unit : no : 1

Double integral :-

1) $$\int_0^1 \int_0^{x^2} (x^2+y^2)\,dy\,dx.$$

Sol:

$$= \int_0^1 \left[x^2y + y^3/3\right]_0^{x^2} dx.$$

$$= \int_0^1 \left[x^2(x^2) + \frac{(x^2)^2}{3}\right] dx$$

$$= \int_0^1 \left(x^4 + \frac{x^6}{3}\right) dx$$

$$= \int_0^1 x^4\,dx + \int_0^1 \frac{x^6}{3}\,dx$$

$$= \left(\frac{x^5}{5}\right)_0^1 + \frac{1}{3}\int (x^6)\,dx$$

$$= \left(\frac{x^5}{5}\right)_0^1 + \frac{1}{3}\left(\frac{x^7}{7}\right)_0^1$$

$= (1/5 - 0) + \frac{1}{3}(1/7 - 0)$

$= 1/5 + 1/21$

$= \frac{1\times 21}{5\times 21} + \frac{1\times 5}{21\times 5}$

$= \frac{21+5}{105} \Rightarrow 26/105.$

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8) $\int_0^{\pi/2}\int_0^{\pi/2} (\sin\theta + \phi)\, d\theta\, d\phi$

Sol:

$= \int_0^{\pi/2} \left[-\cos(\theta+\phi)\right]_0^{\pi/2} d\phi$

$= \int_0^{\pi/2} \left[-\cos(\pi/2+\phi)\right] - \left[-\cos(\theta)\right] d\phi$

$= \int_0^{\pi/2} (\sin\phi + \cos\phi)\, d\phi$

$= \int_0^{\pi/2} \sin\phi\, d\phi + \int_0^{\pi/2} \cos\phi\, d\phi$

$= (-\cos\phi + \sin\phi)_0^{\pi/2}$

$= \left[-\cos(\pi/2) + \sin(\pi/2)\right] - \left[\cos(0) + \sin(0)\right]$

$= (0+1) + (1+0)$

$= (1+1)$

$= 2$

3) Transform the integral into polar co-ordinates and hence evaluate.

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dy\, dx.$$

Sol:

$y = 0$ and $x = a$

$y = 0;\ y^2 = (\sqrt{a^2-x^2})^2$

$y^2 = a^2 - x^2$

$x^2 + y^2 = a^2 \Rightarrow$ Equ. of circle

Transfer into polar form

$x = r\cos\theta \qquad y = r\sin\theta$

$\sqrt{x^2+y^2} = \sqrt{r^2} = r.$

$$dx\,dy = \begin{vmatrix} dx/dr & dy/dr \\ dx/d\theta & dy/d\theta \end{vmatrix}$$

$x = r\cos\theta \qquad y = r\sin\theta$

$\frac{dx}{dr} = (1)\cos\theta \qquad \frac{dy}{dr} = \sin\theta$

$= \cos\theta$

$\frac{dx}{d\theta} = -r\sin\theta \qquad \frac{dy}{d\theta} = r\cos\theta$

$$dxdy = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r(1)$$

$$= r$$

$$dxdy = r\,dr\,d\theta.$$

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dy\,dx.$$

$$x^2+y^2 = r^2$$

$$x^2+y^2 = a^2$$

$$r^2 = a^2$$

$$\boxed{r=a}$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r.\, r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r^2.\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left(\frac{r^3}{3}\right)_0^a d\theta$$

$$= \int_0^{\pi/2} \left(\frac{a^3}{3}\right) d\theta$$

$$= \frac{a^3}{3}(0)_0^{\pi/2}$$

$$= \frac{\pi}{2}\cdot\frac{a^3}{3}$$

$$= \frac{\pi a^3}{6}$$

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4) $\int_0^a \int_0^{\sqrt{ax^2-x}} x^2 dy dx$. (when m.n is even it has $\pi/2$)

<u>sol:</u>

$$= \int_0^a \int_0^{\sqrt{ax^2-x}} x^2 dy dx$$

$$= \int_0^a x^2 (y)_0^{\sqrt{ax^2-x}} dx$$

$$= \int_0^a x^2 \sqrt{ax^2-x}\, dx$$

$$= \int_0^a x^2 x^{1/2} \sqrt{a-x}\, dx$$

$$= \int_0^a x^{5/2} \sqrt{a-x}\, dx$$

put $x = a\sin^2\theta$ (by using polar form)

