Problem:.

A Double pendulum consists of two parti
suspended by a massless rod as shown &
Soln:. Assume that all motion takes place
Plane find the differential equation of
Solution:.

Let 'O' be a fixed point of subpension. Let the rod connecting the upper particle make an angle 'θ' with vertical and the rod connecting lower particle make an angle ϕ with vertical.

To find K.E:.

Absolute velocity of an upper particle

$V_1 = l\dot{\theta}$

Absolute velocity of an lower particle is the vector sum of,

(i) The absolute velocity of upper particle

(ii) the velocity of the lower particle relati the upper particle.

Since the two velocity vectors differ in dir by the angle $\phi - \theta$.

using the cosine law to obtain the magni of the vector sum.

$$|V_1 + V_2| = \sqrt{V_1^2 + V_2^2 + 2V_1V_2\cos\alpha}$$

$\therefore$ The velocity of the lower particle V_2

(i.e) $V_2 = l[\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\theta - \phi)]^{1/2}$

Since the K.E of the upper particle $= \frac{1}{2}$

$= \frac{1}{2} m l^2 \dot{\theta}^2$ $= \frac{1}{8}m$

The K.E of the lower particle $= \frac{1}{2} m V_2^2$

$$= \frac{1}{2} m l^2 [\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\theta-\phi)]$$

$\therefore$ The total K.E $= \frac{1}{2} m v^2 = \frac{1}{2} m (v_1^2 + v_2^2)$

The total K.E $= \frac{1}{2} m [l^2\dot{\theta}^2 + l^2(\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\theta-\phi))]$

$$= \frac{1}{2} m l^2 [2\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\theta-\phi)]$$

To find potential Energy:-

keeping 'O' as fixed point as well as a reference point. Therefore the P.E, $V_1 = -mgl\cos\theta$

$$V_2 = -mg(l\cos\theta + l\cos\phi)$$

$$V = -mgl[2\cos\theta + \cos\phi]$$

The Lagrangean function $L = T - V$

$$L = \frac{1}{2} m l^2 [2\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\phi-\theta)] + [mgl(2\cos\theta + \cos\phi)] \text{---} (1)$$

$$\partial L / \partial \dot{\theta} = ml^2 [2\dot{\theta} + \dot{\phi}\cos(\phi-\theta)] + \dot{\phi}[-\sin(\phi-\theta)(\dot{\phi}-\dot{\theta})]$$

$$\frac{d}{dt}(\partial L/\partial\dot{\theta}) = ml^2[2\ddot{\theta} + \ddot{\phi}\cos(\phi-\theta)]$$

$$= ml^2 [2\ddot{\theta} + \ddot{\phi}\cos(\phi-\theta) - \dot{\phi}^2\sin(\phi-\theta) + \dot{\theta}\dot{\phi}\sin(\phi-\theta)] \rightarrow (2)$$

Now, $\partial L/\partial\dot{\phi} = ml^2[\dot{\phi} + \dot{\theta}\cos(\phi-\theta)]$

$$\frac{d}{dt}(\partial L/\partial\dot{\phi}) = ml^2\{\ddot{\phi} + \ddot{\theta}\cos(\phi-\theta) - \dot{\theta}(\dot{\phi}-\dot{\theta})\sin(\phi-\theta)\} \rightarrow (4)$$

$$\partial L/\partial\theta = -ml^2\dot{\phi}\dot{\theta}\sin(\phi-\theta)(-1) - 2mgl\sin\theta \rightarrow (3)$$

$$\partial L/\partial\phi = \frac{1}{2} m l^2 [2\dot{\phi}\dot{\theta}(-\sin(\phi-\theta))] - mgl\sin\phi \rightarrow (5)$$

Consider the D.E of Lagrange's eqn of motion,

$$\frac{d}{dt}(\partial L/\partial\dot{\theta}) - \partial L/\partial\theta = 0$$

[$\because$ by (2) and (3)]

$$\Rightarrow ml^2[2\ddot{\theta} + \ddot{\phi}\cos(\phi-\theta) - \dot{\phi}\sin(\phi-\theta)]$$

$$-ml^2\dot{\phi}\dot{\theta}\sin(\phi-\theta) + 2mgl\sin\theta = 0 \longrightarrow (6)$$

From (4) and (5), we have

$$ml^2[\ddot{\phi}+\ddot{\theta}\cos(\phi-\theta)-\dot{\phi}\dot{\theta}\sin(\phi-\theta)+\dot{\phi}^2$$

$$+\dot{\theta}\dot{\phi}\sin(\phi-\theta)+mgl\sin\phi=0$$

(2) $\Rightarrow$ $\Rightarrow ml^2[\ddot{\phi}+\ddot{\theta}\cos(\phi-\theta)+\dot{\theta}^2\sin(\phi-\theta)$

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Example :-

A particle of mass m can slide without fr
the inside of a small tube which is bend in
of a circle of radius r. The tube rotates abo
vertical diameter with a constant angular vel
as shown in figure. Find the differential
of motion.

