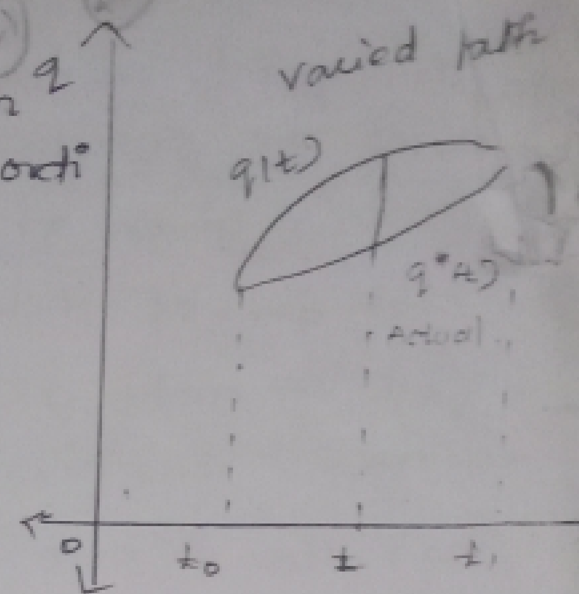
Hamilton's principles for non-holonomonic system (or) 120
Non-holonomic constraints system:-



Let us consider a non-holonomic system suppose there are n-generalized co-ordinates and m-non holonomic constraints equation of the form.

$$\sum_{i=1}^{n} a_{ji}(q.t)\dot{q}_i + a_{jt}(q.t) = 0 \longrightarrow (1)$$

for all $j = 1, 2, \dots m$.

Let us denote the actual path by $q^*(t)$ and varied path $q(t)$ and the variations δq.

$$q_i = q_i^* + \delta q_i \longrightarrow (2),$$

$$\dot{q}_i = \dot{q}_i^* + \delta \dot{q}_i \longrightarrow (3)$$

Now, assume that varied path and actual path both conform to the constraints the eqn (1)

The Taylor's expansion of a's about the reference value at each instant of time neglecting the terms of order higher than the first in the δq's we have,

$$a_{ji}(q,t) = a_{ji}(q^*,t) + \sum_{k=1}^{n} \left(\frac{\partial a_{jt}}{\partial q_k}\right)_0 \cdot \delta q_k \longrightarrow (4)$$

$$a_{jt}(q,t) = a_{jt}(q^*,t) + \sum_{k=1}^{n} \left(\frac{\partial a_{jt}}{\partial q_k}\right)_0 \cdot \delta q_k \longrightarrow (5)$$

where a zero subscript indicates thus a quantity evaluated on the actual path.

Then substituting (3), (4), (5) in (1)

$$\sum_{i=1}^{n} a_{ji}(q^*,t) + a_{jt}(q^*,t) = 0 \longrightarrow (6)$$

$$\sum_{i=1}^{n} \left[a_{ji}(q^*,t) + \sum_{i=1}^{n}\left(\frac{\partial a_{ji}}{\partial q_k}\right)_0 \delta q_k\right]\left(\dot{q}_i^* + \delta \dot{q}_i^*\right)$$

$$+ \left[a_{jt}(q^*,t) + \sum_{i=1}^{n}\left(\frac{\partial a_{jt}}{\partial q_k}\right)\delta q_k\right] = 0$$

For th

$\sum_{i=1}^{n} a_{ji}(q^*, t)\dot{q}_i^* + \sum_{i=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial a_{ji}}{\partial q_k}\right)_0 \delta q_k \cdot \dot{q}_i^* + \sum_{i=1}^{n} a_{ji}(q^*, t)\,\delta\dot{q}_i$

$+ \sum_{i=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial a_{ji}}{\partial q_k}\right) \cdot \delta q_k \cdot \delta\dot{q}_i$

$+ a_{jt}(q^*, t) + \sum_{k=1}^{n}\left(\frac{\partial a_{jt}}{\partial q_k}\right)\boxed{\delta q_k} = 0$ (by (6))

