UNIT - V

Hamilton Jacobi theory:

Prove that the canonical transformation which Preserve the hamiltonian form of the eqn of motion & the variable.

Proof:

we know that complete solution of holonomic system which having n degree of freedom by finding an independent (n) function known as integral of motion.

These funtions are expressed in the form

$$f_i(q,\dot{q},t)=\gamma_i \qquad i=1,2,\dots 2n \longrightarrow (1)$$

Where the constant γ_i are usually evaluated from the initial condition and eliminating $\dot{q}$ interm of P.

We know that $$\dot{q}_i=\sum_{j=1}^{n} b_{ij}\,(p_j-a_j) \longrightarrow (2)$$

Their by producing integrals of motion which are fun of q's and p's.

And hence $$g_i(p,q,t)=\gamma_i \quad , i=1 \text{ to } 2n \longrightarrow (3)$$

Assuming equation (1) and (3) are distinct. Then we have

$$\frac{\partial(g_1,g_2,\dots g_{2n})}{\partial(q_1,q_2,\dots q_n,p_1,p_2,\dots p_n)}\neq 0 \longrightarrow (4)$$

∴ We can solve np's and nq's as the function of 2n and time t.

$$\left.\begin{aligned} q_i&=q_i(\gamma_1,\gamma_2,\dots\gamma_{2n},t)\\ p_i&=p_i(\gamma_1,\gamma_2,\dots\gamma_{2n},t)\end{aligned}\right\}\longrightarrow (5) \quad i=1 \text{ to } n$$

Hence we have a complete solution of the Hamilton canonical equation which is known as the solution of the Hamilton problem.

And view of equation (3) & (5) are representing transformation in 2n space. A transformation betwe

...ffian Differential Form: (5) (6)

A Pfaffian form Ω in m variables $x_1, x_2, \ldots x_m$ ...re written as,

$$\Omega = X_1(x)\,dx_1 + X_2(x)\,dx_2 + \cdots + X_m(x)\,dx_m \longrightarrow (1)$$

above form is similar to a virtual work expression ...which the courses x's as are functions of position. This pfaffian form also leads to a line integral ... a path in x-space let us define,

$$C_{ij} = \frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i} \longrightarrow (2)$$

The pfaffian form is exact differentiation then all the ...s are zero. In the usual form the differential ...rm is not exact //. Now let us consider the ...fferential form.

$$ds = \sum_{i=1}^{n} P_{i1}\,dq_{i1} - \sum_{i=1}^{n} P_{i0}\,dq_{i0} - H_1\,dt_1 + H_0\,dt_0 \longrightarrow (4)$$

...s consists of the difference two pfaffian expression this of the form

$$\sum_{i=1}^{n} P_i\,dq_i - H\,dt$$

here the $2n+1$ variables are p's, q's and t. Hence $n = 2n+1$ and each pfaffian can be written more explicity form.

$$P_1 dq_1 + P_2\,d q_2 + \cdots + P_n\,d q_n + 0\cdot dP_1 + 0\cdot dP_2 + \cdots + 0\cdot dq_n - H(q, P, t)\overset{dt}{t} = 0$$

Another aspect of pfaffian difference form is that, ...it m is odd there is an associated system called the first pfaffian form given by,

$$\sum_{i=1}^{n} C_{ij}\,dx_i = 0, \quad j = 1 \text{ to } m \longrightarrow (3)$$

Applying equation (3) to the above differential form we get.

$$dq_j - \frac{\partial H}{\partial P_j}\,dt = 0 \longrightarrow (4)$$

iii) $-dp_j - \frac{\partial H}{\partial q_j} dt = 0 \rightarrow (5)$

$$\sum_{i=1}^{n} \left(\frac{\partial H}{\partial q_j} dq_j + \frac{\partial H}{\partial p_j} dp_j \right) \rightarrow (6)$$

Equation (4) & (5)

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad j = 1 \text{ to } n.$$

$$\dot{p}_j = \frac{-\partial H}{\partial q_j} \quad j = 1 \text{ to } n.$$

which are the Hamilton canonical equation, taking total derivative of H using equation (6)

$$\dot{H} = \frac{\partial H}{\partial t} \rightarrow (7)$$

In summary we find the ds = the difference betwee the two pfaffian differential form.

