velocity $\dot{\theta}$. similarly $\dot{\phi}$ is the vertical axis associated
therefore P_ϕ is the vertical component of A.M.

UNIT - II

Definition:-

There are two general approaches to the subject of classical dynamics namely vectorial and analytical dynamics. The vectorial dynamics is based on a direct application of Newton's law of motion. It concentrate on the forces and motions associated with the individual part of the system and on the interaction among these parts.

Analytical dynamics is more concerned with the system as a whole and uses descriptive scalar function such as kinetic and potential energy. By performing certain operations on these function. It is often possible to obtain a complete set of equations of motion without solving explicity for the constraint forces acting on the various parts of the system

Sec - kinetic Energy in generalized co-ordinates.

Consider a system of N - particle. Let the

Configuration of the system be given Cartesian co-ordinates $(x_1, x_2, \dots, x_{3N})$ to an inertial frame

The total K.E of the system $T = \frac{1}{2}\sum_{i=1}^{N}$ where $i = 1, 2, \dots 3N$

where $m_1 = m_2 = m_3$ is the mass of the particle and (x_1, x_2, x_3) specify its pos similarly, $m_4 = m_5 = m_6$ is the mass of the particle and so on.

Let the transformation equation conn Cartesian co-ordinates $(x_1, x_2, \dots, x_{3N})$ and generalized co-ordinates $q_1, q_2, \dots q_n$ be $x_i = x_i(q_1, q_2, \dots, q_n, t)$ for all $i = 1, 2, \dots$ where we assume that these functions differentiable with respect to q's and t.

$$\frac{dx_i}{dt} = \frac{\partial x_i}{\partial q_1}\cdot\frac{dq_1}{dt} + \frac{\partial x_i}{\partial q_2}\frac{dq_2}{dt} + \dots + \frac{\partial x_i}{\partial q_n}\cdot$$

$$\dot{x}_i = \frac{\partial x_i}{\partial q_1}\dot{q}_1 + \frac{\partial x_i}{\partial q_2}\cdot\dot{q}_2 + \dots + \frac{\partial x_i}{\partial q_n}\dot{q}_n + \frac{\partial x_i}{\partial t}$$

Since x_i is linear in $\dot{q}$'s and $\frac{\partial x_i}{\partial q_j}$ and functions of the q's and t.

Substitute (2) in eqn (1) we get

$$T = \frac{1}{2}\sum_{i=1}^{3N} m_i \dot{x}_i^2 = \frac{1}{2}\sum_{i=1}^{3N} m_i\left(\sum_{j=1}^{n}\frac{\partial x_i}{\partial q_j}\dot{q}_j + \frac{\partial x_i}{\partial t}\right)$$

$$T = T_2 + T_1 + T_0 \longrightarrow (3)$$

where $T_2 = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{n} m_{ij} \dot{q}_i \dot{q}_j$

where $m_{ij} = m_{ji} = \sum_{k=1}^{3N} m_k \cdot \frac{\partial x_k}{\partial q_i} \cdot \frac{\partial x_k}{\partial q_j} \longrightarrow (4)$

Also, $T_1 = \sum_{j=1}^{n} a_i \cdot \dot{q}_i$

where $a_i = \sum_{k=1}^{3N} M_k \cdot \frac{\partial x_k}{\partial q_i} \cdot \frac{\partial x_k}{\partial t} \longrightarrow ⑤$

Finally,

$$T_0 = \frac{1}{2} \sum_{k=1}^{3N} m_k \cdot \left(\frac{\partial x_k}{\partial t} \right)^2 \longrightarrow (6)$$

$$\dot{x}_i = \sum_{j=1}^{n} \frac{\partial x_i}{\partial q_j} \dot{q}_j + \frac{\partial x_i}{\partial t}$$

Discussion:

Assume that $m_k > 0$ is positive for all k.

∴ eqn (1) ⇒ the total K.E T is positive definite quadratic function of $\dot{x}$.

