$\frac{}{mb^2+I_t}$

UNIT - IV

HAMILTON'S EQUATION.

Hamilton's principle

Definition :-

Stationary values of functions :- (32)

Consider a function $f(q_1, q_2, \ldots q_n)$ is assumed to be continuous through the second order partial derivatives. The first variation of 'f' at the reference point q_0 is

$$\delta f = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \right)_0 \delta q_i \longrightarrow (1)$$

where the δq's are the variations in the individual q's and can be consider as virtual displacement.

The necessary and sufficient condition that f. having a stationary value at q_0 is that $\delta f = 0$. for all geometrical possible δq's where $q = q_0 + \delta q \longrightarrow (2)$

For the case in which the δq's are indepondent an reversible

$$\Rightarrow \left(\frac{\partial f}{\partial q_i} \right)_0 = 0 \longrightarrow (3) \quad \forall \; i = 1, 2, \ldots n$$

consider the 2nd variation of the function f about the stationary point q_0 we have,

$$\delta^2 f = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{\partial^2 f}{\partial q_i \partial q_j} \right)_0 \delta q_i \, \delta q_j \longrightarrow (4)$$

Take

$$k_{ij} = \left(\frac{\partial^2 f}{\partial q_i \partial q_j} \right)_0 \longrightarrow (5)$$

forms the elements of the symmetric the matrix

If k is +ve definite then q_0 is the local minimum.

k is -ve definite q_0 is the local maximum.

If k is indefinite q_0 is a saddle point.

Constrained Stationary values:

Let us consider the condition necessary for a stationary values of the functions $f(q_1, q_2, \cdots q_n)$ subject to the independent constraint equation $\phi_j(q_1, q_2, \cdots q_n) \rightarrow (6)$ for $j = 1, 2, \cdots n$

Where we assume that ϕ's are continuous through the 2nd order partial derivatives.

Since the necessary and sufficient condition that have stationary value at q_0 is $\delta f = 0$ (or)

$$\sum_{i=1}^{n} \left(\partial f / \partial q_i \right)_0 \cdot \delta q_i = 0 \longrightarrow (7)$$

where the refrence configuration q_0 satisfy the eqn (6). The δq's are no longer independent but conform to the m equation.

$$\delta \phi_j = \sum_{i=1}^{n} \left(\frac{\partial \phi_j}{\partial q_i} \right) \cdot \delta q_i = 0, \quad j = 1 \text{ to } m.$$

Note:

Since δq's are not independent $\left(\frac{\partial f}{\partial q_i} \right) = 0$ is longer apply. It is possible using eqn (6) eliminate $m q$'s infavour of the remaining $(n-m)$ q's and find this stationary values of the resulting constrained function.

Using Lagrange's Multipliers Method: (3)

To problems involving constraints maxima (or) minima is to consider a provariation of an argument function.

$$F = F[q_1, q_2, \ldots, q_n, \lambda_1, \lambda_2, \ldots \lambda_m]$$

$$= f(q_1, q_2, \ldots, q_n) + \sum_{i=1}^{n} \lambda_j \phi_j (q_1, q_2, \ldots q_n)$$

$$+ \sum_{j=1}^{m} \lambda_j \phi_j (q_1, q_2, \ldots, q_n)$$

Since n q's and m λ's are independent variables the necessary and sufficient condition for F to be stationary

$$\left(\frac{\partial F}{\partial q_i}\right)_0 = 0, \quad i = 1 \text{ to } n \text{ and}$$

$$\left(\partial F / \partial \lambda_j\right)_0 = 0, \quad \forall\, i = 1 \text{ to } m$$

is equivalently.

$$\sum_{i=1}^{n} \left\{ \left(\frac{\partial f}{\partial q_i}\right)_0 + \sum_{j=1}^{m} \lambda_j \left(\frac{\partial \phi_i}{\partial q_i}\right)_0 \right\} \cdot \delta q_i = 0$$

