Considering only P, P, P from the given forces we see that their resultant is zero since they are equally inclined to one another. So the remaining forces are 2P, P as in the second figure. Their resultant is

$$\sqrt{4P^2+P^2-4P^2\cos 120^\circ} = \sqrt{3}P.$$

If the resultant makes an angle θ with the given forces 3P, then

$$\tan\theta = \frac{P\sin 120^\circ}{2P+P\cos 120^\circ} = \frac{1}{\sqrt{3}} \text{ or } \theta = 30^\circ$$

<u>Resultant of several forces acting on a particle</u>

<u>Bookwork</u>

To find the resultant of coplanar forces using their components.

Let us find the resultant of the forces $\bar{F}_1, \bar{F}_2 \cdots \bar{F}_n$. Now the resultant is

$$\bar{F}_1+\bar{F}_2+\cdots+\bar{F}_n = x\bar{i}+y\bar{j}$$

Multiplying scalarly by $\bar{i}$

$$\bar{F}_1\cdot\bar{i}+\bar{F}_2\cdot\bar{i}+\cdots+\bar{F}_n\cdot\bar{i} = x\,\bar{i}\cdot\bar{i} = x$$

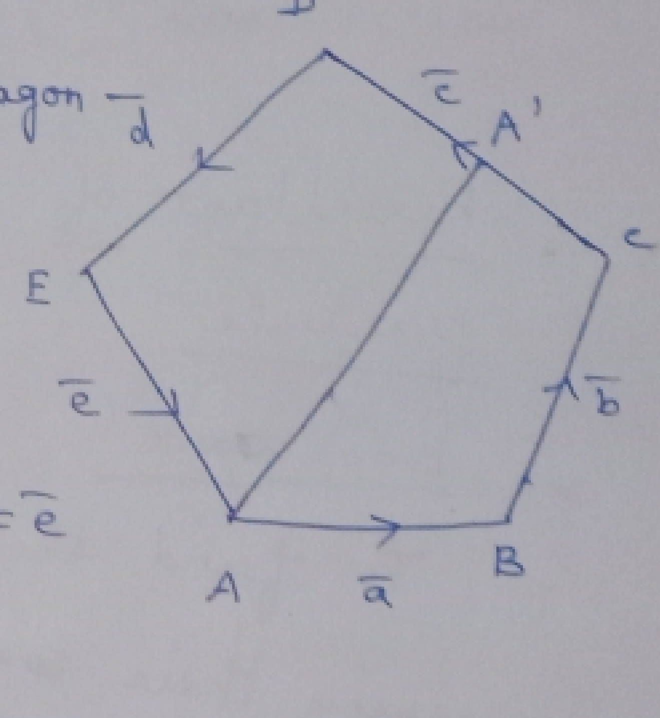
Now $\bar{F}_1\cdot\bar{i}$ is the components of F_1 in the i direction.

Similarly Y is the sum of the components of the forces in the $\bar{j}$ direction. Now the magnitude of the resultant is $|X\bar{i}+Y\bar{j}| = \sqrt{X^2+Y^2}$.

and the angle between the resultant and the $\bar{i}$ direction is $\tan^{-1}\frac{Y}{X}$.

Problem.

Five Forces acting at a point are represented in magnitude and direction by the lines joining the vertices of any pentagon to the midpoints of their opposite sides. Show that they are in equilibrium.



Let ABCDE be the pentagon and A'B', C', D', E' be the midpoints of sides opposite to A, B, C, D, E.

Let $\overline{AB} = \bar{a}$, $\overline{BC} = \bar{b}$, ... $\overline{EA} = \bar{e}$

Then $\bar{a}+\bar{b}+\bar{c}+\bar{d}+\bar{e} = \bar{0}$

Now considering the force $\overline{AA'}$, we have,

$$\overline{AA'} = \overline{AB} + \overline{BC} + \overline{CA} = \bar{a} + \bar{b} + \tfrac{1}{2}\bar{c}$$

$$\Sigma\, \overline{AA'} = \tfrac{5}{2}(\bar{a}+\bar{b}+\bar{c}+\bar{d}+\bar{e}) = \bar{0}$$

Equilibrium of a particle under three forces.

Bookwork

To show that, if three forces keep a particle in equilibrium, then the forces are coplanar.

