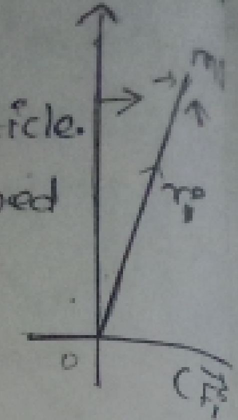
12/11 5H

[D' Alembert's principle :-

Let us consider a system of N particle. Let $\vec{F_i}$ and $\vec{R_i}$ are applied and constrained forces acting on the i^{th} particle of the system.



The equation of the motion of each part be written as

$$m_i \ddot{\vec{r_i}} = \vec{F_i} + \vec{R_i} \rightarrow \text{inertial force.}$$

$$\Rightarrow \vec{F_i} + \vec{R_i} (-m_i \ddot{\vec{r_i}}) = 0 \xrightarrow[\text{asy vector}]{} (1)$$

The forces $-m_i \ddot{\vec{r_i}}$ is called the inertial acting on the i^{th} particle where m_i is constant mass of the i^{th} particle and it's acceleration relative to an inertial. Also $\vec{F_i}, \vec{R_i}$ are called the real (or) actu in contrast to inertial forces.

Hence equation (1) states that the sum the forces real and inertial acting on each of the system is zero. This result is some called. De' Alemberts principles.]

Lagrange's form of De' Alembert principles

[The sum of all the forces acting on each particle be zero, similar to the necessary condition for static equlibrium,

Since the principle of virtual work ap to the system in static equilibrium, let u the principle on this force system incl the inertial forces.

The total work done by all the forces in ar arbitary virtual displacement

$$\sum_{i=1}^{N} (\vec{F_i} + \vec{R_i}) \, \delta\vec{r_i} = 0$$

$$\Rightarrow \sum_{i=1}^{N} (\vec{F_i} + \vec{R_i} - m_i \ddot{\vec{r_i}}) \cdot \delta\vec{r_i} = 0 \longrightarrow (2)$$

$\therefore \delta w = 0$

Assume that $\vec{R_i}$ are workless constraint $\rightarrow \sum_{i=1}^{N} R_i \cdot \delta r_i = 0$.

Now let us assume that $\vec{R_i}$ are motionless constrained force.

$$\therefore \sum_{i=1}^{N} R_i \delta\vec{r_i} = 0$$

$$\therefore \text{eqn}(2) \Rightarrow \sum_{i=1}^{N} (\vec{F_i} - m_i \ddot{\vec{r_i}}) \cdot \delta\vec{r_i} = 0$$

This equation is called Lagrange's form of De' Alembert principle.

Example :- 1

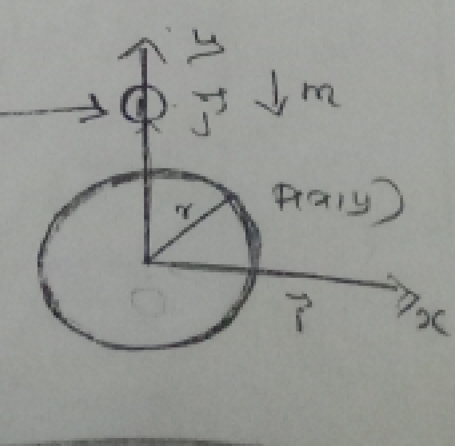
A particle of mass 'm' can slide without friction on a fixed circular wire of radius 'r' which lies in a vertical plane using De' Alemberts principles and the equation of constraint. show that $y\ddot{x} - x\ddot{y} - gx = 0$.

Soln:.

Let 'o' be the centre of the circular wire. Let ox be the horizontal line through 'o' in the plane of wire and oy be the vertical line through 'o'. Let $p(x,y)$ be the position of the particle. Let $\vec{i}$ and $\vec{j}$ are unit vector along the x-axis and y-axis respectively.

Now the applied forces acting on a particle is the gravitational force mg vertically downwards

$$\therefore \vec{F} = -mg\vec{j}$$



since, $\vec{r} = x\vec{i} + y\vec{j}$

$$\Rightarrow \dot{\vec{r}} = \dot{x}\vec{i} + \dot{y}\vec{j}$$

$$\Rightarrow \ddot{\vec{r}} = \ddot{x}\,\vec{i} + \ddot{y}\,\vec{j} \longrightarrow (2)$$

The virtual displacement consistent with the

$$\delta\vec{r} = \delta x\,\vec{i} + \delta y\,\vec{j} \longrightarrow (3)$$

Substituting this in lagrange form of D A principle.

