Example : 10

## Section - I

### Exercise

1) $y = e^x \sin x$

$y_1 = e^x \cos x + \sin x \, e^x$

$y_2 = e^x(-\sin x) + \cos x \, e^x - e^x \cos x + \sin x \, e^x$

$= -e^x \cancel{\sin x} + e^x \cos x + e^x \cos x + \cancel{e^x \sin x}$

$y_2 = 2e^x \cos x \rightarrow ①$

$2xy_1 = 2\cancel{x}e^x \cos x + 2\cancel{x}e^x \sin x \rightarrow ②$

$2y = 2e^x \sin x \rightarrow ③$

① - ② + ③

$y_2 - 2y_1 + 2y = 2e^x \cos x - 2e^x \cos x - 2e^x \sin x$

$= 2e^x \sin x$

$= 0$

2) $y = \cos^{-1} x$

$$y_1 = \frac{1}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\, y_1 = -1.$$

squaring on both sides.

$$(1-x^2) y_1^2 = 1.$$

Diff W.R to "x"

$$(1-x^2)\, 2y_1 y_2 + y_1^2 (-2x) = 0.$$

$$(1-x^2)\, 2y_1 y_2 - 2x y_1^2 = 0$$

$$2y_1 \left[(1-x^2)y_2 - x y_1\right] = 0$$

$$(1-x^2) y_2 - x y_1 = 0.$$

3) $y = e^{m \cos^{-1} x}$

$$y_1 = e^{m\cos^{-1}x} \cdot m \left(\frac{1}{\sqrt{1-x^2}}\right)$$

$$\sqrt{1-x^2}\, y_1 = -m e^{m \cos^{-1} x}.$$

$$\sqrt{1-x^2}\, y_1 = -my.$$

squaring on both sides

$$(1-x^2) y_1^2 = m^2 y^2$$

diff w.R. to x.

$$(1-x^2)\, 2y_1 y_2 + y_1^2 (-2x) = m^2 2yy_1$$

$$2y_1 \left[(1-x^2) y_2 - x y_1\right] = m^2 2yy_1$$

$$\frac{\cancel{2y_1}}{\cancel{2y_1}} \left[(1-x^2)y_2 - xy_1\right] = m^2 y$$

$$(1-x^2)y_2 - x y_1 = m^2 y$$

$$(1-x^2)y_2 - xy_1 - m^2 y = 0.$$

5)

$y = (\log x)^2$

$y = 2\log x \cdot \frac{1}{x}$

$x y_1 = 2\log x$

squaring on both sides

$x^2 y_1^2 = 4(\log x)^2$

$x^2 y_1^2 = 4y$

Diff w.r.to "x"

$x^2\, 2y_1 y_2 + y_1^2 (2x) = 4y_1$

$2y_1 [x^2 y_2 + x y_1] = 2 \times 2 y_1$

$\frac{\cancel{2y_1}}{\cancel{2y_1}} [x^2 y_2 + x y_1] = 2$

$x^2 y_2 + x y_1 = 2.$

6)

$y = ae^{2x} + Bxe^{2x}$

$y_1 = 2ae^{2x} + B\left[x(2e)^{2x} + e^{2x}\right]$

$= 2ae^{2x} + 2Bxe^{2x} + Be^{2x}$

$y_2 = 2a(2e^{2x}) + 2B\left[x(2e^{2x}) + e^{2x}\right] + 2Be^{2x}.$

$= 4ae^{2x} + 4Bxe^{2x} + 2Be^{2x} + 2Be^{2x}.$

$y_2 = 4ae^{2x} + 4Bxe^{2x} + 4Be^{2x} \rightarrow (1)$

$4y_1 = 8ae^{2x} + 8Bxe^{2x} + 4Be^{2x} \rightarrow (2)$

$4y = 4ae^{2x} + 4Bxe^{2x}. \rightarrow (3)$

(1) − (2) + (3)

$y_2 - 4y_1 + 4y = 4ae^{2x} + 4Bxe^{2x} + 4Be^{2x}$
$- 8ae^{2x} - 8Bxe^{2x} - 4Be^{2x} + 4ae^{2x}$

$+ \cancel{4Bxe^{2x}} = 0.$

section : 1

Eg: 4

If $y = a(\cos(\log x) + b\sin(\log x))$. show that $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$.

$$y = a\cos(\log x) + b\sin(\log x)$$

$$\frac{dy}{dx} = \frac{a\sin(\log x)}{x} + \frac{b\sin(\log x)}{x}$$

multiply $x$ with both sides

$$x\left(\frac{dy}{dx}\right) = a\sin(\log x) + b\cos(\log x)$$

Diff W.R.T "$x$"

$$x\left(d^2y/dx^2\right) + dy/dx = \frac{-a\cos(\log x)}{x} - \frac{b\sin(\log x)}{x}$$

multiply $x$ on both sides

$$x\left(d^2y/dx^2\right) + dy/dx^2 = \frac{-a\cos(\log x) - b\sin(\log x)}{x}$$

$$x^2\left(d^2y/dx^2\right) + x\, dy/dx = -a\cos(\log x) - b\sin(\log x)$$

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} = -\left[a\cos(\log x) - b\sin(\log x)\right]$$

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y.$$

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$$

Hence proved.