If for all $\dot{x}$'s are zero ⇒ $T=0$

If any of $\dot{x}$ not zero ⇒ K.E is +ve. since T is a function of q's, $\dot{q}$'s and t.

It is still for any real system that the K.E is $T=0$ then the system is motionless.

since q's are usually choosen such that the $\dot{q}$'s are all zero ⇔ the system is motionless.

T is usually a +ve definite function of the $\dot{q}$'s. But this is not always the case particularly if there are moving constraints.

Let us consider. T_2, if $\frac{\partial x_k}{\partial t} = 0 \Rightarrow T = T_2$.

(i.e) for a system in which any moving constraints of moving ~~refrom~~ rrefrence frame are hold fixed. Then assuming that one or more non-zero $\dot{q}$'s ⇒ The motion of one or more particle's of the system and vice versa.

We conclude that T_2 must be +ve de
quadratic function of $\dot{q}$'s.

The +ve definite nature of T_2 restricts values of the inertia co-efficients m_{ij}

If we consider the symmetric $n \times n$ s inertia matrix 'm', then necessary and condition that T_2 must be +ve definite such that, $m_{11} > 0$,

$$\begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} > 0 \ldots \begin{vmatrix} m_{11} & m \\ m_{21} & m \\ \vdots & \vdots \\ m_{n1} & m \end{vmatrix} > 0$$

This is equivalent to the requirement the determinent of the matrix and all th minus.

For the case of ronomic system $T_1 \neq 0$ it follow that the k.E T of a scleronon system is a homogeneous quadratic f of the $\dot{q}$'s.

In this case eqn (4) that the inertia co m_{ij} are the functions of the q's but not

Since T_1 is linear in the $\dot{q}$'s it is that it can be +ve (or) -ve on the other T_0 is +ve (or) zero.

① 10M (Derivation of Lagrange's Equation for holonomic

✗. (or) ⊗ 10M.

5M

Derive the standard form of Lagrange's e

(or) Prove that $\frac{d}{dt}\left(\partial t/\partial \dot{q}_i\right) - \partial t/\partial q_i = 0 \quad \forall \; i=$

The standard form of Lagrange's equation holonomic system.

Proof:-

Let as consider a system of N particles whose configeuration of the system is specified by 3N cartesian co-ordinates $x_1, x_2, \dots, x_{3N}$.

Let $q_1, q_2, \dots, q_n$ be the n generalized co-ordinates describing the system. The transforming equation,

$$x_i = x_i(q_1, q_2, \dots, q_n, t) \quad \forall\ i = 1,2,\dots 3N \text{ and}$$

$$\dot{x}_i = \left(\frac{\partial x_i}{\partial q_1}\cdot\frac{dq_1}{dt} + \frac{\partial x_i}{\partial q_2}\frac{dq_2}{dt} + \dots + \frac{\partial x_i}{\partial q_n}\frac{dq_n}{dt}\right) + \frac{\partial x_i}{\partial t},$$

where $i = 1,2,\dots 3N$. (or)

$$\dot{x}_i = \sum_{j=1}^{n} \frac{\partial x_i}{\partial q_j}\dot{q}_j + \frac{\partial x_i}{\partial t} \longrightarrow ①$$

Diff with respect to $\dot{q}_j$

Also

$$\frac{\partial \dot{x}_i}{\partial \dot{q}_j} = \sum_{j=1}^{n} \frac{\partial x_i}{\partial q_j} \longrightarrow (2), \forall\ i = 1,2,\dots 3N$$

$$j = 1,2,\dots n.$$

Differentiate eqn ① with respect to q_k.

$$\frac{\partial \dot{x}_i}{\partial q_k} = \sum_{j=1}^{n} \frac{\partial^2 x_i}{\partial q_k \cdot \partial q_j}\dot{q}_j + \frac{\partial^2 x_i}{\partial q_k \partial t}$$

since

$$\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_k}\right) = \partial/\partial q_k\left(\frac{d}{dt}x_i\right) = \frac{\partial \dot{x}_i}{\partial q_k} \longrightarrow (3)$$