$\sum_{i=1}^{n} a_{ji}(q^*, t)\,\delta\dot{q}_i + \sum_{i=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial a_{ji}}{\partial q_k}\right)\delta q_k \cdot \dot{q}_i^* + \sum_{k=1}^{n}\left(\frac{\partial a_{jt}}{\partial q_k}\right)\cdot \delta q_k = 0$ → (7)

We assume that δq's must be instantaneous condition namely

$$\sum_{i=1}^{n} a_{ji}(q^*, t)\,\delta q_i = 0, \quad j = 1 \text{ to } m \longrightarrow (8)$$

Diff eqn (8) w.r.t time t and changing the indicates we can get

$$\sum_{i=1}^{n} \dot{a}_{ji}(q^*, t)\,\delta q_i + \sum_{i=1}^{n} a_{ji}(q^*, t)\,\delta\dot{q}_i = 0 \longrightarrow (9)$$

i is replaced by k,

$$\sum_{k=1}^{n} \dot{a}_{jk}(q^*, t)\,\delta q_k + \sum_{k=1}^{n} a_{jk}(q^*, t)\,\delta\dot{q}_k = 0 \longrightarrow (10)$$

where $\dot{a}_{jk}(q^*, t) = \sum_{i=1}^{n}\left(\frac{\partial a_{jk}}{\partial q_i}\right)_0 \dot{q}_i^* + \left(\frac{\partial a_{jk}}{\partial t}\right)_0 \longrightarrow (11)$

$$\sum_{k=1}^{n}\left\{\sum_{i=1}^{n}\left(\frac{\partial a_{jk}}{\partial q_i}\right)_0 + \left(\frac{\partial a_{jk}}{\partial t}\right)_0\right\} + \sum_{i=1}^{n} a_{jk}(q^*, t)\,\delta\dot{q}_k = 0 \longrightarrow (12)$$

Now equation (7) - (12)

$\Rightarrow \sum_{i=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial a_{ji}}{\partial q_k} - \frac{a_{jk}}{\partial q_j}\right)\dot{q}_i^* \cdot \delta q_k + \sum_{k=1}^{n}\left(\frac{\partial a_{jk}}{\partial q_k} - \frac{\partial a_{ik}}{\partial t}\right)\delta q_k = 0$ → (13)

$\forall\, i, k = 1 \text{ to } n$

$j = 1 \text{ to } m$

In general $\dot{q}_i^* \neq 0$.

Hence the equation to be varied continuously on set of δq which conform to the constraint of an (8), we must have,

Hence, the Necessary

$$\left(\frac{\partial a_{ji}}{\partial q_k} - \frac{\partial a_{jk}}{\partial q_i}\right) = 0 \longrightarrow 3(14)$$

$$\left(\frac{\partial a_{jt}}{\partial q_k} - \frac{\partial a_{jk}}{\partial t}\right) = 0 \longrightarrow 3(15)$$

∴ Equation (13) = 0

Equation (14) & (15) represented exactness conditions for the integrability of equation (1)

In otherwords if these conditions applied the constraints of holonomic.

Thus we have show that if that varied path conform to be actual Constraints and if the δq's are consist with instantaneous constraints then the system and be holonomic.

(ie) Hamilton's priniciple is valied for holonomic system only. The equation

$$\int_{t_0}^{t_1} \left(\delta q + \sum_{i=1}^{n} \dot{q}_i \, \delta q_j\right) dt = 0$$

$$\int_{t_0}^{t_1} \delta(T - V)\, dt = 0$$

applied to non-holonomic System but are not variation principles in the used sense. Because the varied path are not geometrically possible path.

Q4 Derive Hamilton's Equation:- 42

consider the Lagrangian function $L(q, \dot{q}, t)$ is Consist of n-second order diff equation in the nq's and time t.

