~~UNIT II (continued)~~.

~~Ex 3.~~

~~Three particles are~~

Integrals of motion (or) constant of motion:- (or) IOM.

If the configuration of a holonomic is specified by n-generalized co-ordin then the equation of motion is consider n-second order non-linear differential e with time as independent variable,

Any general solution of those differ equation of motion contain 2n constants of which are evaluated from 2n initial conditio

[The general solution is to obtain 2n indep functions of the form $f_j(q,\dot{q},t) = \alpha_j$, $j = 1,2,\ldots$ where α's are arbitary constant.

These 2n functions are called the integra motion (or) constants of motion.]

Each function f_j maintains a constant value α_j motion of the system proceeds the value of depending upon the initial condition

These 2n equation can be solved for the q's and function of t and α_j (ie) $q_i = q_i(\alpha_1, \alpha_2, \ldots$ and hence $\dot{q}_i = \dot{q}_i(\alpha_1, \alpha_2, \ldots, \alpha_{2n}, t)$ for all $i = 1,2,\ldots n$.

5M **Ignorable Co-ordinates** :- Explain ignorabl co-ordinates.

Definition: (Q.1) 2M or 5M,

Consider a holonomic system which can described by the standard form of Lagran equation.

$$\text{i.e)} \quad \frac{d}{dt}\left(\partial L/\partial \dot{q}_i\right) - \partial L/\partial q_i = 0 \text{ for all } i = 1,2,\ldots$$

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Suppose $L(q, \dot{q}, t)$ contain all the $\dot{q}$'s but some of the q's namely $q_1, q_2, \ldots q_k$ are missing from the Lagrangian function. These k co-ordinates are called ignorable co-ordinates For each ignorable co-ordinates, $\partial L / \partial q_i = 0, \forall i = 1, 2, \ldots k$

$$\Rightarrow \frac{d}{dt}\left(\partial L / \partial \dot{q}_i\right) = 0, \forall i = 1, 2, \ldots k$$

$$\Rightarrow \partial L / \partial \dot{q}_i = \beta_i \quad \forall i = 1, 2, \ldots k$$

$$(i.e) \quad p_i = \beta_i \quad \forall i = 1, 2, \ldots k$$

where the β's are constant evaluated from the initial condition. Hence we find that the generalized momentum, corresponding to each ignorable co-ordinates is constant (i.e) It is an Integral of motion.

Example :- The kepler's Problem :-

The motion of a particle of unit mass which is attacted by an inverse square gravitational force to a fixed point using polar co-ordinates the K.E,

$$T = \frac{1}{2} m \left[\dot{r}^2 + (r\dot{\theta})^2\right] \quad (\because m = 1)$$

$$= \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

and the P.E $V = -\mu / r$ where 'μ' is a positive constant known as the gravitation co-efficient.

$\therefore$ The Lagrangian function $L = T - V$

$$= \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \mu / r \rightarrow (1)$$

Here the Lagrangian function does not contain the coordinate θ and hence θ is ignorable co-ordinates.

$\therefore$ The Lagrangian equation of motion, the r equation of motion, $\frac{d}{dt}\left(\partial L / \partial \dot{r}\right) - \partial L / \partial r = 0 \rightarrow (2)$

The θ equation of motion, $\frac{d}{dt}\left(\partial L/\partial \dot\theta\right) - \partial L/\partial\theta$

$\therefore$ Eqn (1) and (2) $\Rightarrow \frac{d}{dt}(\dot r) - \left(r\dot\theta^2 - \mu/r^2\right) = 0$

$$\Rightarrow \ddot r - r\dot\theta^2 + \mu/r^2 = 0$$

From eqn (1) and (3) $\Rightarrow \frac{d}{dt}\left(r^2\dot\theta\right) - 0 = 0 \Rightarrow r^2\dot\theta$

where β is the constant and it is equal angular momentum of the Particle about attracting the centre 'O'.

Therefore eqn (I) and (II) are equatio motions relating to the Kepler's problem

The Routhian function:-

Let the configuration of the holon system be described by 'n' generalized co. $q_1, q_2, \ldots, q_n$. suppose $q_1, q_2, \ldots, q_k$ ignorable co-ordinates.