Since

$$f \cdot \phi_j \Rightarrow F = f + \sum \lambda \phi_j$$

$$\delta F = \delta f + \sum \lambda_j \delta \phi_j \; \delta \phi_j.$$

$$= \sum_{i=1}^{n} \left(\frac{\partial \phi_i}{\partial q_i}\right) \cdot \delta q_i = 0$$

$$\delta f + \sum \lambda_j \delta \phi_j = \sum_{i=1}^{n} \left(\partial F / \partial q_i\right) \delta q_i + \sum_{i=1}^{n} \lambda_j \sum_{i=1}^{n} \left(\frac{\partial \phi_i}{\partial q_i}\right) \delta q_i$$

Since λ is arbitrary we can find a set of n λ's such that the co-efficient if each δq_i is zero.

$$\Rightarrow \left[\left(\partial f / \partial q_i\right)_0 + \sum_{j=1}^{m} \lambda_j \left(\frac{\partial \phi_i}{\partial q_i}\right)\right] = 0$$

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Problem :- (*) 5M.

Find the stationary value of a function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints.

$$\phi(x, y, z) = x + y + z - 1 = 0.$$

Solution :-

since the problem has only one constraint (i.e) Lagrange's Multipliers $F = f + \lambda \phi$.

$$F = x^2+y^2+z^2 + \lambda(x+y+z-1)$$

co-ordinates contains a stationary values.

(i.e) the condition are stationary values are

$\frac{\partial F}{\partial x} = 0 \Rightarrow 2x+\lambda = 0 \Rightarrow x = -\lambda/2 \rightarrow (1)$

$\frac{\partial F}{\partial y} = 0 \Rightarrow 2y+\lambda = 0 \Rightarrow y = -\lambda/2 \rightarrow (2)$

$\partial F/\partial z = 0 \Rightarrow 2z+\lambda = 0 \Rightarrow z = -\lambda/2 \rightarrow (3)$

$\partial F/\partial \lambda = 0 \Rightarrow x+y+z-1 = 0 \longrightarrow (4)$

Substitute (1), (2), (3) in (4)

Equation (4) $\Rightarrow x+y+z-1=0$

$$(-\lambda/2)+(-\lambda/2)+(-\lambda/2)-1=0$$

$$\frac{-\lambda-\lambda-\lambda}{2} = 1 \Rightarrow \frac{-3\lambda}{2} = 1 \Rightarrow \lambda = -\frac{2}{3}$$

Substitute $\lambda = -2/3$ in (1), (2), (3) we get

(1) $\Rightarrow x = -\lambda/2 = -(-2/3) \times 1/2 = 1/3$

$x = 1/3$

(2) $\Rightarrow y = -\lambda/2,\ y = 1/3 \quad (\because \lambda = -2/3)$

(3) $\Rightarrow z = -\lambda/2,\ z = 1/3$.

$\therefore$ The stationary value are $(1/3, 1/3, 1/3)$.

Ex : 4-1

Find the stationary values of a function $f(x,y,z) = z$ subject to the constraints.

$$\phi_1(x,y,z) = x^2+y^2+z^2-4=0$$

$$\phi_2(x,y,z) = xy-1=0.$$

Solution:

This is corresponding to finding the highest and lowest points as the curve formed by the intersection sphere and a hyperbolic cylinder.

The argumented function F is given by

$f + \lambda_1 \phi_1 + \lambda_2 \phi_2$.

$z + \lambda_1 (x^2 + y^2 + z^2 - 4) + \lambda_2 (xy - 1)$

the condition for stationary values are

$\frac{\partial F}{\partial x} = 0 \Rightarrow 2x\lambda_1 + \lambda_2 y = 0 \rightarrow (1)$

$\frac{\partial F}{\partial y} = 0 \Rightarrow 2y\lambda_1 + \lambda_2 x = 0 \rightarrow (2)$

$\frac{\partial F}{\partial z} = 0 \Rightarrow 1 + 2z\lambda_1 = 0 \longrightarrow (3)$

$\frac{\partial F}{\partial \lambda_1} = x^2 + y^2 + z^2 - 4 = 0 \longrightarrow (4)$, $\frac{\partial F}{\partial \lambda_2} = xy - 1 = 0 \longrightarrow (5)$

Equation (1) $\times y \Rightarrow 2xy\lambda_1 + y^2\lambda_2 = 0 \longrightarrow (A)$ (6)