Sec: III

Eg: 1

Find the $n^{th}$ derivative of $x^3 e^{5x}$.

$$D^n(uv) = D^n(u)\,v + nC_1\, D^{n-1}(u)\, D(v) + nC_2\, D^{n-2}(u)\, D^2(v) + \dots$$

$D^n(\underset{u}{x^3}\,\underset{v}{e^{5x}})$ $\quad nC_r = \frac{n!}{r!(n-r)!}$ $\quad nP_r = \frac{n!}{(n-r)!}$

$v = x^3$

$D(v) = 3x^2$

$D^2(v) = 6x$

$D^3(v) = 6$

$D^4(v) = 0$

$$D^n(e^{5x} x^3) = D^n(e^{5x})x^3 + nC_1 D^{n-1}(e^{5x})\, 3x^2 + nC_2\, D^{n-2}(e^{5x})\, 6x + nC_3\, D^{n-3}(e^{5x})\, 6$$

$$D^n(e^{ax}) = a^n e^{ax}.$$

$$= 5^n e^{5x} x^3 + n\,5^{n-1}(e^{5x})\,3x^2 + \frac{n(n-1)}{2!}\,5^{n-2}e^{5x}\,6x + \frac{n(n-1)(n-2)}{3!}\,5^{n-3}e^{5x}\cdot 6.$$

Eg: 2

Find $n^{th}$ derivative of $x^2\cos 2x$.

$D^n(\cos 2x\; x^2)$.

$V = x^2$

$D(V) = 2x$

$D^2(V) = 2$

$D^3(V) = 0$

$$D^n(\cos 2x\; x^2) = D^n(\cos 2x)\,x^2 + nC_1\, D^{n-1}(\cos 2x)\,2x + nC_2\, D^{n-2}(\cos 2x)\,(2).$$

$$\cos(ax+b) = a^n\cos\left(ax+b+\frac{n\pi}{2}\right).$$

$$= 2^n\cos\left(2x+\frac{n\pi}{2}\right)x^2 + n\,2^{n-1}\left(\cos 2x + \frac{(n-1)\pi}{2}\right)2x + \frac{n(n-1)}{2!}\,2^{n-2}\left(\cos 2x + \frac{(n-2)\pi}{2}\right)2.$$

3) If $y = a\cos(\log x) + b\sin(\log x)$ show that

$x^2 y_2 + x y_1 + y = 0$

$D^n(x^2 y_2 + x y_1 + y) = D^n(x^2 y_2) + D^n(x y_1) + D^n(y)$

$~~D^n(x^2y_2) = D~~$

$D^n(uv) = D^n(u)v + nc_1 D^{n-1}(u) D(v) + nc_2 D^{n-2}(u) D^2(v) + \dots$

$D^n(x^2 y_2) = D^n(y_2)x^2 + nc_1 D^{n-1}(y_2)(2x) + nc_2 D^{n-2}(y_2)\,2 + 0$

$D^n(x y_1) = D^n(y_1)x + nc_1 D^{n-1}(y_1)(1)$

$D^n(y) = D^n(y)$

$D^n(x^2 y_2) + D^n(x y_1) + D^n(y) = D^n(y_2)x^2 + nc_1 D^{n-1}(y_2)2x + nc_2 D^{n-2}(y_2)\,2 + D^n(y_1)x + nc_1 D^{n-1}(y_1)(1) + D^n(y)$

$= y_{n+2}(x^2) + n y_{n+1}(2x) + \frac{n(n-1)}{2} y_n(2) + y_{n+1}(x) + n y_n + y_n$

$= y_{n+2}(x^2) + n y_{n+1}(2x) + n(n-1) y_n + x y_{n+1} + n y_n + y_n$ (multiply)

$~~= y_{n+2}(x^2) + n y_{n+1}\,2x~~$

$= y_{n+2}(x^2) + 2x\,n y_{n+1} + n^2 y_n - \cancel{n y_n} + x y_{n+1} + \cancel{n y_n} + y_n$

$= y_{n+2}(x^2) + 2x\,n y_{n+1} + n^2 y_n + x y_{n+1} + y_n$

$= x^2 y_{n+2} + 2xny_{n+1} + xy_{n+1} + n^2 y_n + y_n$

$\Rightarrow x^2 y_{n+2} + xy_{n+1}(2n+1) + y_n(n^2+1)$

$\rightarrow x^2 y_{n+2} + xy_{n+1}(2n+1) + y_n(n^2+1) = 0.$

Eg: 4

If $y = \sin^{-1}x$, prove that $(1-x^2)y_{n+2} - (2n+1) xy_{n+1} - n^2 y_n = 0.$