Thus the Lagrangian function is

$$L = L\left(q_{k+1}, q_{k+2}, \ldots, q_n, \dot q_1, \dot q_2, \ldots, \dot q_n\right)$$

hence $\partial L/\partial q_j = 0$, $j = 1, 2, \ldots k$ and hence the Lagrangian equation becomes,

$$\frac{d}{dt}\left(\partial L/\partial \dot q_j\right) - \partial L/\partial q_j = 0 \Rightarrow \frac{d}{dt}\left(\partial L/\partial \dot q_j\right)$$

where β_j are constants. $\partial L/\partial \dot q_j = \beta_j$

For $\dot q_1, \dot q_2, \ldots \dot q_k$ are functions of $\left(q_{k+1}, q_{k+2}, \ldots q_n, \dot q_{k+1}, \dot q_{k+2}, \ldots \dot q_n, \beta_1, \beta_2\right.$

These expressions for the $\dot q$'s are linear in the β's. now define,

$R = L - \sum_{j=1}^{k} \beta_j \dot{q}_j$. Thus functions is called Routhian function.

Also, $R = R(q_{k+1}, q_{k+2}, \dots q_n, \dot{q}_{k+1}, \dot{q}_{k+2}, \dots \dot{q}_n, \beta_1, \beta_2, \dots \beta_k, t)$.

Derive form of Lagrangian equation with the Routhian function (or) Derive $\frac{d}{dt}\left(\partial R/\partial \dot{q}_i\right) - \partial R/\partial q = 0$ (or) Prove that the Routhian procedure has succeded in eliminating the ignorable co-ordinates from the equation of motion.

Proof:

Since, $R = L - \sum_{j=1}^{k} \beta_j \dot{q}_j$ where,

$R = R(q_{k+1}, q_{k+2}, \dots q_n, \dot{q}_{k+1}, \dot{q}_{k+2}, \dots \dot{q}_n, \beta_1, \beta_2, \dots \beta_k, t)$

and $\beta_i = \partial L/\partial \dot{q}_1$, where $i = 1, 2, \dots n$

An arbitrary variation of all the variables in the Routhian function we have,

$$\delta R = \sum_{i=k+1}^{n} \partial R/\partial q_i \cdot \delta q_i + \sum_{i=k+1}^{n} \partial R/\partial \dot{q}_1 \, \delta \dot{q}_i + \sum_{i=1}^{k} \partial R/\partial \beta_i \, \delta \beta_i + \partial R/\partial t \, \delta t \longrightarrow (1)$$

Since,

$L = L(q_{k+1}, q_{k+2}, \dots q_n, \dot{q}_1, \dot{q}_2, \dots \dot{q}_n, t)$ and hence an arbitrary variation of all the variables is the Lagrangian function, we have.

$$\delta L = \sum_{i=k+1}^{n} \partial L/\partial q_i \cdot \delta q_i + \sum_{i=1}^{k} \partial L/\partial \dot{q}_1 \, \delta \dot{q}_1 + \sum_{i \neq 1}^{k} \partial R/\partial \beta_i \, \delta \beta_i + \partial L/\partial t \cdot \delta t \rightarrow (2)$$

Since, $R = L - \sum_{i=1}^{k} \beta_i \dot{q}_1$ obtain Lagranges equation involving Routhian function in terms of generalized

$$\Rightarrow \delta R = \delta\left(L - \sum_{i=1}^{k} \beta_i \dot{q}_i\right)$$

$$= \delta L - \delta\left(\sum_{i=1}^{k} \beta_i \dot{q}_i\right)$$

$$\delta R = \delta L - \sum_{i=1}^{k} \dot{q}_i \, \delta\beta_i - \sum_{i=1}^{k} \beta_i \, \delta\dot{q}_i \longrightarrow$$

From eqn ①, (2) and (3) $\Rightarrow$

$$\Rightarrow \sum_{i=k+1}^{n} \partial R/\partial q_i \, \delta q_i + \sum_{i=k+1}^{n} \partial R/\partial \dot{q}_i \, \delta q_i + \sum_{i=1}^{k} \partial R/\partial \beta_i \, \delta\beta_i$$

$$= \sum_{i=k+1}^{n} \partial L/\partial q_i \, \delta q_i + \sum_{i=1}^{n} \partial L/\partial \dot{q}_i \, \delta \dot{q}_i + \partial L/\partial t \, \delta t - \sum_{i=1}^{k}$$

$$- \sum_{i=1}^{k} \beta_i \, \delta \dot{q}_i .$$

Comparing the coefficients

(i) $\partial L/\partial q_i = \partial R/\partial q_i$ and $\partial L/\partial \dot{q}_i = \partial R/\partial \dot{q}_i$, $\forall$