$$\Rightarrow \sum_{i=1}^{N} (\vec{F_i} - m_i \ddot{\vec{r_i}}) \cdot \delta\vec{r_i} = 0$$

$$(\vec{F} - m\ddot{\vec{r}}) \cdot \delta\vec{r} = 0 \longrightarrow (4)$$

$$(-mg\vec{j} - m(\ddot{x}\vec{i} + \ddot{y}\vec{j})) \cdot (\delta x\vec{i} + \delta y\vec{j}) = 0$$

$$-mg\,\delta y - m\ddot{x}\,\delta x - m\ddot{y}\,\delta y = 0$$

Divide by '(-m)'

$$g\,\delta y + \ddot{x}\,\delta x + \ddot{y}\,\delta y = 0 \longrightarrow (5)$$

Since $x^2+y^2=r^2$

$$2x\,\delta x + 2y\,\delta y = 2r\,\delta r$$

$$\Rightarrow 2x\,\delta x + 2y\,\delta y = 0 \Rightarrow 2(x\,\delta x + y\,\delta y) = 0$$

$$\delta x = \frac{-y}{x}\,\delta y \rightarrow \text{sub}$$

$$\Rightarrow g\,\delta y + \ddot{x}(-y/x) \cdot \delta y + \ddot{y}\,\delta y = 0$$

Divide by δy,

$$\Rightarrow g + \ddot{x}(-y/x) + \ddot{y} = 0$$

$$\Rightarrow g - \frac{\ddot{x}y}{x} + \ddot{y} = 0$$

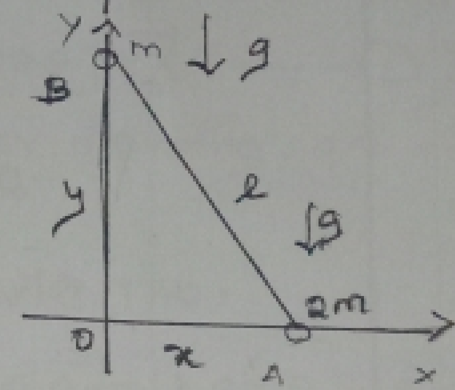
multiply by '$-x$'

$$\Rightarrow -gx + \ddot{x}y - \ddot{y}x = 0$$

$$\Rightarrow \ddot{x}y - \ddot{y}x - gx = 0$$

Hence Proved.

(2) A particle A of mass $2m$ and B of mass m are connected by a massless rigid rod of length 'l'. Particle A is constrained to move along the horizontal x-axis while particle B can move only along the vertical y-axis. What is the equation of the constraint relating x and y using De' Alembert's principle to obtain the equation of motion $2y\ddot{x} - x\ddot{y} - gx = 0$



Solution:

Let $OA = x$ and $OB = y$. From the right angle triangle OAB, $x^2+y^2 = l^2$ which is the required equation of constraint.

Let $\vec{i}$ and $\vec{j}$ are unit vector along the x-axis and y-axis respectively.

$\therefore$ The position vector of $OA = \vec{r_1} = x\vec{i}$ and

The position vector of $OB = \vec{r_2} = y\vec{j}$.

Then the applied forces acting on A and B are

$\vec{F_1} = -2mg\vec{j}$, $\vec{F_2} = -mg\vec{j}$.

Now we consider a virtual displacement of two particles.

$$\delta\vec{r_1} = \delta x\vec{i} \quad \text{and} \quad \delta\vec{r_2} = \delta y\vec{j}$$

$\therefore$ The acceleration of the two particles $\ddot{\vec{r_1}} = \ddot{x}\vec{i}$ and $\ddot{\vec{r_2}} = \ddot{y}\vec{j}$. The Lagrange form of De' Alembert's principle

$$\sum_{i=1}^{N} (\vec{F_i} - m_i\ddot{\vec{r_i}}) \cdot \delta\vec{r_i} = 0$$

$$(\text{i.e}) \sum_{i=1}^{2} (\vec{F_i} - m_i\ddot{\vec{r_i}}) \cdot \delta\vec{r_i} = 0$$

$$\Rightarrow (\vec{F_1} - m_1\ddot{\vec{r_1}}) \cdot \delta\vec{r_1} + (\vec{F_2} - m_2\ddot{\vec{r_2}}) \cdot \delta\vec{r_2} = 0$$

$\Rightarrow (-2mg\vec{j} - 2m\ddot{x}\vec{i})\,\delta\vec{x_i} + (-mg\vec{j} - m\ddot{y}\vec{j})\cdot\delta y$

