15+6-15 & -18-24+30 & -18-12+20 \end{pmatrix}\begin{pmatrix} 3 & 2 & 2 \\ -1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 2 & 2 \\ -2 & 6 & 4 \\ 6 & -12 & -10 \end{pmatrix}\begin{pmatrix} 3 & 2 & 2 \\ -1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -3-2+6 & -2+2+0 & -2+0+2 \\ -6-6+12 & -4+6+0 & -4+0+4 \\ 18+12-30 & 12-12+0 & 12+0-10 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Annihilating

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6.3 ANNIHILATING POLYNOMIALS

The collection of polynomials p, in the sense that $p(T) = 0$ is an ideal in the polynomial algebra $F[x]$.

Defn :- Let T be a linear operator on a finite dimensional vector space V over the field F. The minimal polynomial for T is the (unique) monic generator of the ideal of polynomials over F which annihilate T.

The minimal polynomial p for the operator T is uniquely determined by 3 properties

(i) p is a monic polynomial over the field F.

(ii) $p(T) = 0$

(iii) No polynomial over F which annihilates T has smaller degree than p has.

If A is an $n \times n$ matrix over F, we define the minimal polynomial for A as the unique monic generator of the ideal

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polynomials over F. which annihilate A. the operator T is represented in some ordered basis by the matrix A, then T and A have the same minimal polynomial. if $f(T)$ is represented in the basis by the matrix $f(A)$, so that $f(T)=0 \Leftrightarrow f(A)=0$

Theorem :- (3)

Q. Let T be a linear operator on an n-dimensional vector space V (or) let A be an $n \times n$ matrix. The characteristic and minimal polynomials for T [for A] have the same roots except for multiplicities.

Proof :-

Let $p(x)$ be the minimal polynomial for T. Let c be a scalar.

To p.T $p(c)=0 \Leftrightarrow c$ is a characteristic value of T.

Assume that $p(c)=0$.

To prove c is a characteristic value of T.

If $p(c)=0$, then $p(x)=(x-c)\,q(x)$.

where $q(x)$ is a polynomial.

Since $\deg q(x) < \deg p(x)$ the defn of the minimal polynomial $p(x)$ so that $q(T) \neq 0$.

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choose a vector β such that $q(T)\beta \neq 0$.

Let $\alpha = q(T)\beta$. Then

$$0 = p(T)\beta$$
$$= (T-cI)\,q(T)\beta$$
$$= (T-cI)\alpha$$

$(T-cI)\alpha = 0$

$\therefore$ c is a characteristic value of T.

Assume that c is a characteristic value of T

ie $T\alpha = c\alpha$ with $\alpha \neq 0$.

To p.T $p(c)=0$.

By known lemma,

"Suppose that $T\alpha = c\alpha$. If f is any polynomial then $f(T)\alpha = f(c)\alpha$."

$$p(T)\alpha = p(c)\alpha.$$

Since $p(T)=0$ and $\alpha \neq 0$

$\therefore$ $p(c)=0$.

This completes the proof of the theorem.

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If T is a diagonalizable linear operator then the minimal polynomial for T is a product of distinct linear factors.