Equation (1) containing the initial value and oth values containing p's and q's finally,

Each pfaffion form which is associated which with the set of canonical equation in the given variable

Hence the principle function S is the connecting lin between two sets of cononical variable infact it is the generating function for the canonical transformation betw this variable (or) value

We can generalized the differential form (A) by using the $2n$ parameters $\gamma_1, \gamma_2, \ldots, \gamma_{2n}$ to specify th initial condition,

Let us specified the transformation,

$$q_{i0} = q_{i0}(\gamma_1, \gamma_2, \ldots, \gamma_{2n}) \longrightarrow (8)$$

$$p_{i0} = p_{i0}(\gamma_1, \gamma_2, \ldots, \gamma_{2n}) \longrightarrow (9)$$

when,

$$\frac{\partial(q_{i0}, \ldots, q_{2n})}{\partial(\gamma_1, \gamma_2, \ldots, \gamma_{2n})} \neq 0.$$

$n \quad dq_{i0} = \frac{\partial q_{i0}}{\partial \gamma_1} d\gamma_1 + \frac{\partial q_{i0}}{\partial \gamma_2} d\gamma_2 + \cdots + \frac{\partial q_{i0}}{\partial \gamma_{2n}} d\gamma_{2n}$ 146

$$= \sum_{j=1}^{2n} \frac{\partial q_{i0}}{\partial \gamma_i} d\gamma_i$$

$$P_{i0}\, dq_{i0} = P_{i0} \sum_{j=1}^{2n} \frac{\partial q_{i0}}{\partial \gamma_j} d\gamma_j$$

$$\sum_{i=1}^{n} P_{i0}\, dq_{i0} = \sum_{i=1}^{n} \sum_{j=1}^{2n} \left(P_{i0} \cdot \frac{\partial q_{i0}}{\partial \gamma_j} \right) d\gamma_j .$$

$$= \sum_{j=1}^{2n} \left(\sum_{i=1}^{n} P_{i0} \frac{\partial q_{i0}}{\partial \gamma_j} \right) d\gamma_j - H(q, P, t)\, dt = 0$$

$$\sum_{i=1}^{n} P_{i0}\, q_{i0} = \sum_{j=1}^{2n} \Gamma_j(\gamma)\, d\gamma_j \quad \text{where } \Gamma_j(\gamma) = \sum_{i=1}^{n} P_{i0} \frac{\partial q_{i0}}{\partial \gamma_j} \rightarrow (10)$$

Using pfaffian theorem $2n\gamma$'s can be replaced by $n\alpha$'s and $n\beta$'s where the function,

$$\alpha_i = \alpha_i(\gamma_1, \gamma_2, \ldots, \gamma_{2n}) \longrightarrow (11)$$

$\beta_i = \beta_i(\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_{2n})$ are choosen such that

$$\sum_{i=1}^{n} \beta_i\, d\alpha_i = \sum_{j=1}^{2n} \Gamma_i(\gamma)\, d\gamma_j \longrightarrow (12)$$

Then in the same manner

$$\Rightarrow \Gamma_j(\gamma) = \sum_{i=1}^{n} \beta_i \frac{\partial \alpha_i}{\partial \gamma_j}$$

$\Rightarrow \alpha$'s and β's are not unique

Comparing equation (10) & (12)

$$\sum_{i=1}^{n} P_{i0}\, dq_{i0} = \sum_{i=1}^{n} \beta_i\, d\alpha_i \longrightarrow (13)$$

where α's and β's are another representation of initial condition. From equation (1) we can find (q_0, P_0) and (α, β) are connected by a homogeneous canonical transformation at the given initial time t_1 and t_0.