Instead of $L(q, \dot{q}, t)$, we fix a new function $H[q, P$ is called Hamilton's function which will need to a set of 2n first order differential equation get from H of known as Hamilton canonical equation or Hamilton equation of motion.

a holonomic system which has a lagrange's eqn

$$\dot{l}_i) - \frac{\partial L}{\partial q_i} = 0 \longrightarrow (1), \quad i = 1, 2, \ldots n$$

noralized momentum conjugate to q_i is given by

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \longrightarrow (2)$$

Now put eqn (2) in (1)

$$\Rightarrow \frac{d}{dt}\left(p_i\right) - \frac{\partial L}{\partial q_i} = 0, \quad i = 1 \text{ to } n.$$

$$\text{(ie)} \quad \dot{p}_i = \frac{\partial L}{\partial q_i} \longrightarrow (3)$$

let us define hamilton's function

$$H(q, p, t) = \sum_{i=1}^{n} p_i \dot{q}_i - L(q, \dot{q}, t) \longrightarrow (4)$$

Here H is the explict functions of p's and q's and time

since the R.H.S of equation (4) contains $\dot{q}_i$ we must eliminate and can be expressed interms of p. this is of the form.

$$p_i = \sum_{j=1}^{n} m_{ij}(q, t)\dot{q}_j + a_i(q, t)$$

Then we solve for the q's and contained

$$\dot{q}_i = \sum_{j=1}^{n} b_{ij}(p_j - a_j)$$

where $b_{ij}(q, t)$ be the element of matrix $b = m^{-1}$

since the ~~matrix~~ inertia matrix m can also inver-tible and it is also positive definite.

Since H is a function of (q, p, t)

$\therefore$ The first difference

$$\delta H = \sum_{i=1}^{n} \frac{\partial H}{\partial q_i}\delta q_i + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\delta p_i + \frac{\partial M}{\partial t}\delta t \longrightarrow (5)$$

Now, from equation (4),

$$\delta H = \sum_{i=1}^{n} p_i \delta \dot{q}_i + \sum_{i=1}^{n} \dot{q}_i \delta p_i - \sum_{i=1}^{n} \frac{\partial L}{\partial q_i}\delta q_i - \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial t}\delta t.$$

$$= \sum_{i=1}^{n}\left(p_i - \frac{\partial L}{\partial \dot{q}_i}\right)\delta\dot{q}_i + \sum_{i=1}^{n}\dot{q}_i\,\delta p_i - \sum_{i=1}^{n}\frac{\partial L}{\partial q_i}\delta q_i - \frac{\partial L}{\partial t}\delta t.$$

$$= \sum_{i=1}^{n} 0\cdot\delta\dot{q}_i + \dots \text{ (by (2))} \longrightarrow (6)$$

compare equation (5) & (6)

$$\Rightarrow \text{(i) } \frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i} \Rightarrow \dot{p}_i = -\frac{\partial H}{\partial q_i} \longrightarrow (7)$$

$$\text{(ii) } \frac{\partial H}{\partial p_i} = \dot{q}_i \longrightarrow (8)$$

$$\text{(iii) } \frac{\partial H}{\partial t} = \frac{-\partial L}{\partial t}$$

Hamilton canonical equation of motion.

Suppose there are generalised forces are not all deri from a potential function.

If the forces of Q_i' s the lagrange's equation are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i'$$

Equivalently $\dot{p}_i - \frac{\partial L}{\partial q_i} = Q_i'$

$$\text{(ie) } \dot{p}_i = Q_i' + \frac{\partial L}{\partial q_i}$$

The Hamilton equation for the system

$$\therefore \dot{p}_i = Q_i' - \frac{\partial H}{\partial q_i} \text{ and } \dot{q}_i = \frac{\partial H}{\partial p_i}$$

consider a non-holonomic system having 'm' constraints satisfying

$$\sum_{i=1}^{n} a_{ji}\dot{q}_i + a_{jt} = 0, \quad i = 1 \text{ to } n; \quad j = 1 \text{ to } m.$$

The corresponding lagrange's equation for this syste

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial Q_i} = \sum_{j=1}^{n}\lambda_j a_{ji} + Q_i' \text{ (or)}$$

Equivalently

$$\dot{p}_i = \sum_{j=1}^{n}\lambda_j a_{ji} + Q_i' + \frac{\partial L}{\partial q_i}$$

a written to the original function of the original active variable. The result set of variable)

ther variational principle: 4.3 (14)

Modified Hamilton's principle:-

Consider a holonomic system having 'n' independent q's consider a hamilton function. H is defined as,

$$H = \sum_{i=1}^{n} p_i \dot{q}_i - L \ \& \ L = \sum_{i=1}^{n} p_i \dot{q}_i - H$$

We canchoose the relation in the Hamilton's principle.