$-2m(g\vec{j} + \ddot{x}\vec{i})\cdot\delta\vec{x_i} - m(g\vec{j} + \ddot{y}\vec{j})\cdot\delta y\,\vec{j} = 0$

$-2m\ddot{x}\delta x - mg\,\delta y - m\ddot{y}\,\delta y = 0$

$\sin 3x = \frac{\sqrt{2}}{2}$

since,

$x^2 + y^2 = l^2$

$2x\,\delta x + 2y\,\delta y = 0 \Rightarrow \delta x = \frac{-y}{x}\delta y$

$-2m\ddot{x}(-y/x)\,\delta y - mg\,\delta y - m\ddot{y}\delta y = 0$

Divide by δy,

$-2m\ddot{x}(-y/x) - mg - m\ddot{y} = 0$

$2m\ddot{x}(y/x) - mg - m\ddot{y} = 0$

$\Rightarrow m(2\ddot{x}(y/x) - g - \ddot{y}) = 0$

$+2\ddot{x}(y/x) - g - \ddot{y} = 0$

multiply by 'x'

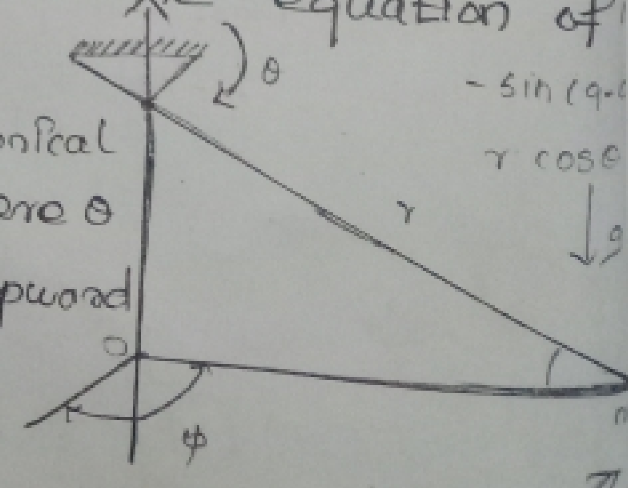
$2\ddot{x}y - gx - \ddot{y}x = 0$

$\Rightarrow 2y\ddot{x} - x\ddot{y} - gx = 0.$

Hence Proved.

V.I (3) 10M. & 5M A particle of mass 'm' is suspended by a wire of length $r = a + b\cos\omega t$, $a > b > 0$ to f spherical pendulum. Find the equation of

Soln:



Let us use the spherical co-ordinates θ and ϕ where θ is measured from the upward vertical as shown in the figure.

The angle ϕ is measured between a vertical reference plane passing through the

$e_\theta \times e_r = 0$

$e_\phi \times e_r = 0$

Support point 'o' and the vertical plane containing the pendulum.

The general expression for the acceleration of a particle whose spherical co-ordinates (r, θ, ϕ) is as follows.

$$\ddot{\vec{r}} = [\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2 \sin^2\theta]\,\vec{e_r} + [r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2 \sin\theta\cos\theta]\,\vec{e_\theta}$$
$$+ [r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta]\,\vec{e_\phi} \longrightarrow (1)$$

where $\vec{e_r}$, $\vec{e_\theta}$ and $\vec{e_\phi}$ are unit vector forming an orthogonal traid.

A virtual displacement consist with the instantaneous constraints is $\delta\vec{r} = r\delta\theta\,\vec{e_\theta} + r\sin\theta\,\delta\phi\,\vec{e_\phi}$

Furthermore, the applied gravitational force

$$\vec{F} = -mg\cos\theta\,\vec{e_r} + mg\sin\theta\,\vec{e_\theta}$$

Now the Lagrange form of De'Alembert's principle

$$\left[\sum_{i=1}^{n} (F_i - m_i\ddot{\vec{r_i}}) \cdot \delta\vec{r_i} = 0\right.$$

$$\Rightarrow (F - m\ddot{\vec{r}}) \cdot \delta\vec{r} = 0$$

$$\Rightarrow -mg\cos\theta\,\vec{e_r} + mg\sin\theta\,\vec{e_\theta} - m[(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta)\vec{e_r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\,\vec{e_\theta} + (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)\,\vec{e_\phi}] \cdot (r\delta\theta\,\vec{e_\theta} + r\sin\theta\,\delta\phi\,\vec{e_\phi}) = 0$$

$$\Rightarrow [mg\sin\theta - m(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)]\,r\delta\theta = 0$$
$$-m(r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)\,r\sin\theta\,\delta\phi = 0$$

since $\delta\theta$ and $\delta\phi$ are independent. Therefore the co-efficient of $\delta\theta$ and $\delta\phi$ is equal to zero.