Soln:

Absolute velocity of the particle is
the vector sum of

(i) The absolute velocity of the particle alo
radial direction $r\dot{\theta}=v_1$

(ii) The absolute velocity of the particle
to the rotation of the tube with the
velocity $\omega = r\omega\sin\theta = v_2$.

Therefore, the total K.E $T=\frac{1}{2}mv^2$

$$\therefore T=\frac{1}{2}m[(r\dot{\theta})^2+(r\omega\sin$$

$$=\frac{1}{2}mr^2[\dot{\theta}^2+\omega^2\sin^2\theta]$$

Potential energy :-

Potential energy $P.E = V = mg(ON)$

$$= mg(r\cos\theta)$$

$$V = mg(r\cos\theta)$$

$\therefore$ The Lagrangian function $L=T-V$

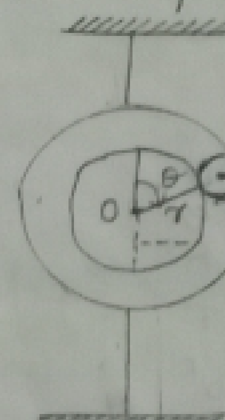
$$\therefore L = \frac{1}{2} m r^2 [\dot{\theta}^2 + \omega^2 \sin^2\theta] - mgr\cos\theta$$

This is reheonomic because any set of equation giving the inertial cartesian co-ordinates of the particles interms of its single generalized co-ordinates θ must involve time explicity.

The tube is a moving constrain which does work on the particle in an actual displacement

Neverthless no work is done by the constrain forces in a virtual displacement and the constraint is classed as a workless constraint.

Since the only generalized forces acting on the system is ~~dir~~ derivates from a potential function. The standard form of Lagrangian equation,



$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = mr^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = mr^2\omega^2\sin\theta\cos\theta \mathbin{\cdot} mgr\sin\theta$$

$$mr^2\ddot{\theta} - (mr^2\omega^2\sin\theta\cos\theta + mgr\sin\theta) = 0$$

$$mr[r\ddot{\theta} - r\omega^2\sin\theta\cos\theta - g\sin\theta] = 0$$

$$r\ddot{\theta} - ((r\omega^2\cos\theta + g)\sin\theta) = 0 //$$

Example:

A particle of mass m is connected by a massless spring of stiffness k and unstressed length r_0 to a point p which is moving along a circular path of radius 'a' at a uniform ~~Anaty~~ angular rate ω assuming that the particle moves without friction on a horizontal plane. Find the differential

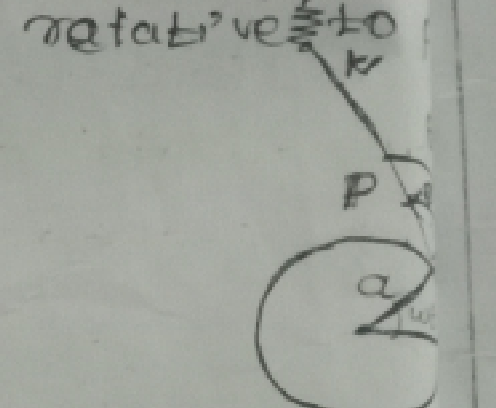
Soln:

This is a ~~sch~~ rheonomic system with 2 degrees of freedom corresponding to the independent generalised co-ordinates (r, t)

we know that the expression for the K.E into the motion with respect to an arbitrary point p is

$$T = \frac{1}{2} m \dot{\vec{r}}_p^2 + \frac{1}{2} m \dot{\vec{P}}^2 \left(\dot{\vec{R}}^2\right) + \dot{\vec{r}}_p \cdot m \dot{\vec{P}}$$

where $\dot{\vec{r}}_p$ is the absolute velocity of the point, $\dot{\vec{P}}$ is velocity of the particle relative to



we see that,

$$\dot{\vec{r}}_p = a^2 \omega^2 , \quad \vec{r} = a\omega t$$

$$\dot{\vec{r}} = a\omega$$

$$\dot{\vec{P}} = \dot{\vec{r}} + r^2 \dot{\theta}^2$$

Hence, T_0 = the K.E due to a particle having m an moving with the reference point $p = \frac{1}{2}$