Let the forces $\bar{F_1}, \bar{F_2}, \bar{F_3}$ keep a particle in equilibrium. Then the resultant forces on the particle is

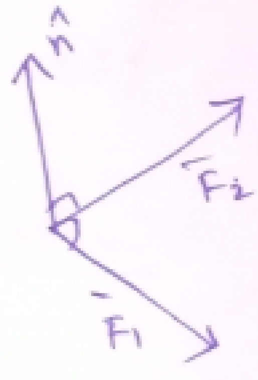
$$\bar{F}_1+\bar{F}_2+\bar{F}_3$$

since the particle is in equilibrium,

$$\bar{F}_1+\bar{F}_2+\bar{F}_3=\bar{0}. \longrightarrow (1)$$

Let $\hat{n}$ be the unit vector perpendicular to both $\bar{F_1}$ and $\bar{F_2}$. Then $\hat{n}\cdot\bar{F}_1=0$, $\hat{n}\cdot\bar{F}_2=0$.

Multiplying (1) scalarily $\hat{n}$ we have,



$$\hat{n}\cdot(\bar{F}_1+\bar{F}_2+\bar{F}_3)=0.$$

or $\hat{n}\cdot\bar{F_1}+\hat{n}\cdot\bar{F_2}+\hat{n}\cdot\bar{F_3}=0$.

$\therefore$ $\bar{F_1}, \bar{F_2}$ and $\bar{F_3}$ are coplanar.

Triangle of forces

If three forces acting on a particle can be represented in magnitude and direction by the sides of a triangle taken in order, then the forces keep the particle in equilibrium.

Let the given forces can be represented in magnitude and direction by the sides

AB, BC, CA of a triangle ABC.

Then the forces are $\overline{AB}$, $\overline{BC}$, $\overline{CA}$.

Their resultant is $\overline{AB} + \overline{BC} + \overline{CA}$. But by vector theory, $\overline{AB} + \overline{BC} + \overline{CA} = 0$.

Hence the resultant forces acting on the particle is zero. So the particle is in equilibrium.

Polygon of forces

It can be easily seen that if several coplanar forces acting on a particle can be represented in magnitude and direction by the sides of a polygon, taken in order, the forces keep the particle in equilibrium, This result is called the polygon of forces.

Lami's theorem:

If a particle is in equilibrium under the action of three forces $\overline{P}, \overline{Q}, \overline{R}$ then to show that $\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma}$.

where α is the angle between $\overline{Q}$ and $\overline{R}$, β is the angle between $\overline{R}$ and $\overline{P}$, and γ is the angle between $\overline{P}$ and $\overline{Q}$ and $|\overline{P}| = P$, etc.

The forces keep the particle in equilibrium. So they are coplanar and their resultant is zero.

Hence $\bar{P} + \bar{Q} + \bar{R} = 0$.

Multiplying this vectorially by $\bar{P}$

$$\bar{P}\times\bar{P} + \bar{P}\times\bar{Q} + \bar{P}\times\bar{R} = 0 \longrightarrow (1)$$

Let $\hat{n}$ be the unit vector perpendicular to the forces such that $\bar{P}, \bar{Q}, \hat{n}$ form a right-handed triad.

Then (1) becomes

$$\bar{0} \times PQ\sin\gamma\,\hat{n} + PR\sin\beta(-\hat{n}) = \bar{0}$$

$$(PQ\sin\gamma - PR\sin\beta)\hat{n} = \bar{0}$$

$$PQ\sin\gamma = PR\sin\beta.$$

$$\frac{Q}{\sin\beta} = \frac{R}{\sin\gamma} \longrightarrow (2)$$

III^{ly}, the vectorial multiplication of $\bar{P} + \bar{Q} + \bar{R} = \bar{0}$ by $\bar{Q}$ gives

$$\frac{P}{\sin\alpha} = \frac{R}{\sin\gamma} \longrightarrow (3)$$

From (2) and (3) we get

$$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma}$$

1. I is the incentre of a triangle ABC. If forces of magnitudes P, Q, R acting along the bisectors IA, IB, IC are in equilibrium. Show that

$$\frac{P}{\cos A/2} = \frac{Q}{\cos\frac{B}{2}} = \frac{R}{\cos\frac{C}{2}}$$

The forces P, Q, R act at I and are in equilibrium. So we shall use Lami's theorem.

The angles opposite to P, Q, R are $\angle BIC$, $\angle CIA$, $\angle AIB$

$$\therefore \frac{P}{\sin BIC} = \frac{Q}{\sin CIA} = \frac{R}{\sin AIB}$$

Now, from ΔBIC, $\angle BIC = 180^\circ - \left(\frac{B}{2} + \frac{C}{2}\right)$

$$\sin BIC = \sin\left(\frac{B}{2} + \frac{C}{2}\right)$$

$$= \sin\left(90 - \frac{A}{2}\right)$$

$$= \cos \frac{A}{2}$$

|||ly $\sin CIA = \cos \frac{B}{2}$, $\sin AIB = \cos \frac{C}{2}$.

Then (1) becomes,

$$\frac{P}{\cos \frac{A}{2}} = \frac{Q}{\cos \frac{B}{2}} = \frac{R}{\cos \frac{C}{2}}$$