Ex :-

4. In Ex. 3 to find the minimal polynomial for the operator in Ex. 3.

$$A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$$

Solution :-

In Ex. 3 we found that T is a diagonalizable with characteristic polynomial $f(x) = (x-1)(x-2)^2$.

By a note a minimal polynomial for T is $p(x) = (x-1)(x-2)$.

To find the minimal polynomial for the operator in Ex. 2.

Solution :-

Ex. 2. we found that T is not diagonalizable with characteristic polynomial.

Since T is not diagonalizable when the minimal polynomial for T is of the form $(x-1)^K (x-2)^l$, $K \geq 1$, $\lambda \geq 1$.

But the minimal polynomial the degree atleast 3.

From $(x-1)(x-2)^2$ we get the characteristic polynomial thus the minimal polynomial is the characteristic polynomial.

Q. 10M

Cayley - Hamilton Theorem :-

Let T be a linear operator on a fi dimensional vector space V. If f is the cha stic polynomial for T then $f(T) = 0$, in oth words, the minimal polynomial divides th characteristic polynomial for T.

Proof :-

Given (i) V is a finite dimension space.

(ii) T is linear operator on

(iii) f characteristic polynomi

To P.T $f(T) = 0$.

Let K be the commutative ring with identity consisting of all polynomials in

is K is a commutative algebra identity over the scalar field.

Choose an ordered basis $\{\alpha_1, \ldots,$ for V.

Let A be the matrix which represent

in the given basis. Then

$$T\alpha_i = \sum_{j=1}^{n} A_{ji}\alpha_j , \quad 1 \le j \le n.$$

This eqn may be written in the equivalent form,

$$\sum_{j=1}^{n} \delta_{ij} T\alpha_i = \sum_{j=1}^{n} A_{ji} I\alpha_j , \quad 1 \le i \le n.$$

$$\sum_{j=1}^{n} (\delta_{ij} T - A_{ji} I)\alpha_j = 0 , \quad 1 \le i \le n$$

$i = j$

Let B denote the element of $K^{n\times n}$ with entries $B_{ij} = \delta_{ij} T - A_{ji} I$.

If $n = 2$,

$$B = \begin{bmatrix} \delta_{11} T - A_{11} I & \delta_{21} T - A_{21} I \\ \delta_{12} T - A_{12} I & \delta_{22} T - A_{22} I \end{bmatrix}$$

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$= \begin{bmatrix} 1.T - A_{11} I & 0.T - A_{21} I \\ 0.T - A_{12} I & 1.T - A_{22} I \end{bmatrix}$$

$$= \begin{bmatrix} T - A_{11} I & -A_{21} I \\ -A_{12} I & T - A_{22} I \end{bmatrix}$$

$$\det B = \begin{bmatrix} T - A_{11} I & -A_{21} I \\ -A_{12} I & T - A_{22} I \end{bmatrix}$$

$$= (T - A_{11} I)(T - A_{22} I) - A_{12} A_{21} I$$

$$= T^2 - A_{11} T - A_{22} T + A_{11} A_{22} I - A_{12} A_{21} I$$

$$= T^2 - (A_{11} + A_{22}) T + (A_{11} A_{22} - A_{12} A_{21}) I$$

$$= f(T)$$

where f is the characteristic polynomial

$$f = x^2 - (\text{trace } A)x + \det A$$

When $n > 2$, it is also clear that

$$\det B = f(T).$$

Since f is the determinant of the matrix $xI - A$ whose entries are the polynomials

$$(xI - A)_{ij} = \delta_{ij} x - A_{ji}.$$

To S.T $f(T) = 0$.

i.e. $f(T)$ be the zero operator.

It is necessary and sufficient that

$(\det B)\alpha_k = 0$ for $k = 1, 2, \ldots, n$. column

By the defn of B, the vectors $\alpha_1, \ldots, \alpha_n$ satisfy the eqns

$$\sum_{j=1}^{n} B_{ij}\alpha_j = 0 , \quad 1 \le i \le n \quad \text{— ①.}$$

When $n = 2$, $\begin{bmatrix} T - A_{11}I & -A_{21}I \\ -A_{12}I & T - A_{22}I \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ — ②