Std $\delta \int_{t_0}^{t_1} L \cdot dt = 0$ which is valid for holonomic system.

and $\Rightarrow \delta \int^{t_1} \left(\sum_{i=1}^{n} p_i \dot{q}_i - H \right) dt = 0 \longrightarrow (1)$

So

$$= \int_{t_0}^{t_1, t_0} \left[\sum_{i=1}^{n} \left(p_i \cdot \delta \dot{q}_i + \dot{q}_i \delta p_i \right) - \frac{\partial H}{\partial p_i} \delta p_i - \frac{\partial H}{\partial q_i} \delta q_i \right] dt$$

also we use this relation. $\delta \dot{q}_i = \frac{d}{dt} (\delta q_i) \longrightarrow (2)$ which valid for the system. Now eqn (1) becomes

$$\int_{t_0}^{t_1} \left[\sum_{i=1}^{n} \left(p_i \delta \dot{q}_i + \dot{q}_i \delta p_i \right) - \frac{\partial H}{\partial p_i} \delta p_i - \frac{\partial H}{\partial q_i} \delta q_i \right] dt \longrightarrow (3)$$

The time t is hold constant during each variation (i.e) $\delta t = 0$

From the above result also,

$$\int_{t_0}^{t_1} p_i \cdot \delta \dot{q}_i \, dt = \int_{t_0}^{t_1} p_i \frac{d}{dt} (\delta q_i) \, dt$$

$$= \int_{t_0}^{t_1} p_i \, d(\delta q_i) = (p_i \, \delta q_i)_{t_0}^{t_1} - \int_{t_0}^{t_1} \dot{p}_i$$

... coefficient using the place

$$= 0 - \int_{t_0}^{t_1} \dot{p}_i \, \delta q_i$$

Equation (3) $\Rightarrow \int_{t_0}^{t_1} \Sigma \left(- \dot{p}_i \delta q_i + \dot{q}_i \delta p_i - \frac{\partial H}{\partial p_i} \delta p_i - \frac{\partial H}{\partial q_i} \delta q_i \right)$

$$= \int_{t_0}^{t_1} \left\{ \Sigma \left(\dot{q}_i - \frac{\partial H}{\partial p_i} \right) \delta p_i - \Sigma \left(\dot{p}_i - \frac{\partial H}{\partial q_i} \right) \delta q_i \right\} dt = 0$$

By the canonical equation,

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \; ; \; \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

The co-efficient of δp_i and δq_i are zero.

$\Rightarrow$ we may consider δp_i and δq_i are independent to eqn.

We define phrase space of 2n dimension as a space in which n. q's and n. p's of the 2n co-ordinate of the point.

$\therefore$ The modified Hamilton principle says that the actual point in such that

$\delta \int \Sigma (p_i \dot{q}_i - H) \, dt = 0$ is stationary for arbitrary variation in phase space with the restriction that δq's vanish at the fixed time t_0 and t_1.

The δp's need not be zero at the end points.

Definitions:

(13)

Contemporaneous:-

In Hamilton Principle we consider a type of variation Δq_i of the generalized co-ordinates are contemporaneous.

(i.e) a point $(q + \delta q, t)$ on the varied path correspond to a point (q, t) on the actual path. Hence the variation are assume to occur without the passage of time as in a virtual displacement.

Action of the integral:-

rangian 'L' we define action of the integral. 134

δq_i), $$A = \int_{t_0}^{t_1} \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} q_i \, dt = \int_{t_0}^{t_1} \sum_{i=1}^{n} p_i q_i \, dt.$$

General variation in configuration space let us $dt=0$ ine an integral I.

$$I = \int_{t_0}^{t_1} L(q, \dot{q}, t) \, dt.$$

e general non-contemporaneous variances to this tegral is given by

ani. ce of

$$\delta I = \int_{t_0}^{t_1} \sum \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i\right) dt + \int_{t_0}^{t_1} \left[\frac{\partial L}{\partial t} \delta t - \left[\sum \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L\right] \frac{d}{dt} \delta t\right] dt$$
$$- \int_{t_0}^{t_1} \sum \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}\right) \delta q_i \, dt.$$

...ciples of least Action:- (12)

Statement: (5x) 5M.