From equation (5) $\Rightarrow xy = 1$

$$2\lambda_1 + y^2\lambda_2 = 0$$

$$y^2 = \frac{-2\lambda_1}{\lambda_2}$$

Equation (2) $\times x \Rightarrow 2xy\lambda_1 + x^2\lambda_2 = 0$

put $xy = 1$

$$2\lambda_1 + x^2\lambda_2 = 0 \Rightarrow x^2 = \frac{-2\lambda_1}{\lambda_2} \longrightarrow (7)$$

Equation (6) and (7), becomes $\Rightarrow x^2 = y^2$

Equation (5) $\Rightarrow xy = 1$

$x^2y^2 = 1$ (put $y^2 = x^2$)

$x^4 = 1$

$x = \pm 1$

III^{ly} $\Rightarrow xy = 1 \Rightarrow x^2y^2 = 1 \Rightarrow y^4 = 1$, $\therefore y = \pm 1$

Equation (4) $\Rightarrow x = \pm 1$ & $y = \pm 1$ then $1 + 1 + z^2 - 4 = 0$

$z^2 - 2 = 0$

$\Rightarrow z^2 = 2$

$z = \pm\sqrt{2}$

$\therefore$ The stationary points are $(1, 1, \sqrt{2})$; $(-1, -1, \sqrt{2})$, $(1, 1, -\sqrt{2})$, $(-1, -1, -\sqrt{2})$.

The first two points are constrained maximum of f. The other 2 point are constrained minimum of f.

The Lagrangian Multipliers are.

Equation (5) => $1+2z\lambda_1=0 \Rightarrow \lambda_1=-\frac{1}{2z}$

$$\lambda_1=\pm\frac{1}{2\sqrt{2}}$$

(4) => $2\lambda_1+\lambda_2=0$

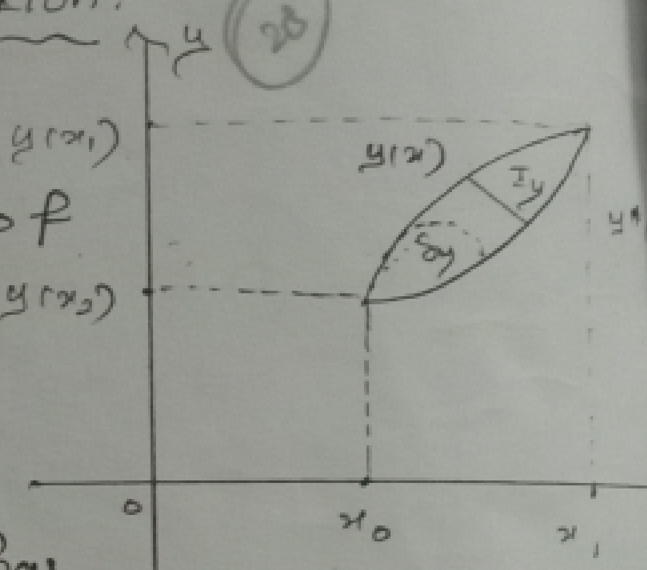
$$\lambda_2=-2\lambda_1$$

$$\lambda_2=\pm\frac{1}{\sqrt{2}}.$$

(X) Find the Stationary values of definite integral (or) derived Euler's - Lagrange's equation:

10M.

Proof:



Let us find the stationary value of definite integral

$$I=\int_{x_0}^{x_1} f(y(x), y'(x), x)\,dx \longrightarrow (1)$$

where $y'(x)=\frac{dy}{dx}$ & $f(y, y', x)$ has two continuous derivative in each of it's argument and the limit x_0, x_1 are fixed.

Now we have to find the function $y^*(x)$ which the stationary value of I as compared to it's value other neighbouring function.

Thus we have $y(x)=y^*(x)+\delta y(x) \longrightarrow (2)$

where $\delta y(x)$ is the small variation in y. It is convenient to represents δy in the form $\delta y=\alpha\eta(x) \longrightarrow (3)$

where η is the arbitrary function having the requi smoothness and α is a parameters independent of x.