In this case, the classical adjoint, adj B is the matrix

$$\tilde{B} = \begin{bmatrix} T - A_{22}I & A_{21}I \\ A_{12}I & T - A_{11}I \end{bmatrix}$$

and $$\tilde{B}B = \begin{bmatrix} T - A_{22}I & A_{21}I \\ A_{12}I & T - A_{11}I \end{bmatrix} \begin{bmatrix} T - A_{11}I & -A_{21}I \\ -A_{12}I & T - A_{22}I \end{bmatrix}$$

$$= (T - A_{22}I)(T - A_{11}I) - (A_{21}I)(A_{12}I)$$

$$[(T - A_{11}I)(T - A_{22}I) - (-A_{21}I)(-A_{12}I)]$$

$$= \begin{bmatrix} \det B & 0 \\ 0 & \det B \end{bmatrix}$$

$$= \det B \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

$$= B\tilde{B} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

$$= \tilde{B}\left(B \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}\right)$$

$$= \tilde{B} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{by ②}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In the general case, let $\tilde{B} =$ adj

by ① $\sum_{j=1}^{n} \tilde{B}_{ki} B_{ij} \alpha_j = 0$ for each

and summing on i, we have

$$0 = \sum_{i=1}^{n} \sum_{j=1}^{n} \tilde{B}_{ki} B_{ij} \alpha_j$$

$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \tilde{B}_{ki} B_{ij} \right) \alpha_j \quad \text{— ③}$$

Now $$\tilde{B}B = \begin{bmatrix} \det B & 0 \\ 0 & \det B \end{bmatrix}$$

$$= \det B \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \det B \, I$$

so that $\sum_{i=1}^{n} \tilde{B}_{ki} B_{ij} = \delta_{kj} \det B$.

③ ⇒ $$0 = \sum_{j=1}^{n} \delta_{kj} (\det B) \alpha_j.$$

$$0 = (\det B)\alpha_k, \quad 1 \leq k \leq n.$$

$\alpha_k \neq 0$

Hence the minimal polynomial is $p(x)$, $x(x+2)$, $x(x-2)$, ~~$x(x-2$~~), (x^2-4).

The minimal polynomial is not of degree 2 because $A^2 \neq -2A$, $A^2 \neq 2A$, $A^2 \neq 4I$.

$\therefore$ $p(x)$ is the minimal polynomial for A.

In particular 0, 2, −2 are the characteristic values of A.

$\therefore$ One of the factors $x, x-2, x+2$ must be repeated twice in the characteristic polynomial.

Clearly $\text{rank}(A) = 2$. Consequently there is a two dimensional space of characteristic vectors associated with the characteristic value zero.

By a theorem, the characteristic polynomial of $A = x\,p(x)$

$$= x(x^3 - 4x) = x^4 - 4x^2.$$ //

6.6 DIRECT - SUM DECOMPOSITIONS

Defn:-

Let $W_1, \ldots, W_k$ be subspaces of the vector space V. We say that $W_1, \ldots, W_k$ are independent if $\alpha_1 + \cdots + \alpha_k = 0$, $\alpha_i \in W_i$.

$\Rightarrow$ each α_i is 0.

NOTE:-

For $k = 2$, the meaning of independ is $\{0\}$ intersection. if W_1 and W_2 are independ $\Longleftrightarrow$ $W_1 \cap W_2 = \{0\}$.

If $k > 2$, the independence of $W_1, \ , W_k$ much more than $W_1 \cap \ldots \cap W_k = \{0\}$.

Each W_j intersects the sum of the othe subspaces W_i only in the zero vector.

Remark:-

1. If $W_1, \ldots, W_k$ are independent, then $\alpha \in W$ be <u>uniquely</u> <u>expressed</u> as $\alpha = \alpha_1 + \cdots + \alpha_k$, α_i

Proof:-

Given that $W_1, \ldots, W_k$ are independen

Let $W = W_1 + \cdots + W_k$ be the subspace spa by $W_1, \ldots, W_k$.

$$\therefore \alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_k, \quad \alpha_i \in W_i.$$

Also let $\alpha = \beta_1 + \beta_2 + \cdots + \beta_k$, $\beta_i \in W_i$

$$\Rightarrow \alpha_1 + \alpha_2 + \cdots + \alpha_k = \beta_1 + \beta_2 + \cdots + \beta_k$$

$$\Rightarrow (\alpha_1 - \beta_1) + (\alpha_2 - \beta_2) + \cdots + (\alpha_k - \beta_k) = 0, \quad \alpha_i - \beta_i$$

[$\because$ W_i is a subspa

(i.e. $W_1, \dots, W_k$ are independent)

$\Rightarrow \alpha_i = \beta_i$

$\therefore \alpha$ is unique.