Now consider the generalized momentum $p_j = \frac{\partial T}{\partial \dot{q}_j}$

$$p_j = \partial/\partial \dot{q}_j\left(\frac{1}{2}\sum_{i=1}^{3N} m_i(\vec{\dot{x}}_i)^2\right)$$

$$= 1/2 \sum_{i=1}^{3N} m_i\, \partial/\partial \dot{q}_j (\vec{\dot{x}}_i)^2$$

$T = \left(\frac{1}{2}\sum_{i=1}^{3N} m_i (\dot{x}_i)^2\right)$

$$= 1/2 \sum_{i=1}^{3N} m_i\, 2\vec{\dot{x}}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_j}$$

$\frac{\partial \dot{x}_i}{\partial \dot{q}_j} = \frac{\partial x_i}{\partial q_j}$

$$p_j = \sum_{i=1}^{3N} m_i \dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_j} \longrightarrow (4) \text{ (by eqn (2))}$$

$$\therefore \frac{d}{dt}(p_j) = d/dt\left(\partial T/\partial \dot{q}_j\right)$$

$$= d/dt\left[\sum_{i=1}^{3N} m_i \dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_j}\right] \quad \text{(by eqn (4))}$$

$$= \sum_{i=1}^{3N} m_i \left[\ddot{\vec{x}}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_j} + \dot{\vec{x}} \frac{d}{dt}\left(\frac{\partial x_i}{\partial q_j}\right)\right]$$

$$\frac{d}{dt}(p_j) = \sum_{i=1}^{3N} m_i \ddot{\vec{x}}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_j} + \sum_{i=1}^{3N} \dot{\vec{x}}_i\, m_j \frac{\partial \dot{x}_i}{\partial \dot{q}_j} \quad \text{—} ⑤$$

$\frac{\partial \dot{x}_i}{\partial q_k} = \frac{d}{dt}\left(\frac{\partial x_i}{\partial q_k}\right)$

But $$\partial T/\partial q_j = \partial/\partial q_j\left(\frac{1}{2}\sum_{i=1}^{3N} m_i (\dot{\vec{x}}_i)^2\right)$$

$$\partial T/\partial q_j = \cancel{1/2}\sum_{i=1}^{3N} m_i \cancel{2}\, \dot{\vec{x}}_i \cdot \frac{\partial \dot{x}_i}{\partial q_j} \longrightarrow (6)$$

From eqn (5) and (6) we have,

$$\frac{d}{dt}(p_j) - \partial T/\partial q_j = \sum_{i=1}^{3N} m_i \ddot{\vec{x}}_i \frac{\partial x_i}{\partial q_j}$$

(i.e) $$\frac{d}{dt}\left(\partial T/\partial \dot{q}_j\right) - \partial T/\partial q_j = \sum_{i=1}^{3N} m_i \ddot{\vec{x}}_i \frac{\partial x_i}{\partial q_j} -$$

$j=1$

Since the generalized force Q_j is, $\sum_{i=1}^{3N}$

since $\vec{F}_i$ is the applied force component with x_i then the De' Alembert's princ

$$\sum_{i=1}^{3N} (F_i - m_i \ddot{\vec{x}}_i)\, \delta x_i = 0 \longrightarrow (9)$$

since $$\delta x_i = \sum_{j=1}^{n} \frac{\delta x_i}{\partial q_j} \cdot \delta q_j$$

$\therefore$ Eqn (9) $\Rightarrow$ $$\sum_{i=1}^{3N}\left(\vec{F}_i - m_i \ddot{\vec{x}}_i\right)\left(\sum_{j=1}^{n} \frac{\partial x_i}{\partial q_j} \delta q\right.$$

$$\Rightarrow \sum_{j=1}^{n}\left[\sum_{i=1}^{3N} F_i \cdot \frac{\partial x_i}{\partial q_j} - \sum_{i=1}^{3N} m_i \ddot{\vec{x}}_i \frac{\partial x_i}{\partial q_j}\right] \delta q_j =$$

∴ Eqn (7)(8) and (10) $\Rightarrow \sum_{j=1}^{3} \left[Q_j - \frac{d}{dt}\left(\partial T/\partial \dot{q}_j\right) - \partial T/\partial q_j \right) \delta q_j = 0$

which is essentially restatement of the lagrangean ⑪ form of De'Alemberts principle generalize co-ordinates.