Hence for any given $\eta(x)$ we can consider the varied curve y be a function of α and x.

$$\therefore y(\alpha, x)=y^*(x)+\alpha\eta(x) \longrightarrow (4)$$

...us make the additional assumptions the variation $\delta y = 0$ ...the end points of x_0 and x_1 which given.

$$\alpha\eta(x_0) = 0 \ \& \ \alpha\eta(x_1) = 0.$$

$$\text{(i.e)} \quad \eta(x_0) = 0 \ \& \ \eta(x_1) = 0 \rightarrow (5)$$

...integral that, integral I is a function of α for any ...$\eta(x)$.

...us a necessary condition that $y^*(x)$ results in a stationary ...e of I, variation $\delta I = 0$.

(or)

$$\text{(i.e)} \ \delta f = 0, \ \& \ \delta I = 0$$

$$\left(\frac{dI}{d\alpha}\right)_{\alpha=0} \delta r = 0$$

...ce x_0 and x_1 are independent of α.

$$I = \int_{x_0}^{x_1} f(y, y', x)\,dx$$

$$\therefore \frac{dI}{d\alpha} = \int_{x_0}^{x_1} \left(\frac{\partial f}{\partial y} \cdot \frac{dy}{d\alpha} + \frac{\partial f}{\partial y'} \cdot \frac{dy'}{d\alpha} \right) dx = 0$$

$$= \int_{x_0}^{x_1} \left(\frac{\partial f}{\partial y} \eta(x) + \frac{\partial f}{\partial y'} (\eta'(x)) \right) dx = 0$$

$$\frac{dI}{d\alpha} = \int_{x_0}^{x_1} \frac{\partial f}{\partial y} \eta(x)\,dx + \int_{x_0}^{x_1} \frac{\partial f}{\partial y'} \eta'(x)\,dx = 0 \longrightarrow (6)$$

Consider the 2^{nd} integral

$$\int_{x_0}^{x_1} \frac{\partial f}{\partial y'} \eta'(x)\,dx = \int_{x_0}^{x_1} \frac{\partial f}{\partial y'}\, d(\eta(x))$$

$$= \left[\frac{\partial f}{\partial y'} \eta(x) \right]_{x_0}^{x_1} - \int_{x_0}^{x_1} \eta(x) \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) dx$$

$$= -\int_{x_0}^{x_1} \eta(x) \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) dx \rightarrow (7) \quad [\because \text{using } (5)]$$

Equation (6) & (7) becomes

$$\Rightarrow \int_{x_0}^{x_1} \frac{\partial f}{\partial y} \eta(x)\,dx - \int_{x_0}^{x_1} \eta(x) \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) dx = 0$$

$$\Rightarrow \int_{x_0}^{x_1} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] \eta(x)\, dx = 0$$

Since $\eta(x)$ is an arbitrary.

$\Rightarrow$ The necessary condition for the integral to be zero that

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \longrightarrow (8)$$

For any curve $y = y^*(x)$.

Equation (8) is known as "Euler's Lagrange's equation".

State and prove Brachistochrone Problem using Calculus of Variation : (x) 10M

Statement :

For the curve going two point along which is a particle starting from ~~lost~~ rest and sliding down the curve without friction under the influence of the uniform gravitational force will reach the end of the curve in a minima of a time.

Proof :-

Take the stationary point as the origin 'O'. ox in the direction of gravitational force. Let oy in the horizontal in the plane of motion.

Let (x_1, y_1) be the another given point where the particle and with the motion. Let v be the velocity of the particle at time T. Let (x, y) be the position at the time t and it is the co-ordinate then the kinetic energy $T = \frac{1}{2} m v^2$ & the potential energy $P = -mgx$.