Lemma :-

Let V be a finite-dimensional vector space. Let $W_1, \dots, W_k$ be subspaces of V and let $W = W_1 + \dots + W_k$. The following are equivalent.

(a) $W_1, \dots, W_k$ are independent.

(b) For each j, $2 \leq j \leq k$, we have

$$W_j \cap (W_1 + \dots + W_{j-1}) = \{0\}.$$

(c) If $\mathcal{B}_i$ is an ordered basis for W_i, $1 \leq i \leq k$, then the sequence $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_k)$ is an ordered basis for W.

Proof :- (a) $\Rightarrow$ (b) :-

Assume that $W_1, \dots, W_k$ are independent.

Let α be a vector in the intersection

$W_j \cap (W_1 + \dots + W_{j-1})$.

ie $\alpha \in W_j \cap (W_1 + \dots + W_{j-1})$

$\therefore \alpha \in W_j$ and $\alpha \in (W_1 + \dots + W_{j-1})$.

Then there are vectors $\alpha_1, \dots, \alpha_{j-1}$ with $\alpha_i \in W_i$

Since $\alpha_1 + \dots + \alpha_{j-1} + (-\alpha) + 0 + \dots + 0 = 0$.

Since $W_1, W_2, \dots, W_k$ are independent, it must be that $\alpha_1 = \alpha_2 = \dots = \alpha_{j-1} = \alpha = 0$.

$\therefore W_j \cap (W_1 + \dots + W_{j-1}) \ \{0\}$

$\therefore$ (a) $\Rightarrow$ (b).

Now to prove (b) $\Rightarrow$ (a).

Suppose $\alpha_1 + \dots + \alpha_k = 0$, $\alpha_i \in W_i$.

Let j be the largest integer i such that $\alpha_i \neq 0$. Then $\alpha_1 + \dots + \alpha_j = 0$, $\alpha_j \neq 0$

$\Rightarrow \alpha_j = -\alpha_1 - \alpha_2 - \dots - \alpha_{j-1}$ is a non-zero vector in $W_j \cap (W_1 + \dots + W_{j-1})$.

ie $\alpha_j \in W_j$ and $\alpha_j \in (W_1 + \dots + W_{j-1})$

$\therefore \alpha_j = 0$.

which is a contradiction.

Hence each $\alpha_i = 0$

$\therefore W_1, \dots, W_k$ are independent.

(b) $\Rightarrow$ (a).

Now to prove (a) $\Rightarrow$ (c).

Assume that $W_1, \dots, W_k$ are independent.

Let $\mathcal{B}_i$ be a basis for W_i, $1 \le i \le k$, and let $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_k)$.

Any linear relation between the vectors in $\mathcal{B}$ will have the form $\beta_1 + \dots + \beta_k = 0$. where β_i is some linear combination of the vectors in $\mathcal{B}_i$.

Since $W_1, \dots, W_k$ are independent each β_i is 0. since each $\mathcal{B}_i$ is independent.

$\therefore$ $\mathcal{B}$ is a basis for W.

$\therefore$ (a) $\Rightarrow$ (c).

To prove (c) $\Rightarrow$ (a) :-

Let $\mathcal{B}_i$ is an ordered basis for W_i, $1 \le i \le k$ then the sequence $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_k)$ is an ordered basis for W.

Since $W = W_1 + W_2 + \dots + W_k$.

Any $w \in W$ can be written as $w = \beta_1 + \dots + \beta_k$ where $\beta_i \in \mathcal{B}_i$. ($\because$ $\mathcal{B}_i$ is an ordered basis for w_i and $\mathcal{B}$ is a ordered basis for W).

$W = 0 \Rightarrow \beta_1 + \dots + \beta_k = 0$

$\Rightarrow \beta_i = 0, \forall i$

$\Rightarrow w_1, w_2, \dots, w_k$ are independent.

Defn :- If V is a vector space, a projection of V is a linear operator E on V such that E^2

NOTE :- suppose that E is a projection. Let R be the null space range of E and let N be the null space

(i) The vector β is in the range $\overset{E}{R}$ $\Leftrightarrow$ E

Proof :- If $\beta = E\alpha$ then $E\beta = E(E\alpha) = E^2\alpha = E\alpha$

$\Rightarrow E\beta = \beta$.

conversely, if $\beta = E\beta$, then β is in the of E.