Now let us assume that the system is holonomic and its configuration is prescribed by the set of independent generalized co-ordinates.

(i.e) δq_j are independent.

$\Rightarrow$ The co-efficient of $\delta q_j = 0 \quad \forall\ j = 1, 2, \dots n$

$$\Rightarrow \frac{d}{dt}\left(\partial T/\partial \dot{q}_j\right) - \partial T/\partial q_j = Q_j \rightarrow (12) \quad \forall\ j = 1, 2, \dots n$$

These 'n' eqns are called Lagrangean equations.

Now let us make the additional assumption that all the generalized forces are derivable from a potential function $v(q,t)$. Then $Q_j = -\partial v/\partial q_j$, $j = 1$ to n.

$$\therefore \text{Eqn}(12) \Rightarrow \frac{d}{dt}\left(\partial T/\partial \dot{q}_j\right) - \partial T/\partial q_j = -\partial v/\partial q_j \rightarrow (13)$$

$$(13) \Rightarrow \frac{d}{dt}\left(\partial T/\partial \dot{q}_j\right) = \frac{dT}{\partial q_j} - \frac{\partial v}{\partial q_j} \qquad \partial v/\partial q_j = 0$$

$$= \partial/\partial q_j (T - v)$$

$$\frac{d}{dt}\left(\partial T/\partial \dot{q}_j\right) = \partial L/\partial q_j$$

$\left(\because L = T - v \right.$, also v is a function of q, t $\therefore \left. \frac{\partial v}{\partial \dot{q}_j} = 0 \right)$

$$\therefore \frac{d}{dt}\left(\partial T/\partial \dot{q}_j - \partial v/\partial \dot{q}_j\right) = \frac{\partial L}{\partial q_j}$$

$$\Rightarrow \frac{d}{dt}\left(\partial/\partial \dot{q}_j (T - v)\right) = \frac{\partial L}{\partial q_j}$$

$$\therefore \frac{d}{dt}\left(\partial L/\partial \dot{q}_j\right) - \partial L/\partial q_j = 0$$

This is the standard form of Lagrangean equatio for holonomic system.

Another form of Lagrangean's equation:-

Suppose the generalized forces Q_j is given by

$$Q_j = -\frac{\partial V}{\partial q_j} + Q_j'$$, where Q_j' is the generalized forces are not derivable from a potential func

$\therefore$ Eqn (12) $\Rightarrow$ $$\frac{d}{dt}\left(\partial T/\partial \dot{q}_j\right) - \partial T/\partial q_j = -\frac{\partial V}{\partial q_j} + Q_j' \; j = 1,$$

$$\frac{d}{dt}\left(\partial T/\partial q_j'\right) - \partial/\partial q_j (T-$$

$$\Rightarrow \frac{d}{dt}\left(\partial L/\partial \dot{q}_j\right) - \partial L/\partial q_j = Q_j' \longrightarrow (14)$$

5M ⓧ Derive the Lagrangean equation for non-holonomi system:-

<u>Proof</u>:-

For a non-holonomic system however there is be more generalized co-ordinates then the numb of degrees of freedom.