$(\because \dot{\vec{r}}$

$$P = \frac{1}{2} m \dot{\vec{r}}_p^2$$

$$= \frac{1}{2} m a^2 \omega^2$$

T_1 = The scalar product of the velocity of the reference point and linear momentum of the relative to the reference point $= \dot{\vec{r}}_p \, m \dot{\vec{P}}$

$$= ma\omega [\dot{r} \sin(\theta - \omega t) +$$

T_2 = The K.E of the system due to its motion relative to p

$(\theta$

$$= \frac{1}{2} m \dot{\vec{P}}$$

$$= \frac{1}{2} m (\dot{\vec{r}} + r^2 \dot{\theta}^2)$$

Also the P.E, $V = \frac{1}{2} k (r - r_0)^2$.

therefore, the Lagrangian function $L = T - V$

tangent and normal $= V = \dot{s}\vec{T} = a\dot{\theta}\vec{T}$

Radial and transverse $= V = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$

$$L = \frac{1}{2} ma^2\omega^2 + ma\omega\left[\dot{r}\sin(\theta-\omega t) + r\dot{\theta}\cos(\theta-\omega t)\right] + \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{2}k(r-r_0)^2$$

$$= \frac{1}{2} m\left[a^2\omega^2 + 2a\omega\left[\dot{r}\sin(\theta-\omega t) + r\dot{\theta}\cos(\theta-\omega t)\right] + \dot{r}^2 + r^2\dot{\theta}^2\right] - \frac{1}{2}k(r-r_0)^2$$

The differential equation of motion are,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

$$m\ddot{r} + ma\omega\dot{\theta}\cos(\theta-\omega t) - ma\omega^2\cos(\theta-\omega t) - \left[(mr\dot{\theta}^2 + ma\omega\dot{\theta}\cos(\theta-\omega t) - k(r-r_0)\right] = 0$$

$$m\ddot{r} + ma\omega\dot{\theta}\cos(\theta-\omega t) - ma\omega^2\cos(\theta-\omega t) - mr\dot{\theta}^2 - ma\omega\dot{\theta}\cos(\theta-\omega t) + k(r-r_0) = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}\left[mr^2(2\dot{\theta}) + ma\omega r\cos(\theta-\omega t)\right]$$

$$= mr^2\dot{\theta} + ma\omega\cos(\theta-\omega t)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} + ma\omega\dot{r}\cos(\theta-\omega t) + ma\omega r\left[-\sin(\theta-\omega t)\right]\left[\dot{\theta}-\omega\right]$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} + ma\omega\dot{r}\cos(\theta-\omega t) - ma\omega r\sin(\theta-\omega t)\dot{\theta} + ma\omega^2 r\sin(\theta-\omega t)$$

$$\frac{\partial L}{\partial \theta} = ma\omega\left[\dot{r}\cos(\theta-\omega t) - r\dot{\theta}\sin(\theta-\omega t)\right]$$

$$= \left[ma\omega\dot{r}\cos(\theta-\omega t) - ma\omega r\dot{\theta}\sin(\theta-\omega t)\right]$$

$\therefore$ The Lagrangian equation of motion rotive to θ is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} + ma\omega\dot{r}\cos(\theta-\omega t) - ma\omega r\sin(\theta-\omega t)\dot{\theta} + ma\omega^2 r\sin(\theta-\omega t) - \left[ma\omega\dot{r}\cos(\theta-\omega t) - ma\omega r\dot{\theta}\sin(\theta-\omega t)\right] = 0$$

$$\Rightarrow 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} + ma\omega\dot{r}\cos(\theta-\omega t) - ma\omega r\sin(\theta-\omega t)\dot{\theta} + ma\omega^2 r\sin(\theta-\omega t) - ma\omega\dot{r}\cos(\theta-\omega t) + ma\omega r\dot{\theta}\sin(\theta-\omega t) = 0$$

$\Rightarrow 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} + maw^2 r\sin(\theta - \omega t) = 0$ //

(8) Problem:

2 particle are connected by a rigid m rod of length 'l' which rotates in a horizon with constant angular velocity ω knife edge at the 2 particles prevent either particle having a velocity component along the rod. B particles can slide without friction in a Perpendicular to the rod. Find the differen equation of motion. Solve for x, y and the force as function of time. If the centre mass initially at the origin and has velocity in the positive y direction.