(ii) $V = R \oplus N$

(iii) The unique expression for α as of vectors in R and N is $\alpha = E\alpha + (\alpha -$

Defn :- If R and N are subspaces of V

$V = R \oplus N$, there is one and only one operator E which has range R and null

That operator is called the projection on N.

Theorem :- 9.

If $V = W_1 \oplus \dots \oplus W_k$, then the k linear operator $E_1, \dots, E_k$ on V such

Part II :- We assume that the minimal polynomial for T is of the form

$$p(x) = (x-c_1)\dots(x-c_k) \text{ where } c_1,\dots,c_k$$

are distinct elements of F.

T.P.T T is diagonalizable.

Let W be the subspace spanned by all of the characteristic vectors of T

Suppose $W \neq V$.

By known lemma, there is a vector α not in W. and a characteristic value c_j of T such that the vector $\beta = (T - c_j I)\alpha$ —① lies in W.

Since β is in W then,

$$\beta = \beta_1 + \beta_2 + \dots + \beta_k$$

where $T\beta_i = c_i \beta_i$, $1 \le i \le k$.

∴ The vector $h(T)\beta = h(c_1)\beta_1 + \dots + h(c_k)\beta_k$ is in W for every polynomial $h(x)$

Now, $p(x) = (x-c_j)\,q(x)$ for some polynomial $q(x)$ also $q(x) - q(c_j) = (x-c_j)\,h(x)$

$$q(T)\alpha - q(c_j)\alpha = h(T)(T-c_jI)\alpha$$

$$= h(T)\beta$$

But $h(T)\beta$ lies in W.

since $0 = p(T)\alpha$

$$= [(T-c_j)\,q(T)]\alpha$$

$$= (T-c_jI)(q(T)\alpha)$$

The vector $q(T)\alpha$ is in W.

∴ $q(c_j)\alpha$ is in W.

Since α is not in W, we have

$q(c_j) = 0$

That contradicts the fact that p has distinct roots.

6.5 SIMULTANEOUS TRIANGULATION ; SIMULTANEOUS DIAGONALIZATION

Defn :- Let V be a finite dimensional space and let 𝓕 be a family of linear operators on V. The subspace W is invariant under 𝓕 if W is invariant under each operator in 𝓕.

Lemma :- Let 𝓕 be a commuting family of triangulable linear operators on V. Let W be

Proper subspace of V which is invariant

(62) under $\mathcal{F}$. $\exists$ a vector α in V such that

(a) α is not in W

(b) for each T in $\mathcal{F}$, the vector $T\alpha$ is in the subspace spanned by α and W.

Proof:- Given

(i) $\mathcal{F}$ be a commuting family of triangulable linear operators on V.

(ii) Let W be a proper subspace of V, which is invariant under $\mathcal{F}$.

To P.T $\exists$ α in V,

(a) α is not in W

(b) For each T in $\mathcal{F}$, the vector $T\alpha$ is in the subspace spanned by α & W.

Without loss of generality, we assume that $\mathcal{F}$ contains a only a finite number of operators.

Let $\{T_1, T_2, \ldots, T_r\}$ be a maximum linearly independent subset of $\mathcal{F}$.

is a basis for the subspace spanned by $\mathcal{F}$.

If α is a vector such that (b)

(63) for each T_i then (b) will hold for

operator which is a linear combination of $T_1, T_2, \ldots, T_r$.

By the known lemma, we can find vector β_1 not in W and a scalar c_1 such

$(T_1 - c_1 I)\beta_1 \in W$.

Let V_1 be the collection of all vect β in V such that $(T_1 - c_1 I)\beta \in W$. Then V_1 is a subspace of V which is properly larger than W.

Further V_1 is invariant under $\mathcal{F}$.

If T commutes with T_1 then

$$(T_1 - c_1 I)(T\beta) = ((T_1 - c_1 I)T)\beta$$
$$= (T(T_1 - c_1 I))\beta$$
$$= T(T_1 - c_1 I)\beta \quad \text{— ①}$$

If β is in V_1 then $(T_1 - c_1 I)\beta$ is in W since W is invariant under each T in $\mathcal{F}$ then by ① $T(T_1 - c_1 I)\beta$ is in W.

is $T\beta$ is in V_1, $\forall$ $\beta \in V_1$. & all $T \in \mathcal{F}$

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