Hamilton Jacobi Equation: 5.2 3 4

Derive Hamilton Jacobi equation (or) prove that a complete soln of the hamilton Jacobi equation gives the principle function $S(q, \alpha, t)$ with provides the path of the system in phase space merely by differentiation and algebraic manipulation;

Proof :-

Let us consider the differential form

$$ds = \sum_{i=1}^{n} p_{i_1} dq_{i_1} - \sum_{i=1}^{n} p_{i0}\, dq_{i0} - H_1 dt_1 + H_0 dt_0 \longrightarrow (1)$$

which is associated with a canonical transformation relating the initial and final point of a path in phase space. Let the initial condition be specified by $n\alpha$ and $n\beta$'s where we have,

$$\left.\begin{aligned} \alpha_i &= \alpha_i(q_{10}, \dots q_{n0}, p_{10}, \dots, p_{n0}) \\ \beta_i &= \beta_i(q_{10}, \dots, q_{n0}, p_{10}, \dots, p_{n0}) \end{aligned}\right\} \longrightarrow (2)$$

and further

$$\sum_{i=1}^{n} p_{i0}\, dq_{i0} = \sum_{i=1}^{n} \beta_i\, d\alpha_i \longrightarrow (3)$$

The above function is not arbitrary but they represent homogeneous canonical transformation using equation (1) and (3)

$$\Rightarrow ds = \sum_{i=1}^{n} p_{i_1} dq_{i_1} - \sum_{i=1}^{n} \beta_i\, d\alpha_i - H_1 dt_1 + H_0 dt_0 \longrightarrow (4)$$

where s is a function of $q_{i_1}, \alpha_i, t_0, t_1$.

$$S = s(q_{i_1}, \alpha_i, t_1, t_0)$$

we have

$$ds = \sum_{i=1}^{n} \frac{\partial s}{\partial q_{i_1}} dq_{i_1} + \sum_{\alpha=1}^{n} \frac{\partial s}{\partial \alpha_i} d\alpha_i + \frac{\partial s}{\partial t_1} dt_1 + \frac{\partial s}{\partial t_0} dt_0 \rightarrow$$

And let us assume that the $\left| \frac{\partial^2 s}{\partial q_{i_1} \partial \alpha_i} \right| \neq 0 \longrightarrow (6)$

Then we can solve for α's interms of $\frac{\partial s}{\partial q_{i_1}}$, $i = 1$ to n

The determinant (6) is actually the Jacobian,

$\frac{\partial(p_{i_1}, p_{i_2}, \dots, p_{in})}{\partial(\alpha_1, \alpha_2, \dots, \alpha_n)}$, where p_i's are consider to be a function of $p_i = p_i(q_{i_1}, \alpha_i, t_1, t_0)$

There exist no relation involving q_1 and α's but the p_i //

Thus the P_1 and α's are independent variables hence equating the corresponding co-efficient in (4) and (5)

$$\Rightarrow \quad P_i = \frac{\partial s}{\partial q_{i_1}} \longrightarrow (7)$$

$$-\beta_i = \frac{\partial s}{\partial \alpha_i} \longrightarrow (8)$$

$$\left.\begin{aligned} -H_1 &= \frac{\partial s}{\partial t_1} \\ H_0 &= \frac{\partial s}{\partial t_0} \end{aligned}\right\} \longrightarrow (9)$$

Simplifying the equation (4) by arbitrary $t_0 = c$ then $dt_0 = 0$.

$$(4) \Rightarrow ds = \sum_{i=1}^{n} P_i \, dq_{i_1} - \sum_{i=1}^{n} \beta_i \, d\alpha_i - H_1 \, dt_1 \longrightarrow (10)$$

Let us consider, another more general approach here we consider, the corresponding trajectories in the (α, β) and (q, p) in phase space as a functions of common time t.