Proof:

This is a non-holonomic Rheonomic system the cartesian co-ordinates of given 2 partic and (x_2, y_2)

Since the equation of holonomic constrains

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = l^2 \longrightarrow (1) \text{ and } ?$$

$$\tan\omega t = \frac{y_2 - y_1}{x_2 - x_1} \longrightarrow (2).$$

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The non-holonomic constrained equation is,

$$\dot{x}_1\cos\omega t + \dot{y}_1\sin\omega t + \dot{x}_2\cos\omega t + \dot{y}_2\sin\omega t = 0.$$

(i.e), $(\dot{x}_1 + \dot{x}_2)\cos\omega t + (\dot{y}_1 + \dot{y}_2)\sin\omega t = 0 \longrightarrow$

Here 4 cartesian co-ordinates and three indepen equation of constraint. Therefore, the system 1 degrees of freedom.

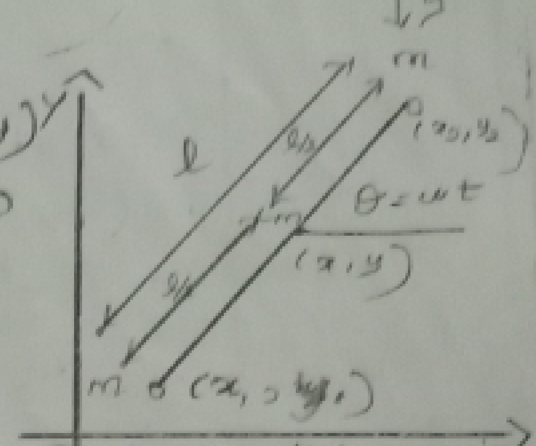
Choose the generalized co-ordinates such that system has only one non-holonomic constraint. which is no holonomic constrain.

The Cartesian Co-ordinates (x, y) is the centre of mass as the generalized co-ordinate.

Since the transformation equation

$$x_1 = x - \frac{l}{2}\cos\omega t \quad , \quad x_2 = x + \frac{l}{2}\cos\omega t$$
$$y_1 = y - \frac{l}{2}\sin\omega t \quad , \quad y_2 = y + \frac{l}{2}\sin\omega t$$

$$\left.\begin{aligned} \Rightarrow \dot{x}_1 &= \dot{x} + \frac{l}{2}\omega\sin\omega t \\ \dot{x}_2 &= \dot{x} + \frac{l}{2}\omega(-\sin\omega t) \Rightarrow \dot{x} - \frac{l}{2}\omega\sin\omega t \\ \dot{y}_2 &= \dot{y} + \frac{l}{2}\omega\cos\omega t \\ \dot{y}_1 &= \dot{y} - \frac{l}{2}\omega\cos\omega t \end{aligned}\right\} \rightarrow (4)$$



A. Non-Holonomic Rheonomic system

Now from the equation (3) and eqn (4)

$$\Rightarrow 2\dot{x}\cos\omega t + 2\dot{y}\sin\omega t = 0$$
$$\Rightarrow \dot{x}\cos\omega t + \dot{y}\sin\omega t = 0$$

Since the total kinetic energy.

$$T = \frac{1}{2}m[\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2]$$

$$= \frac{1}{2}m\Big[\Big(\dot{x}^2 + \frac{l^2}{4}\omega^2\sin^2\omega t + \cancel{2}\dot{x}\frac{l}{\cancel{2}}\omega\sin\omega t\Big)$$
$$+ \Big(\dot{y}^2 + \frac{l^2}{4}\omega^2\cos^2\omega t - \cancel{2}\dot{y}\frac{l}{\cancel{2}}\omega\cos\omega t\Big)$$
$$+ \Big(\dot{x}^2 + \frac{l^2}{4}\omega^2\sin^2\omega t - \cancel{2}\dot{x}\frac{l}{\cancel{2}}\omega\sin\omega t\Big)$$
$$+ \Big(\dot{y}^2 + \frac{l^2}{4}\omega^2\cos^2\omega t + \cancel{2}\dot{y}\frac{l}{\cancel{2}}\omega\cos\omega t\Big)\Big]$$

$$= \frac{1}{2}m\Big[2\dot{x}^2 + 2\dot{y}^2 + 2\Big(\frac{l^2}{4}\omega^2\Big)\Big]$$

$$T = m(\dot{x}^2 + \dot{y}^2) + m\frac{l^2}{4}\omega^2 \longrightarrow (6)$$

$\therefore$ The total K.E $T = T_{tran} + T_{rot}$

Since the moment of inertia of the rigid rod about the centre of mass = mass of the rigid rod $\times \left(\frac{\text{length}}{2}\right)^2$

$$= 2m\left(\frac{l}{2}\right)^2 = 2m\left(\frac{l^2}{4}\right)$$
$$= m\frac{l^2}{2}$$

The P.E $\Rightarrow V = 0$

$\therefore$ The Lagrangian function $L = T - V$

$(i.e)\ L = T - 0$

$L = T$

Now the Lagrangian equation of motion.