(ii) $\dot{q}_i = -\partial R/\partial \beta_i$, $i = 1, 2, \ldots k \longrightarrow$ ⑤

(iii) $\partial L/\partial t = \partial R/\partial t \longrightarrow$ ⑥

we know that, the Lagrangian equation of motion is,

$$\frac{d}{dt}\left(\partial L/\partial \dot{q}_i\right) - \partial L/\partial q_i = 0, \ \forall \ i = 1, 2, \ldots n.$$

Therefore by equation (4) and $\Rightarrow$

$$\frac{d}{dt}\left(\partial R/\partial \dot{q}_i\right) - \partial R/\partial q_i = 0, \ \forall \ i = k+1$$

These equations are of the form of Lagrangian equation with the Routhian function use in the of the Lagrangian function. That their are only n-k second order differential equations in the ignorable variables.

Thus the Routhian Procedure has succeded eliminating the ignorable co-ordinates from the of motion with the number of degrees of freedom has been reduced to n-k.

Frequently there is no need to solve for the ignorable co-ordinates. But if eqn (7) has been solved for n-k non-ignorable co-ordinates then eqn (5)

$\Rightarrow \dot{q}_i = -\partial R/\partial \beta_i \,, \; i = 1 \text{ to } k.$

i.e) $\dfrac{d q_i}{dt} = -\partial R/\partial \beta_i \,, \; i = 1 \text{ to } k$

$\Rightarrow \; d q_i = -\partial R/\partial \beta_i \cdot dt \,, \; i = 1 \text{ to } k$

$\Rightarrow \; q_i = -\int \partial R/\partial \beta_i \, dt \,, \; \forall\, i = 1, 2, \dots k.$

Kepler's problem with Routhian method: Rewrite the earlier kepler's Problem.

we start this problem

(II) $\Rightarrow r^2 \dot{\theta} = \beta$

Here clearly, θ is the ignorable co-ordinates.

since, $R = L - \sum_{i=1}^{k} \beta_i \dot{q}_i$

$r^2\dot{\theta} = \beta$
$\Rightarrow \dot{\theta} = \beta/r^2$

$R = L - \beta\dot{\theta}$

[$\because \dot{\theta}$ is corresponding ignorable co-ordinate θ]

$\therefore R = \frac{1}{2}\dot{r}^2 + \frac{1}{2} r^2 \dot{\theta}^2 + \mu/r - \beta\dot{\theta}$

$= \frac{1}{2}\dot{r}^2 + \frac{1}{2} r^2\dot{\theta}^2 + \mu/r - \beta(\beta/r^2)$

$= \frac{1}{2}\dot{r}^2 + \frac{1}{2} r^2 (\beta/r^2)^2 + \mu/r - \beta(\beta/r^2)$

$= \frac{1}{2}\dot{r}^2 + \frac{1}{2} r^2 \beta^2/r^4 - \beta^2/r^2 + \mu/r$

$R = \frac{1}{2}\dot{r}^2 - \frac{1}{2}\beta^2/r^2 + \mu/r$

since the differential equation of the Routhian function,

$\dfrac{d}{dt}\left(\partial R/\partial \dot{q}_i\right) - \partial R/\partial q_i = 0$

$\Rightarrow -\frac{1}{2}\beta^2/r^2 + \mu/r$
$\Rightarrow -\frac{1}{2}\beta^2 r^{-2} + \mu r^{-1}$

$\Rightarrow \dfrac{d}{dt}\left(\partial R/\partial \dot{r}\right) - \partial R/\partial r = 0$

$\partial R/\partial \dot{r} = \frac{2}{2}\dot{r} = \dot{r} \Rightarrow \dfrac{d}{dt}\left(\partial R/\partial \dot{r}\right) = \ddot{r}$

$-\frac{1}{2}\beta^2 r^{-2}$
$-\frac{1}{2}\beta^2 \cdot -2 r^{-2-1}$
$+\beta r^{-3} = \beta/r^3$
$\mu r^{-1} = -1 \mu r^{-1-1}$