This variables are related by a canonical transformation which is given by equating total difference of two Paffian form,

$$\sum_{i=1}^{n} P_i \, dq_i - H \, dt$$

$$\sum_{i=1}^{n} \beta_i \, d\alpha_i - k \, dt \quad , \quad k = k(\alpha, \beta, t)$$

Resulting from the consideration α and β as variable Hence we get

$$ds = \sum_{i=1}^{n} P_i \, dq_i - H \, dt - \left(\sum_{i=1}^{n} \beta_i \, d\alpha_i - k \, dt \right)$$

$$= \sum P_i \, dq_i - \sum_{i=1}^{n} \beta_i \, d\alpha_i - H \, dt + k \, dt \longrightarrow (11)$$

Even through α's and β's are taken as variable in the Hamiltonian formulation we would like to have they to be require on constraints of motion.

(i.e) We want to have the entire trajectories in the phase space consist of a single fixed point. For this let us take k identically equal to 0. $(k=0)$

The according to the canonical equation we get $\dot{\alpha}_i$ $\dot{\beta}_i = 0$.

$\Rightarrow$ α and β are constants.

From (10) and (11)

(10) which is equality to (11)

consider the principle functions of the form $s(q,$

$$\therefore ds = \sum_{i=1}^{n} \frac{\partial s}{\partial q_i} \cdot dq_i + \sum_{i=1}^{n} \frac{\partial s}{\partial \alpha_i} \cdot d\alpha_i + \frac{\partial s}{\partial t} \cdot dt \longrightarrow (12)$$

and
$$\left| \frac{\partial^2 s}{\partial q_i \partial \alpha_i} \right| \neq 0 \longrightarrow (13)$$

Now equating (10) and (12)

$$-\beta_i = \frac{\partial s}{\partial \alpha_i} \longrightarrow (14)$$

$$p_i = \frac{\partial s}{\partial q_i} \longrightarrow (15)$$

$$\therefore \frac{\partial s}{\partial t} = -H \longrightarrow (16)$$

Equation (14) can be solved for the q's, α, t. This giv the soln of the lagrange problem. This is possible ex

Then substitute the soln for the q's into the equa (15) we obtain the expression for the p's as functi of (α, β, t)

ie) $p_i = p_i(\alpha, \beta, t)$

There by completing the soln of Hamilton.

But the hamiltonian function usually denoted by functions of (q, p, t). If we substitude for the p's fro (15) & (16) we get,

$$\frac{\partial s}{\partial t} + H\left(q, \frac{\partial s}{\partial q}, t\right) = 0 \longrightarrow (17)$$

thus equation (17) this 1^{st} order partial differential
ition is called Hamilton Jacobi equation. This equation
a single dependent variable S and $n+1$ independent
variable (q, t)

$\therefore$ The complete soln of this equation has $n+1$ arbitrary constant. Thus we conclude that a complete soln of hamilton Jacobi equation.

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial(u,v)}{\partial(x,y)}$$

Jacobi's theorem:

Statement:

If $S(q, \alpha, t)$ is any complete solution of the Hamilton Jacobi equation

$x_i = x_i(q_1, q_2, \ldots q_n)$

$dx = \frac{\partial x_i}{\partial q_1} dq_1 + \partial$

$$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}, t\right) = 0 \longrightarrow (1)$$ and if the equation.

$\frac{\partial H}{\partial q_i}(1) + \frac{\partial H}{\partial q_i} d\left(\frac{\partial S}{\partial q_i}\right) + \frac{\partial H}{\partial q_i}$

$$-\beta_i = \frac{\partial S}{\partial \alpha_i}, \quad i = 1 \text{ to } n \longrightarrow (2) \quad ; \quad P_i = \frac{\partial S}{\partial q_i}, \quad i = 1 \text{ to } n \rightarrow (3)$$

where the β's are arbitrary constants, are used to solve for $q_i(\alpha, \beta, t)$ and $P_i(\alpha, \beta, t)$ then those expressions provide the general solution of the canonical equations associated with the Hamiltonian function $H(q, P, t)$.