$$y = \sqrt{ax-x^2}$$

Squaring on both sides

$$y^2 = ax - x^2$$

$$x^2 + y^2 = ax$$

$$x^2 + y^2 = a$$

$$x = a \sin\theta$$

$$x = a \sin^2\theta$$

$$\frac{dx}{d\theta} = a \, 2\sin\theta\cos\theta$$

$$\frac{dx}{d\theta} = 2a\sin\theta\cos\theta$$

$$dx = 2a\sin\theta\cos\theta \cdot d\theta$$

$x = 0,$

$$0 = a\sin^2\theta$$

$$a\sin^2\theta = 0$$

$$\sin^2\theta = 0/a$$

$$\sin^2\theta = 0$$

$$\sin\theta = 0$$

$$\theta = \sin^{-1}(0)$$

$$\theta = 0.$$

$x = a,$

$$a\sin^2\theta = a$$

$$\sin^2\theta = a/a$$

$$\sin^2\theta = 1$$

$$\sin\theta = 1$$

$$\theta = \sin^{-1}(1)$$

$$\theta = \pi/2$$

$$= \int_0^{\pi/2} (a\sin^2\theta)^{5/2} \sqrt{a - a\sin^2\theta} \; 2a\sin\theta\cos\theta \, d\theta$$

$$\int_0^{\pi/2} (a^{5/2})(\sin^5\theta) \sqrt{a(1-\sin^2\theta)} \; 2a\sin\theta\cos\theta \, d\theta.$$

$$1 - \sin^2\theta = \cos^2\theta.$$

$$= \int_0^{\pi/2} (a^{5/2})(\sin^5\theta)\sqrt{a}\sqrt{\cos^2\theta}\ 2a\sin\theta\cos\theta\, d\theta.$$

$$= 2\int_0^{\pi/2} a^{5/2}. a^{1/2} \sin^6\theta \cos^2\theta\, d\theta \cdot$$

$$= 2\int_0^{\pi/2} a^4 \sin^6\theta \cos^2\theta\, d\theta$$

$$= 2a^4 \int_0^{\pi/2} \sin^{\underline{m}6}\theta \cos^{\underline{n}2}\theta\, d\theta$$

$$\int_0^{\pi/2} \sin^m \cos^n dx = \frac{(m-1)(m-3)(n-1)(n-2)\,/\,\cancel{\pi/2}}{(m+n)(m+n-2)} (\pi/2)$$

$$= 2a^4 \frac{(6-1)(6-3)(6-5)\cdots(2-1)}{(6+2)(6+2-2)(6+2-4)(6+2-6)} \left(\frac{\pi}{2}\right)$$

$$= 2a^4 \frac{(5)(\cancel{3})(1)(1)}{(8)(\cancel{6})(4)(\cancel{2})} \left(\frac{\pi}{2}\right)$$

$$= \frac{a^4 5\pi}{128}$$

$$= \frac{5\pi a^4}{128}$$

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5) Evaluate $\iint (x^2+y^2)\,dx\,dy$ over the area bounded by the curves $y = 4x$, $x+y=3$, $y=0$, $y=2$.

Sol:

$$y = 4x \longrightarrow ①$$

$$x + y = 3 \longrightarrow ②$$

$$(1) \Rightarrow 4x = y$$

$$x = y/4$$

$$(2) \Rightarrow x = 3 - y$$

$$= \int_0^2 \int_{y/4}^{3-y} (x^2+y^2)\, dx\, dy$$

$$= \int_0^2 \left[\frac{x^3}{3} + y^2 x\right]_{y/4}^{3-y} dy$$

$$= \int_0^2 \left[\frac{(3-y)^3}{3} + y^2(3-y)\right] - \left[\frac{(y/4)^3}{3} + y^3(y/4)\right] dy$$

$$= \int_0^2 \left[\frac{(3-y)^3}{3} + 3y^2 - y^3 - \frac{y^3}{192} - \frac{y^3}{4}\right] dy$$

$$= \int_0^2 \left[\frac{(3-y)^4}{12}(-1) + \frac{3y^3}{3} - \frac{y^4}{4} - \frac{y^4}{768} - \frac{y^4}{16}\right]_0^2$$

$$= \left[\frac{(3-y)^4}{12} + (2)^4 - \frac{(2)^4}{4} - \frac{(2)^4}{768} - \frac{(2)^4}{16}\right]_0^2$$

$$= \left[-\frac{(3-2)^4}{12} + (2)^4 - \frac{(2)^4}{4} - \frac{(2)^4}{768} - \frac{(2)^4}{16}\right]$$

$$= -\frac{1}{12} + 8 - 4 - \frac{1}{48} - 1 + \frac{27}{4}$$

$$= \frac{-4 + 144 - 1 + 324}{48}$$

$$= \frac{463}{48}$$