$\partial R/\partial r = +\beta^2/r^3 - \mu/r^2$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

$$\boxed{\ddot{r} - \frac{\beta^2}{r^3} + \frac{\mu}{r^2} = 0}$$

$$\Rightarrow \ddot{r} - r\dot{\theta}^2 + \mu/r^2 = 0 \qquad \because \dot{\theta} = \beta$$

Conservative system:- (k.E + p.E is constant)

A conservative forces field has the prop that,

(1) The generalized force component are obt from the potential energy function $V(q, t)$ by

$Q_i = \partial V/\partial q_i$ → generalized force, where $V(q)$ is a function of the con-ration only,

(2) The integral $W = \int_A^B Q \cdot dq = \sum_{i=1}^{n} \int_{A_i}^{B_i} Q_i\, dq_i$ is independ the path taken between the given 'n' point in space.

If no other forces do work on the system the total mechanical energy is conserved and hence the system is called conservative system.

In this case, the total energy $E(q, \dot{q}) = T + V$ is integral of motion.

Definition:- Conservative.

A system to be conservative if it satisfy the following conditions,

(i) The standard form of Lagranges equation (Holo-mic or non-holonomic) applies.

(ii) The Lagrangian function L is not an explicit fu of time 't'

(iii) Any constrained equation can be expressed in differential form $\sum_{t=1}^{n} a_{jt} \cdot dq_t = 0 \quad \forall\ j = 1$ to m

i.e), All the co-efficient $a_{jt} = 0\ \forall j$ an

Derive the Jacobi Integral or Energy Integral (or)

Prove that $\sum_{i=1}^{n} \partial L/\partial \dot{q}_i \cdot \dot{q}_i - L = h$, where h is a constant

(or)

obtain the energy integral for a conservative system.

Proof:

Since the standard non-holonomic form of Lagranges equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \partial L/\partial q_i = \sum_{j=1}^{m} \lambda_j \cdot a_{ji} \longrightarrow (1) \ \forall\ i = 1 \text{ to } n.$$

where $L(q, \dot{q})$ is not an explicit function of time 't';

The non, ie, m-equation of constrained in the form can be written as,

$$\sum_{i=1}^{n} a_{ji}\, dq_i = 0 \longrightarrow (2)$$

where a's are functions of q and t. Also since the Lagrangian $L = L(q, \dot{q})$

i.e) L is a function of q and $\dot{q}$.

Therefore the total derivative $\frac{dL}{dt} = \sum_{i=1}^{n} \partial L/\partial q_i \cdot \frac{dq_i}{dt} + \sum_{i=1}^{n} \partial L/\partial \dot{q}_i \frac{d\dot{q}_i}{dt}$.

$$= \sum_{i=1}^{n} \partial L/\partial q_i \cdot \dot{q}_i + \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \longrightarrow (3)$$

From (1) $\Rightarrow$ $\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \sum_{j=1}^{m} \lambda_j \cdot a_{ji} \longrightarrow (4)$

Sub (4) in (3), we have

$$\frac{dL}{dt} = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i + \sum_{i=1}^{n} \left\{ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \sum_{j=1}^{m} \lambda_j a_{ji} \right\} \dot{q}_i$$

$$= \sum_{i=1}^{n} \partial L/\partial \dot{q}_i\, \ddot{q}_i + \sum_{i=1}^{n} \frac{d}{dt}\left(\partial L/\partial \dot{q}_i\right) \dot{q}_i - \sum_{i=1}^{n} \left\{ \sum_{j=1}^{m} \lambda_j a_{ji} \right\} \dot{q}_i \quad (5)$$

consider double summation in (5),

$$\sum_{j=1}^{m} \lambda_j \sum_{i=1}^{n} a_{ji} \dot{q}_i \longrightarrow (6)$$

consider the equation (2), $\sum_{i=1}^{n} a_{ji} \dot{q}_i = 0$

$\therefore$ Equation (2) and (6) $\Rightarrow \sum_{j=1}^{m} \lambda_j \sum_{i=1}^{n} a_{ji} \dot{q}_i = 0$

i.e) $\sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_j a_{ji} \dot{q}_i = 0$

$\therefore$ Eqn(5) $\Rightarrow$ $\frac{dL}{dt} = \sum_{i=1}^{n} \partial L/\partial \dot{q}_i \ddot{q}_i + \sum_{i=1}^{n} \frac{d}{dt} \left(\partial L/\partial \dot{q}_i \right.$