$\therefore$ δq's are no longer independent, if we as a virtual displacement consistent with the con

For example,

if there are m- non-holonomic constra equation of the form,

$$\sum_{i=1}^{n} a_{ji}\, dq_i + a_{jt}\, dt = 0 \longrightarrow ① \quad \forall j = 1, 2, \ldots$$

then,

$$\sum_{i=1}^{n} a_{ji}\, \delta q_i = 0 \longrightarrow (2), \; \forall j = 1 \text{ to } m.$$

Assume each generalized applied force Q_j is obtained from the potential function $V(q$

$$Q_j = -\partial V/\partial q_j \quad \forall j = 1, 2, \ldots m \longrightarrow (3)$$

Now the constraints are assumed to be workless so the generalized constrained forces c_i Satisfies the condition,

$$\sum_{i=1}^{n} c_i \, \delta q_i = 0 \longrightarrow (4)$$

For any virtual displacement consistent with the constraints

multiply the eqn (2) by the factor the Lagrangean multiplies λ_j and obtain m. equation;

$$\lambda_j \sum_{i=1}^{n} a_{ji} \, \delta q_i = 0 \quad \forall \; j = 1, 2, \dots m$$

Add some of these m-equation

$$\sum_{j=1}^{m} \lambda_j \sum_{i=1}^{n} a_{ji} \, \delta q_i = 0 \quad \forall \; j = 1 \text{ to } m.$$

$$\Rightarrow \sum_{i=1}^{n} \left\{ \sum_{j=1}^{m} \lambda_j \, a_{ji} \right\} \delta q_i = 0 \longrightarrow (5)$$

Now, consider eqn (4) $\overset{\text{minus}}{-}$ eqn (5),

$$\Rightarrow \sum_{i=1}^{n} \left\{ c_i - \sum_{j=1}^{m} \lambda_j \, a_{ji} \right\} \overset{\text{independent}}{\delta q_i} = 0 \longrightarrow (6)$$

The λ's are arbitaray while the δq's must conform to the constraints of eqn (2), but choose λ such that

$$c_i = \sum_{j=1}^{m} \lambda_j \, a_{ji} \;, \forall \; i = 1, 2, \dots m \longrightarrow (7)$$

Then the co-efficient of the δq's are zero and eqn (6) will apply for any set of δq's

In otherwords, δq's can be choosen independently, with these assumption we can equate the generalized force c_i with Q_i' and using

$$\frac{d}{dt}\left(\partial L / \partial \dot{q}_i \right) - \partial L / \partial q_i = Q_i$$

$$= c_i$$

$$= \sum_{j=1}^{m} \lambda_j a_{ji}$$

$$\Rightarrow \frac{d}{dt}\left(\partial L / \partial \dot{q}_i\right) - \partial L / \partial q_i = \sum_{j=1}^{m} \lambda_j a_{ji}$$

This is the standard form Lagrangean e
for a non-holonomic system.

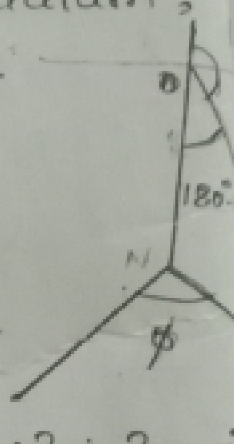
5M ⊗ T **

Example :-

Find the differential equation of motio
spherical pendulum of length 'l'.

Soln:-

Let 'm' be the mass of the particle s
by a massless wire of length 'l'. From a
point 'o' to form a spherical pendulum.



Let us use the spherical co-ordinates θ and ϕ then the cartesian co-ordinates with respect to axes.

$x = l \sin\theta \cos\phi$

$y = l \sin\theta \sin\phi$, $T = \frac{1}{2} m (l^2 \dot{\theta}^2 + l^2 \sin^2\theta \, \dot{\phi}^2$

$z = -l \cos\theta$

Since the total K.E, $T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

where,

$\dot{x} = l \{ -\sin\theta \sin\phi \, \dot{\phi} + \cos\theta \cos\phi . \dot{\theta} \}$

$\dot{y} = l \{ \sin\theta \cos\phi \, \dot{\phi} + \cos\theta \sin\phi \, \dot{\theta} \}$