$\frac{d}{dt}\left(\partial L/\partial \dot{x}\right) - \partial L/\partial x = 0$ and $\frac{d}{dt}\left(\partial L/\partial \dot{y}\right) - \partial L$

$(i.e)\ \frac{d}{dt}\left(\partial T/\partial \dot{x}\right) - \partial T/\partial x = 0 \Rightarrow \lambda_1 \cos\omega$

$\frac{d}{dt}\left(\partial T/\partial \dot{y}\right) - \partial T/\partial y = 0 \Rightarrow \lambda_1 \sin\omega t$

$\partial T/\partial \dot{x} = 2m\dot{x}\ ;\ \partial T/\partial x = 0$

$\frac{d}{dt}\left(\partial T/\partial \dot{x}\right) = 2m\ddot{x}$

$\partial T/\partial \dot{y} = 2m\dot{y}\ ,\ \partial T/\partial y = 0$

$\frac{d}{dt}\left(\partial T/\partial \dot{y}\right) = 2m\ddot{y}$

$\Rightarrow 2m\ddot{x} = \lambda_1 \cos\omega t$ and $2m\ddot{y} = \lambda_1 \sin\omega t$

Eqn (7) $\Rightarrow \frac{\ddot{y}}{\ddot{x}} = \tan\omega t$

$\Rightarrow \ddot{y} = \ddot{x} \tan\omega t \longrightarrow (8)$

From eqn (5)

$\Rightarrow \dot{x}\cos\omega t + \dot{y}\sin\omega t = 0$

$\dot{y}\sin\omega t = -\dot{x}\cos\omega t$

$\frac{-\dot{y}\sin\omega t}{\cos\omega t} = \dot{x} \Rightarrow \dot{x} = \dot{y}\tan\omega$

$\Rightarrow \tan\omega t = -\frac{\dot{x}}{\dot{y}} \longrightarrow (9)$

The eqn (8) and (9)

$\Rightarrow \frac{\ddot{y}}{\ddot{x}} = \frac{-\dot{x}}{\dot{y}}$

$\Rightarrow \ddot{y}\dot{y} + \ddot{x}\dot{x} = 0$

$\Rightarrow \frac{1}{2}\frac{d}{dt}(\dot{y})^2 + \frac{1}{2}\frac{d}{dt}(\dot{x})^2 = 0.$

$$\Rightarrow \frac{d}{dt}(\dot{x}^2+\dot{y}^2)=0$$

$$\Rightarrow \dot{x}^2+\dot{y}^2=c$$

where 'c' is the integrating constant

let $c=V_0^2$

$$(\text{i.e}) \quad \dot{x}^2+\dot{y}^2=V_0^2$$

(i.e) The centre of mass with constant velocity V_0. since the direction of motion is always $\perp r$ to the rod. Therefore, $\dot{x}=-V_0 \sin\omega t \, dt$. and $\dot{y}=V_0 \cos\omega t$

$$\Rightarrow dx=-V_0 \sin\omega t \, dt \;\cdot\; \text{and} \;\; dy=V_0 \cos\omega t \, dt \longrightarrow (10)$$

Again integrate on both sides

$$x=-V_0\int_0^t \sin\omega t \, dt$$

$$=V_0\left(\frac{\cos\omega t}{\omega}\right)_0^t=\frac{V_0}{\omega}(\cos\omega t-1) \quad (\cos 0=1)$$

$$x=\frac{V_0}{\omega}(\cos\omega t-1)$$

$|||^{ly}$

$$y=\frac{V_0}{\omega}(\sin\omega t)$$

$\therefore$ The eqn (10) $\Rightarrow$
$$\left.\begin{aligned}\ddot{x}&=-V_0\cos\omega t\cdot\omega\\ \ddot{y}&=-V_0\sin\omega t\cdot\omega\end{aligned}\right\}\longrightarrow (11)$$

$\therefore$ eqn (7) and (11) $\Rightarrow 2m(-V_0\omega\cos\omega t)=\lambda_1\cos\omega t$

$$\Rightarrow \lambda_1=-2m\,V_0\,\omega$$

$\therefore$ The Lagrangian multipliers is $\lambda_1=-2mV_0\omega$

The equation of motion.

eqn (7) $\Rightarrow 2m\ddot{x}=\lambda_1\cos\omega t$

$$\Rightarrow 2m\ddot{x}=-2mV_0\omega\cos\omega t$$

$$\Rightarrow \ddot{x}=-V_0\omega\cos\omega t$$

$|||^{ly} \quad \ddot{y}=-V_0\omega\sin\omega t.$