$= \sum_{i=1}^{n} \frac{d}{dt} \left[\partial L/\partial \dot{q}_i \dot{q}_i \right)$ → uv form

$\frac{dL}{dt} = \frac{d}{dt} \left\{ \sum_{i=1}^{n} \partial L/\partial \dot{q}_i \dot{q}_i \right\}$

$\int^{ing}$ with respect to t,

$dL = \frac{d}{dt} \left\{ \sum_{i=1}^{n} \partial L/\partial \dot{q}_i \dot{q}_i \right\} \cdot dt \Rightarrow \int dL = \int d \left(\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \right.$

$\Rightarrow L = \sum_{i=1}^{n} \partial L/\partial \dot{q}_i \dot{q}_i + C$

$\Rightarrow \sum_{i=1}^{n} \partial L/\partial \dot{q}_i \dot{q}_i - L = h$

where $h = -C$ is a constant

Thus we have obtained a constant of the motion which is know as a Jacobi integral or Energy integral.

Those integral of motion exists for all cons system.

(*) Prove that for any conservative system

Liouville's system :-

As we consider the problem of finding the integral of motion or constant of motion, a question aries consorning what characteristic of a system make it possible complotoly solve for its motion by a quadrature.

The general answer of this question is not know but there are examples of some system which are seporable and therefore are capable of being solved by quadrature.

First, since a system having n degrees of freedom required 2n integrals of motion for a complete solution

For a standard holonomic system the presence of ignorable co-ordinates permit the reduction of the number of degrees of freedom by the Routhian procedure.

If a system is conservative, the energy integral 'h' is immediately available. Hence it can been seen that any conservative holonomic system with 'n' degrees of freedom, n-1 ignorable coordinate can be integral complotoly by quadrature.

we find that 2(n-1) constant are obtained by ignoration of co-ordinate and the energy integral given an equation of the form $\dot{q} = f(q)$ which can be integrated to get a complete solution.

[If a conservative holonomic system does not have a sufficient number of ignorable co-ordinates it may be seperable, if it is an orthogonal system]

As example, suppose that $T = \frac{1}{2} f \sum_{i=1}^{n} \dot{q}_i^2$ — (1) and the potential energy $V = \frac{1}{f} \sum_{i=1}^{n} v_i(q_i)$ — (2) where,

$$f = \sum_{i=1}^{n} f_i = \sum_{i=1}^{n} f_i(q_i) \quad \text{—— (3)}$$

We will show ~~that~~ the system separable.

Let the Lagrangian function and Lagrangian equation.

of motion we have the form.

$$\frac{d}{dt}\left(\partial L/\partial \dot{q}_1\right) - \partial L/\partial q_1 = 0 \rightarrow (4)$$

$$\Rightarrow \frac{d}{dt}\left(\partial T/\partial \dot{q}_1\right) - \partial T/\partial q_1 + \partial V/\partial q_1 = 0 \quad -(5)$$

From eqn (1), (2) and (3) we have

$$\partial T/\partial \dot{q}_1 = \frac{1}{\cancel{2}} f \cancel{2} \dot{q}_1 = f\dot{q}_1$$

$$\partial T/\partial q_i = \frac{1}{2}\frac{\partial f_i}{\partial q_i}\sum_{i=1}^{n} \dot{q}_i^2$$

$$\partial V/\partial q_i = \frac{1}{f}\partial V_i/\partial q_i - \frac{1}{f^2}\frac{\partial f_i}{\partial q_i}\sum_{i=1}^{n} V_i(q_i)$$

$$= 1/f\frac{\partial V_i}{\partial q_i} - 1/f\frac{\partial f_i}{\partial q_i}\left(\frac{1}{f}\sum_{i=1}^{n} V_i(q_i)\right)$$

$$= \frac{1}{f}\frac{\partial V_i}{\partial q_i} - \frac{1}{f}\frac{\partial f_i}{\partial q_i}\cdot V. \Rightarrow \frac{1}{f}\frac{\partial V_i}{\partial q_i} - \frac{V}{f}$$