$$\Rightarrow mgr\sin\theta - mr(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta) = 0$$

$$mr\,[(g\sin\theta - r\ddot{\theta} - 2\dot{r}\dot{\theta} + r\dot{\phi}^2\sin\theta\cos\theta)] = 0$$

$$g\sin\theta - r\ddot{\theta} - 2\dot{r}\dot{\theta} + r\dot{\phi}^2\sin\theta\cos\theta = 0 \longrightarrow (A)$$

$\Rightarrow -m(r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)r\sin\theta$

$-mr\sin\theta(r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)$

$r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta = 0$

This is the required equation of motion

$r = a + b\cos\omega t$ $\quad -[(a+b\cos\omega t)\theta - 2b\omega\sin\omega(-\theta$ $\quad \phi^2\sin\theta\cos\theta = -g\sin$

$\dot{r} = -b\sin\omega t \cdot \omega$ $\quad +[(a+b\cos\omega t)\ddot{\theta} - 2b\omega\sin\omega$ $\quad -(a+b)\cos t)\dot{\phi}^2\sin$

$\dot{r} = -b\omega\sin\omega t$

$\ddot{r} = -b\omega\cos\omega t \cdot \omega$

$\ddot{r} = -b\omega^2\cos\omega t$

(A) $\Rightarrow$ $g\sin\theta - (a+b\cos\omega t)\ddot{\theta} - 2(-b\omega\sin\omega t)\dot{\theta} + ($

$\cdot\,\dot{\phi}^2\sin\theta\cos\theta = 0$

$\Rightarrow (a+b\cos\omega t)\ddot{\theta} - 2b\omega\dot{\theta}\sin\omega t - (a+b\cos\omega t)$

$= g$

(B) $\Rightarrow (a+b\cos\omega t)\ddot{\phi}\sin\theta + 2(-b\omega\sin\omega t)\dot{\phi}\sin$

$+ 2(a+b)\cos\omega t$

$\Rightarrow (a+b\cos\omega t)\ddot{\phi}\sin\theta - 2b\omega\dot{\phi}\sin\omega t\sin\theta +$

$2(a+b\cos\omega t)\dot{\theta}\dot{\phi}$

These are the differential equation of mo the system.

Generalized forces:.

Defn:.

[Find the total workdone interms of Generalized forces]

Consider a system of N particles, upon by the forces at component $F_1, F_2, \ldots$ configuration of a system is given by the

Cartesian co-ordinates $x_1, x_2, \ldots, x_{3N}$.
Then the virtual work of this forces,

$$\delta w = \sum_{j=1}^{3N} F_j \, \delta x_j \longrightarrow (1)$$

Now let us suppose that the $3N$ ordinary cartesian co-ordinates $x_1, x_2, \ldots x_{3N}$ are related to N generalized co-ordinates $q_1, q_2, \ldots q_n, t$ by the transformation equation $x_i = x_i(q_1, q_2, \ldots q_n, t)$ where $i = 1, 2, \ldots 3N$

Differentiate this equation and set $\delta t = 0$

$$\Rightarrow \delta x_j = \sum_{i=1}^{n} \frac{\partial x_j}{\partial q_j} \delta q_i \text{, where } j = 1, 2, \ldots 3N \quad (2)$$

where the co-efficient $\frac{\partial x_j}{\partial q_i}$ are in general functions of the q's and t.

Substituting these expression δx_j into eqn (1) we obtain,

$$\delta W = \sum_{j=1}^{3N} F_j \left(\sum_{i=1}^{N} \frac{\partial x_j}{\partial q_i} \cdot \delta q_i \right)$$

$$= \sum_{i=1}^{N} \left(\sum_{j=1}^{3N} F_j \frac{\partial x_j}{\partial q_i} \right) \delta q_i \longrightarrow (3)$$

F = applied forces
r = constrained forces
x = cartesian co-ordinate
Q : Generalized co-ordinate

Let us define the generalized force,

$$Q_i = \sum_{j=1}^{3N} F_j \cdot \frac{\partial x_j}{\partial q_i} \longrightarrow (4) \text{ where } i = 1, 2, \ldots n$$

Substituting (4) in (3)

$$\Rightarrow \delta w = \sum_{i=1}^{n} Q_i \, \delta q_i \longrightarrow (5)$$

Note : 1

Comparing equation (1) and (5) we see that They are of the same mathematical form since F's are ordinary force components applied