The actual path of a conservative holonomic system is $\ni$ the action is stationary with respect to varied path carry the same energy integral H and the same end points in q-space.

Proof:-

Consider a holonomic system, We assume that δq's are consistent with constraints

We consider the most general variation for I is,

$$I = \int_{t_0}^{t_1} L(q, \dot{q}, t) \, dt.$$

$$\delta I = \int_{t_0}^{t_1} \sum \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i\right) dt + \int_{t_0}^{t_1} \left[\frac{\partial L}{\partial t} \delta t - \left[\sum \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L\right] \frac{d}{dt}(\delta t)\right] dt$$
$$- \int_{t_0}^{t_1} \sum \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}\right] \delta q_i \, dt \quad \to (1)$$

If all the applied force are derivable from the potential function $V(q, t)$.

$$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

$\therefore$ The third integral of eqn (1) vanishes, If the varied path have the fixed and fixed end in the configuration space.

$$\int_{t_0}^{t_1} \sum \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i\right) dt = \left[\frac{\partial L}{\partial \dot{q}_i} \cdot \delta q_i\right]_{t_0}^{t_1} = 0$$

In eqn (1) I integral vanish

Suppose we define ourself to varied path having energy integral

$$\sum \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L = h, \text{ where } h \text{ is constant.}$$

Assume that the variation of non-contemporaneous in use $\delta t \neq 0$

$\because \frac{\partial L}{\partial t} = 0$ for the Conservative system

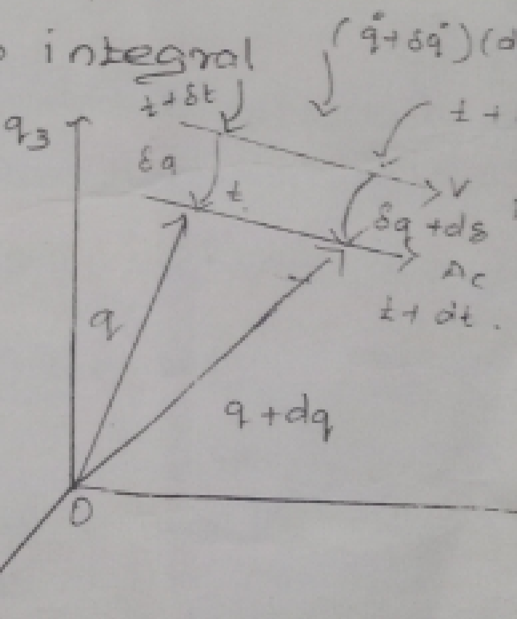
(1) reduce to $\delta I = -\int_{t_0}^{t_1} h \cdot \frac{d}{dt}(\delta t)\, dt$

$$= -h(\delta t)_{t_0}^{t_1} = -h[\delta t_1 - \delta t_0] \longrightarrow (2)$$

Let us define the action of the integral

$$A = \int_{t_0}^{t_1} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i\, dt = \int_{t_0}^{t_1} \sum_{i=1}^{n} p_i \dot{q}_i\, dt$$

$$A = \int_{t_0}^{t_1} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i\, dt = \int_{t_0}^{t_1} (L+h)\, dt$$



$$\therefore \delta A = \delta\int_{t_0}^{t_1} (L+h)\, dt = \delta\int_{t_0}^{t_1} L\, dt + \delta\int_{t_0}^{t_1} h\, dt$$

$$= \delta I + \delta h \int_{t_0}^{t_1} dt$$

$$= -h(\delta t_1 - \delta t_0) + \delta h(t_1 - t_0) + h(\delta t_1 - \delta t_0)$$

$$= \delta h(t_1 - t_0)$$

Place

135

a restrict the varied path to those for which hence
the same value as the actual path