Substitute these values to eqn (4), we get

$$(4) \Rightarrow \frac{d}{dt}\left(f\dot{q}_i\right) - \frac{1}{2}\frac{\partial f_i}{\partial q_i}\sum_{i=1}^{n}\dot{q}_i^2 + \frac{1}{f}\frac{\partial V_i}{\partial q_i} - \frac{V}{f}\frac{\partial f_i}{\partial q_i}$$

Now it is a natural system so that its ene integral is constant

ie) $T+V = h$ (a constant)

$$\frac{1}{2}f\sum_{i=1}^{n}\dot{q}_i^2 + V = h$$

$$\Rightarrow \frac{1}{2}\sum_{i=1}^{n}\dot{q}_i^2 = \frac{h-V}{f} \longrightarrow (6)$$

Substitute eqn (6) in equation (5),

$$(5) \Rightarrow \frac{d}{dt}\left(f\dot{q}_i\right) - \partial f_i/\partial q_i\left(\frac{h-V}{f}\right) + \frac{1}{f}\frac{\partial V_i}{\partial q_i} - \frac{V}{f}\frac{\partial f_i}{\partial q_i}$$

$$\Rightarrow \frac{d}{dt}\left(f\dot{q}_i\right) - \frac{\partial f_i}{\partial q_i}\frac{h}{f} + \frac{\partial f_i}{\partial q_i}\frac{V}{f} + 1/f\frac{\partial V_i}{\partial q_i} - \frac{V}{f}\frac{\partial f_i}{\partial q_i}$$

$$\Rightarrow \frac{d}{dt}(f\dot{q}_i) - \frac{\partial f_i}{\partial q_i}\frac{h}{f} + \frac{1}{f}\frac{\partial v_i}{\partial q_i} = 0 \quad \text{——— (7)}$$

Now i^{th} eqn is multiplied by $2f\dot{q}_i$

$$\Rightarrow 2f\dot{q}_i \frac{d}{dt}(f\dot{q}_i) - 2f\dot{q}_i \frac{h}{\cancel{f}}\frac{\partial f_i}{\partial q_i} + 2f\dot{q}_i\left(+\frac{1}{\cancel{f}}\frac{\partial v_i}{\partial q_i}\right) = 0$$

$$\frac{d}{dt}(f^2\dot{q}_i^2) = 2h\frac{\partial f_i}{\partial q_i}\cdot\frac{dq_i}{dt} - 2\frac{\partial v_i}{\partial q_i}\cdot\frac{dq_i}{dt}$$

$$= 2h\frac{d}{dt}f_i - 2\frac{d}{dt}v_i$$

$$\frac{d}{dt}(f^2\dot{q}_i^2) = 2\frac{d}{dt}[hf_i - v_i]$$

$$\boxed{\begin{aligned} &d/dt(f\dot{q}_i) = f\frac{d\dot{q}_i}{dt} + \dot{q}_i\frac{df}{dt} \\ &2f\dot{q}_i\, d/dt(f\dot{q}_i) = 2f\dot{q}_i\frac{d\dot{q}_i}{dt} + 2\dot{q}_i^2 f\frac{df}{dt} \\ &= d/dt(f^2\dot{q}_i^2) \end{aligned}}$$

Integrating on both sides, we get,

$$f^2\dot{q}_i^2 = 2[hf_i - v_i + c_i] \;\forall\; i = 1 \text{ to } n \quad \text{——— (8)}$$

where c_i is a constant of integration.

To show that :- $\sum_{i=1}^{n} c_i = 0$

Take the summation of equation (8),

$$\sum_{i=1}^{n} f^2\dot{q}_i^2 = 2[h\sum_{i=1}^{n} f_i - \sum_{i=1}^{n} v_i + \sum_{i=1}^{n} c_i] \quad (\text{by equation (2)})$$

$$f\,[f\sum_{i=1}^{n}\dot{q}_i^2] = 2[hf - fv + \sum_{i=1}^{n} c_i]$$

$$2Tf = 2[h - v]f + 2\sum_{i=1}^{n} c_i \quad (\text{by eqn (1)})$$

$$\Rightarrow 2\sum_{i=1}^{n} c_i = 0$$

$$\Rightarrow \sum_{i=1}^{n} c_i = 0$$

The sum of all c_i is zero and hence 'c_i' and 'h' are together comprise 'n' independent constant

$$\text{Eqn (8)} \Rightarrow \dot{q}_i^2 = \frac{2[hf_i - v_i + c_i]}{f^2}$$