$\dot{z} = l \sin\theta \, \dot{\theta}$ $\sin^2\theta$

$$T = \frac{1}{2} m \Big[(l(-\sin\theta \sin\phi \, \dot{\phi} + \cos\theta \cos\phi \, \dot{\theta}))^2 + (l(\sin\theta \cos\phi \, \dot{\phi} + \cos\theta \sin\phi \, \dot{\theta}))^2 + (l s$$

$$= \frac{1}{2} m \Big[\{ l^2 (\sin^2\theta \sin^2\phi \, \dot{\phi}^2 + \cos^2\theta \cos^2\phi \, \dot{\theta}^2 + 2(-\sin\theta \sin\phi \, \dot{\phi})(\cos\theta \cos\phi \, \dot{\theta}))$$

$\{l^2(\sin^2\theta\cos^2\phi\dot\phi^2+\cos^2\theta\sin^2\phi\dot\theta^2+2(\sin\theta\cos\phi\dot\phi)$

$(\cos\theta\sin\phi\dot\theta))\}+l^2\sin^2\theta\dot\theta^2]$

$=\frac{1}{2}m[l^2(\sin^2\theta\sin^2\phi\dot\phi^2+\cos^2\theta\cos^2\phi\dot\phi^2-2\sin\theta\cos\theta\sin\phi\dot\phi\cos\phi\dot\theta+$

$\sin^2\theta\cos^2\phi\dot\phi^2+\cos^2\theta\sin^2\phi\dot\theta^2+2\sin\theta\cos\phi\cos\phi\dot\phi\sin\phi\dot\phi+\sin^2\theta\dot\theta^2]$

$T=\frac{1}{2}m[l^2\sin^2\phi\dot\phi^2(\sin^2\theta+\cos^2\theta)\cos^2\phi\dot\theta^2+\cos^2\phi\dot\phi^2(\sin^2\theta+\cos^2\theta)$

$\sin^2\phi\dot\theta^2+\sin^2\theta\dot\theta^2$

$T=\frac{1}{2}m[l^2\dot\theta^2+l^2\sin^2\theta\dot\phi^2]$

$=\frac{1}{2}ml^2[\sin^2\theta\dot\phi^2+\cos^2\theta\dot\theta^2+\sin^2\theta\dot\theta^2]$.

$=\frac{1}{2}ml^2[\sin^2\theta\dot\phi^2+\dot\theta^2]$

P.E $=-mgh$

$=-mg(ON)$

$=-mg(l\cos(180-\theta))$

$V=mgl\cos\theta$.

$L=T-V$

$=\frac{1}{2}m(l^2\dot\theta^2+l^2\sin^2\theta\dot\phi^2)-mgl\cos\theta$

The required diff. eqn of motions are

$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right)-\frac{\partial L}{\partial\theta}=0$ and

$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\phi}\right)-\frac{\partial L}{\partial\phi}=0$.

$\frac{\partial L}{\partial\dot\theta}=ml^2\ddot\theta$

$\frac{\partial L}{\partial\theta}=ml^2\dot\phi^2\sin\theta\cos\theta+mgl\sin\theta$

$\frac{\partial L}{\partial\dot\phi}=ml^2\dot\phi\sin^2\theta$

$\frac{\partial L}{\partial\phi}=0$.

$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right)=ml^2\ddot\theta$

$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\phi}\right)=ml^2[\ddot\phi\sin^2\theta+2\dot\phi\sin\theta\cos\theta\dot\theta]$

$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right)-\frac{\partial L}{\partial\theta}=$

$=ml^2\ddot\theta-[ml^2\dot\phi^2\sin\theta\cos\theta]+mgl\sin\theta=0$

$\Rightarrow m[l^2\ddot\theta-l^2\dot\phi^2\sin\theta\cos\theta-gl\sin\theta]=0$

$l^2\ddot\theta-l^2\dot\phi^2\sin\theta\cos\theta-gl\sin\theta=0$

$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\phi}\right)-\frac{\partial L}{\partial\phi}=0$

$ml^2(\ddot\phi\sin^2\theta+2\dot\phi\sin\theta\cos\theta\dot\theta)-0=0$.

$\ddot\phi\sin^2\theta+2\dot\phi\sin\theta\cos\theta\dot\theta=0$