Hence $\delta h = 0$

end

It follows that, $\delta A = \delta \int \sum p_i \dot{q}_i \, dt = 0$

This is the principles of least action.

x: 4.7 V.V.I (2) 5M.

acobi form of the principles of the least Action. (11)

In general $\sum_{i=1}^{n} p_i \dot{q}_i = 2T_2 + T_1$ for the natural system

$T_1 = 0$.

rving

$$\therefore \sum_{i=1}^{n} p_i \dot{q}_i = 2T_2.$$

$\therefore$ The principle of least action

$$\delta A = \delta \int \sum p_i \dot{q}_i \, \delta t = 0$$

$$= \delta \int 2T_2 \, dt = 0$$

neo

$$\Rightarrow \delta \int 2T \, dt = 0 \longrightarrow (1)$$

For the natural system we have,

$$\therefore (1) \Rightarrow \delta \int 2T \, dt = 0$$

$$= \delta \int 2(h-V) \, dt = 0 \text{ depends} \longrightarrow (2)$$

$$ds^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} \dot{q}_i \dot{q}_j \, dt^2$$

Since, $T = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} \dot{q}_i \dot{q}_j$

$$ds^2 = 2T dt^2$$

$$ds = \sqrt{2T} \, dt.$$

$$\delta A = 0$$

ive

$$\delta \int 2(h-V) \cdot \frac{ds}{\sqrt{2T}} = 0$$

ro

$$\Rightarrow \delta \int \sqrt{2(h-V)} \, ds = 0$$

on

This is called the Jacobi from of principles of least action.

This method is help us to obtained the path in configuration space without expressing in motion a function of time.

4.4 am (8) 5M.

Phase space:

The equation of the motion for the standared holonomic system can be express in the form,

$\dot{x} = X(x,t)$, where x is the two n-dimension vector consisting of nq's and np's. This two n-dim space in which $q_1, q_2, \ldots q_n$ and $P_1, P_2, \ldots P_n$ can be taken as a point is called the phase space.

Note:

The soln of the diff eqn consist of n-function $q_i(t)$ describing a holonomic system can be considered as a path trajories traced by a moving point in configuration space.

Equilibrium point:

A point in phase space in which all $\dot{q}$'s and $\dot{p}$'s are zero is called equilibrium point (or) singular point.

Extended phase space:-

For a non-conservative system $q_{n+1}(t)$ - considering a time t as a additional dependent variable. The soln of the corresponding diff eqn for the system yields trajeries at (q, P)'s space of $2n+2$ dimension consist of $(n+1)$ q's and $(n+1)$ p's which is called Extended phase space.

Liouvelli's theorem: (10)

Statement:

A volume of any domain in phase space remain constant the volume is having along with the phase in accorded space canonical equation.

Proof:

Consider a holonomic system represented n-dependent q's the follows a group of phase point the describing trajeries in phase space of 2n dimension.

The point with is a small volume element

$$dv = dq_1, dq_2, \ldots dq_n \cdot dp_1, \ldots, dp_n.$$

Can be considered as moving Particle of a fluid called whose fluid with phase velocity $\overline{V}$ of a fluid particle has a component $(\dot{q}_i, \dot{p}_i)$ - These can be expressed in terms of q's and p's and by canonical equation,

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = \frac{-\partial H}{\partial q_i}$$

as the volume element of phase fluid as changes its sp shape.

But neighbouring Particle will remain closed to each other,

Now,
$$\nabla \cdot \overline{V} = \sum_{i=1}^{n} \left(\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i} \right)$$

$$= \sum_{i=1}^{n} \left(\frac{\partial}{\partial q_i} \left(\frac{\partial H}{\partial p_i} \right) + \frac{\partial}{\partial p_i} \left(-\frac{\partial H}{\partial q_i} \right) \right)$$

$$= \sum_{i=1}^{n} \left(\frac{\partial^2 H}{\partial q_i \partial p_i} - \frac{\partial^2 H}{\partial p_i \partial q_i} \right) = 0$$

These show that phase fluid incomprasable. The volume of each fluid element is constant during the motion.