By the principle of conservation of energy =

kinetic energy (T) + potential energy (P)

= E (Constant)

$$\therefore \frac{1}{2} m v^2 - mgx = c \text{ (a constant)} \qquad (\because v = 0,\ x = c$$

$$= 0 \text{ (at origin o)}$$

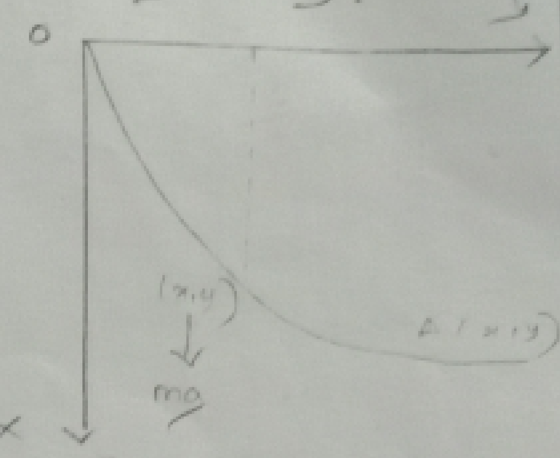
$\Rightarrow \frac{1}{2} m v^2 = mgx$

$v^2 = 2gx$, $v = \sqrt{2gx} \longrightarrow (1)$

we know that,

$v = \frac{ds}{dt} \longrightarrow (2)$ (rate of change of displacement)

(i) The length of the arc the part is given by.



$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$ds = \sqrt{1+y'^2}\,dx \longrightarrow (3)$$

Equation (2) $\Rightarrow dt = \frac{ds}{v} \Rightarrow t = \int \frac{ds}{v}$

$$t = \int_0^x \frac{\sqrt{1+y'}}{\sqrt{2gx}}\,dx$$

$$I = \int f(y, y', x)\,dx$$

$$\frac{df}{dy} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

Since the time equal to reach the point x, y, is

$$t = \int_0^x \sqrt{\frac{1+y'^2}{2gx}}\,dx$$

Thus we have, $f(y, y'x) = \sqrt{\frac{1+y'^2}{2gx}}$.

∴ For minimum value for the corresponding the stationary value. Hence by Euler lagrange equation,

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \longrightarrow (4)$$

Now, $\frac{\partial f}{\partial y} = 0$

$$\frac{\partial f}{\partial y'} = \frac{1}{\sqrt{2gx}} \cdot \left(+\frac{1}{2}\right)(1+y'^2)^{\frac{1}{2}-1} \cdot (\cancel{2}y'y'') = \frac{y'y''}{\sqrt{2gx}\sqrt{1+y'^2}}$$

Equation (4) $\Rightarrow \frac{d}{dx}\left(\frac{y'y''}{\sqrt{2gx}\sqrt{1+y'^2}}\right) = 0$

$$\Rightarrow \frac{d}{dx}\left(\frac{y'}{\sqrt{2gx(1+y'^2)}}\right) = 0$$

$$\Rightarrow \frac{y'}{\sqrt{2gx(1+y'^2)}} = c$$

$$y' = c\sqrt{2gx(1+y'^2)}$$

$$y'^2 = c^2(2gx)(1+y'^2)$$

$$y'^2 = c^2 2gx + c^2 2gx\, y'^2$$

$$y'^2(1-2c^2gx) = c^2 2gx \Rightarrow y'^2 = \frac{c^2 2gx}{(1-2gc^2x)}$$

$$y' = \sqrt{\frac{2c^2gx}{1-2c^2gx}}$$

$$= \sqrt{\frac{2c^2gx}{2c^2g\left(\frac{1}{2gc^2}-x\right)}}$$

$$= \sqrt{\frac{x}{\frac{1}{2gc^2}-x}}$$

$$\left(\because 2a = \frac{1}{2gc^2}\right) \qquad = \sqrt{\frac{x}{2a-x}}$$

$$\frac{dy}{dx} = \sqrt{\frac{x}{2a-x}}$$

$$dy = \sqrt{\frac{x}{2a-x}}\,dx \Rightarrow$$

$$y = \int_0^x \sqrt{\frac{x}{2a-x}}\,dx$$

$x = a(1-\cos\theta)$, $dx = +a\sin\theta\, d\theta$

$$y = +\int \sqrt{\frac{a(1-\cos\theta)}{2a-a(1-\cos\theta)}}\, a\sin\theta\, d\theta = \int \sqrt{\frac{a(1-\cos\theta)}{a(1+\cos\theta)}} \cdot a\sin\theta$$

$$= a\int \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \cdot \sqrt{\sin^2\theta}\, d\theta$$