Proof:

Taking a partial derivative of the Hamilton-Jacobi equation with respect to α_i.

$P_i = -\frac{\partial H}{\partial q_i}$

$$\frac{\partial^2 S}{\partial \alpha_i \partial t} + \sum_{j=1}^{n} \frac{\partial H}{\partial P_j} \cdot \frac{\partial P_j}{\partial \alpha_i} = 0 \quad \cancel{\dot{q}_i = \frac{\partial H}{\partial P}} \rightarrow (4)$$

where P_j as a function of q, α, t.

$$P_j = P_j(q, \alpha, t).$$

Take the total time derivative of equation (2)

$$\frac{\partial^2 S}{\partial t \partial \alpha_i} + \sum_{j=1}^{n} \frac{\partial^2 S}{\partial q_j \partial \alpha_i} \cdot \dot{q}_j = 0 \longrightarrow (5)$$

where we note that $\frac{\partial S}{\partial \alpha_i}$ is a function of (q, α, t)

order of differential is immaterial because

∴ Equation (3) in (4) we get

$$\Rightarrow \frac{\partial^2 s}{\partial t\,\partial \alpha_i} + \sum_{j=1}^{n} \frac{\partial H}{\partial p_j} \cdot \frac{\partial^2 s}{\partial q_j\,\partial \alpha_i} = 0 \longrightarrow (6)$$

since, (3) $\Rightarrow$ $\frac{\partial p_j}{\partial \alpha_i} = \frac{\partial^2 s}{\partial \alpha_i\,\partial q_j}$

Subtracting (6) from (5) we obtain

$$\sum_{j=1}^{n}\left(\dot{q}_j - \frac{\partial H}{\partial p_j}\right)\frac{\partial^2 s}{\partial q_j\,\partial \alpha_i} = 0 \quad (i = 1, 2, \ldots n) \longrightarrow (7)$$

The co-efficient $\frac{\partial^2 s}{\partial q_j\,\partial \alpha_i}$ are the element of the determinant

$$\left|\frac{\partial^2 s}{\partial q_j\,\partial \alpha_i}\right| \neq 0$$

∴ We see that $\dot{q}_j = \frac{\partial H}{\partial p_j}$ $(j = 1, 2, \ldots n) \longrightarrow (8)$

which gives the first Hamilton's equations.

Differentiating Hamilton-Jacobi equation again partially with respect to q_j.

Assuming that p_i is a function of (q, α, t) we have,

$$\frac{\partial^2 s}{\partial q_j\,\partial t} + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \cdot \frac{\partial p_i}{\partial q_j} + \frac{\partial H}{\partial q_j} = 0 \longrightarrow (9)$$

Taking the total time derivative of equation (3)

$$\dot{p}_j - \frac{\partial^2 s}{\partial t\,\partial q_j} - \sum_{i=1}^{n} \frac{\partial^2 s}{\partial q_i\,\partial q_j} \cdot \dot{q}_i = 0 \longrightarrow (10)$$

We note each α_i is constant along a soln path adding equation (9) & (10)

$$\dot{p}_j + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \cdot \frac{\partial p_i}{\partial q_j} + \frac{\partial H}{\partial q_j} - \sum_{i=1}^{n} \frac{\partial^2 s}{\partial q_i\,\partial q_j} \cdot \dot{q}_i = 0 \longrightarrow (11)$$

using equation (3) and (8)

$$\dot{p}_j + \frac{\partial H}{\partial q_j} = 0 \quad (\text{or}) \quad \dot{p}_j = -\frac{\partial H}{\partial q_j} \quad (j = 1 \text{ to } n) \longrightarrow (12)$$

which is the second canonical equation.