$$= a\int \sqrt{\frac{1-\cos\theta}{1+\cos\theta}(1-\cos^2\theta)}\, d\theta$$

$$= a\int_0^\theta \sqrt{\frac{1-\cos\theta}{\cancel{1+\cos\theta}}(1-\cos\theta)(\cancel{1+\cos\theta})}\, d\theta$$

$$= a\int_0^\theta \sqrt{(1-\cos\theta)^2}\, d\theta$$

$a\int_0^{\theta}(1-\cos\theta)\,d\theta \Rightarrow a[\theta-\sin\theta]+c_1$

When $x=0$, $a(1-\cos\theta)=0$

$\cos\theta = 1$ $\qquad y=0$

$\theta=\cos^{-1}(1)$ $\qquad c_1=0$

$\theta=0$ $\qquad \therefore y=+a[\theta-\sin\theta]$.

$y=+a[\theta-\sin\theta]+c_1$

which is the path of the equation and hence the parametric representation of the curve

$$x=a(1-\cos\theta)$$
$$y=a(\theta-\sin\theta)$$
$$dx=a\sin\theta\,d\theta \text{ which is cycloid}$$

$\therefore$ The curve of minimum time is cycloid.

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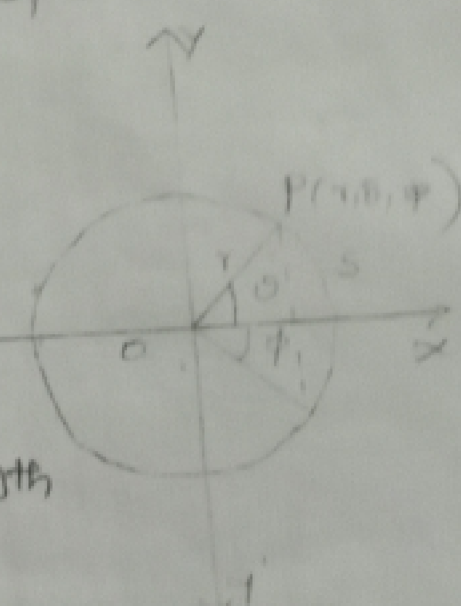
State and prove Geodesic problem using Calculus variation (or) Find the shortest path between 2 points in a given space (or) Find the path of minimum length between two given points on the two-dimensional of a sphere of radius r.



Proof:

Let (r,θ,ϕ) be the spherical polar co-ordinates of any point on the surface of a sphere with centre at origin and radius r.

The differential elements of the arc length is given by,

$$ds^2 = r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2$$
$$= r^2 d\theta^2\left[1+\sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2\right]$$
$$ds = r\sqrt{1+\sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2}\,d\theta$$
$$s = r\int_{\theta_0}^{\theta}\sqrt{1+\sin^2\theta\,\phi'^2}\,d\theta \quad \text{where } \phi'=\frac{d\phi}{d\theta}$$
$$f(y,y',x)=\sqrt{1+\sin^2\theta\,\phi'^2}$$
$$f(r,\theta,\phi)=\sqrt{1+\sin^2\theta\,\phi'^2}$$

Euler's -lagrange's equation for the stationary values of definite integral is,

$$\frac{\partial f}{\partial \phi}-\frac{d}{d\theta}\left(\frac{\partial f}{\partial \phi'}\right)=0 \quad , \frac{\partial f}{\partial \phi}=0$$

$$\frac{\partial f}{\partial \phi'}=\frac{1}{2}(1+\sin^2\theta\,\phi'^2)^{-1/2}\,2\phi'\phi''$$

$$=\frac{\phi'\phi''\sin^2\theta}{\sqrt{1+\sin^2\theta\cdot\phi'^2}}$$

$$\frac{d}{d\theta}\left[\frac{\partial f}{\partial \phi'}\right]=0 \qquad \because \partial f/\partial\phi=0$$

$$\therefore \frac{d}{d\theta}\left(\frac{\phi'\sin^2\theta}{1+\sin^2\theta\cdot\phi'^2}\right)=0$$

Integrating on both sides, $\frac{\phi'\sin^2\theta}{\sqrt{1+\sin^2\theta\phi'^2}}=c$

$$\phi'\sin^2\theta=c\sqrt{1+\sin^2\theta\,\phi'^2}$$

$$\phi'^2\sin^4\theta=c^2(1+\sin^2\theta\,\phi'^2)$$

$$\phi'^2\sin^2\theta(\sin^2\theta-c^2)=c^2$$

$$\phi'^2=\frac{c^2}{\sin^2\theta(\sin^2\theta-c^2)}=\frac{c^2\operatorname{cosec}^2\theta}{(\sin^2\theta-c^2)}=\frac{c^2\operatorname{cosec}^2\theta}{\sin^2\theta(1-c^2\operatorname{cosec}^2\theta)}$$

$$\phi'^2=\frac{c^2\operatorname{cosec}^4\theta}{1-c^2\operatorname{cosec}^2\theta}$$

$$\phi'^2=\frac{c^2\operatorname{cosec}^4\theta}{1-c^2\operatorname{cosec}^2\theta}$$

$$\phi'=\frac{c\operatorname{cosec}^2\theta}{\sqrt{1-c^2\operatorname{cosec}^2\theta}} \qquad \left[\because \phi'=\frac{d\phi}{d\theta}\right]$$

$$\frac{d\phi}{d\theta}=\frac{c\operatorname{cosec}^2\theta}{\sqrt{1-c^2\operatorname{cosec}^2\theta}} \Rightarrow d\phi=\frac{c\operatorname{cosec}^2\theta}{\sqrt{1-c^2\operatorname{cosec}^2\theta}}\,d\theta$$

$$\phi=\int_{\theta_0}^{\theta_1}\frac{c\operatorname{cosec}^2\theta}{\sqrt{1-c^2\operatorname{cosec}^2\theta}}\,d\theta=\int_{\theta_0}^{\theta_1}\frac{d(c\cot\theta)}{\sqrt{1-c^2(1+\cot^2\theta)}}\,d\theta$$

Values of

$$= \int_{\theta_0}^{\theta_1} \frac{d(-c\cot\theta)}{(\sqrt{1-c^2}) - (c\cot\theta)^2}$$

$$\int \frac{-dx}{\sqrt{a^2-x^2}} = \cos^{-1}\left(\frac{x}{a}\right)$$

$$\phi = \cos^{-1}\left(\frac{c\cot\theta}{\sqrt{1-c^2}}\right) + \phi_0 .$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$$

$$\phi - \phi_0 = \cos^{-1}\left(\frac{c\cot\theta}{\sqrt{1-c^2}}\right) + \phi_0 .$$

$$\cos(\phi-\phi_0) = \frac{c(\cot\theta)}{\sqrt{1-c^2}}$$

$$\cos\phi\cos\phi_0 + \sin\phi\sin\phi_0 = \frac{c\cot\theta}{\sqrt{1-c^2}}$$

tiply both sides by $r\sin\theta$,

$$\therefore r\sin\theta\cos\phi\cos\phi_0 + r\sin\theta\sin\phi\sin\phi_0 = \frac{c\, r\sin\theta\cot\theta}{\sqrt{1-c^2}}$$

$$x\cos\phi_0 + y\sin\phi_0 = \frac{cz}{\sqrt{1-c^2}}$$

where,

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

($ax+by+cz+d=0$ Equation of plane)

hich is the equation of the plane passes through the origin

$$\left(x\cos\phi_0 + y\sin\phi_0 - \frac{cz}{\sqrt{1-c^2}} = 0\right)$$

is plane intersect the sphere along the ~~great~~ great circle us we say that long the great circle. Thus we say that ve join 2 points on the sphere gives the stationary lues of the integral is a great circle.

The great circle gives the curve of the minimum length only when the length of the great circle is less an radius.

Hamilton's principle :- 4.1 (23), (24)

Statement :- (X) 10M.

The actual path in configuration space followed by a holonomic dynamical system during the fixed interval t_0 to t_1 is such as the integral $I = \int_{t_0}^{t_1} L\, dt$ is stationary w.r.t to the path variation which vanishes at the end points.

Proof:-