$y = \sin^{-1}x$

$$y_1 = \frac{1}{\sqrt{1-x^2}}$$

$\sqrt{1-x^2}\, y_1 = 1.$

squaring both sides

$(1-x^2) y_1^2 = 1$

Diff w.r.to "x"

$(1-x^2)\, 2y_1 y_2 + y_1^2(-2x) = 0$

$(1-x^2)\, 2y_1 y_2 - 2y_1^2 x = 0$

$2y_1\left[(1-x^2) y_2 - xy_1\right] = 0$

$(1-x^2) y_2 - xy_1 = 0.$

$D^n\left[(1-x^2)y_2 - xy_1\right]$

$= D^n\left[\underset{v}{(1-x^2)}\,\underset{u}{y_2}\right] - D^n\left[\underset{v}{x}\,\underset{u}{y_1}\right]$

$D^n\left[(1-x^2)y_2\right] = D^n(y_2)(1-x^2) + nC_1\, D^{n-1}(y_2)(-2x) +$
$nC_2\, D^{n-2}(y_2)(-2).$

$$D^n(xy_1) = D^n(y_1)x + nC_1 D^{n-1}(y_1)(1)$$

$$D^n[(1-x^2)y_2 - xy_1] = D^n(y_2)(1-x^2) + nC_1 D^{n-1}(y_2)(-2x)$$
$$+ nC_2 D^{n-2}(y_2)(-2) + D^n(y_1)x +$$
$$nC_1 D^{n-1}(y_1)(1).$$

$$= y_{n+2}(1-x^2) + ny_{n+1}(-2x) + \frac{n(n-1)}{2}y_n(-2)$$
$$+ y_{n+1}(x) + ny_n = 0.$$

$$=(1-x^2)y_{n+2} - 2xny_{n+1} - n(n-1)y_n +$$
$$xy_{n+1} + ny_n = 0.$$

$$= (1-x^2)y_{n+2} - 2xny_{n+1} - [n^2y_n - ny_n] +$$
$$xy_{n+1} + ny_n = 0.$$

$$= (1-x^2)y_{n+2} - 2xny_{n+1} - n^2y_n + ny_n +$$
$$xy_{n+1} + ny_n = 0.$$

$$= (1-x^2)y_{n+2} - 2xny_{n+1} + xy_{n+1} - n^2y_n +$$
$$ny_n + ny_n = 0.$$

$$= \cancel{(1-x^2)y_{n+2} - 2xy_{n+1}(2n-1)}$$

$$= (1-x^2)y_{n+2} - xy_{n+1}(2n-1) + y_n(-n^2 +$$
$$n + n) = 0.$$

$$=(1-x^2)y_{n+2} - (2n-1)xy_{n+1}$$

Section

~~1) If $y = e^{3}\sin x$~~

4) $y = x\sin(\log x) + x\log x$, $x^2y_2 - xy + 2y = x\log$

Sol:

$$y = x\sin(\log x) + x\log x$$

$$y_1 = \frac{x\cos(\log x)}{x} + \sin(\log x) + x\cdot 1/x + \log x$$

$$y_1 = \cos(\log x) + \sin(\log x) + \log x + 1$$

$$y_2 = -\frac{\sin(\log x)}{x} + \frac{\cos(\log x)}{x} + 1/x$$

$$xy_2 = -\sin(\log x) + \cos(\log x) + 1$$

$$xy_2 = -\sin(\log x) + y_1 - \sin(\log x) + \log x$$

$$xy_2 = -2\sin(\log x) + y_1 + \log x$$

Multiple both sides by $x$..

$$x^2y_2 = -2x\sin(\log x) + xy_1 + x\log x$$

$$x^2y_2 = -2y + xy_1 + x\log x$$

$$x^2y_2 - xy_1 + 2y = x\log x$$

Hence proved.

1) Sin 2x

Sol:

cos(

2)

Section - 2

1) $\sin 2x \sin 3x$.

Sol:

$\cos(A-B) - \cos(A+B) = \cos A \cos B - \sin A \sin B - \cos$ -

$\cos A \cos B - \sin A \sin B$

$\cos(A-B) = \cos(A+B) = 2\sin A \sin B$

$$\frac{\cos(A-B) - \cos(A+B)}{2} = \sin A \sin B$$

$$\frac{\cos(3x-2x) - \cos(3x+2x}{2} = \sin 2x \sin 3x$$

$$\frac{1}{2}\left[\cos(x) - \cos(5x)\right]$$

$$\frac{1}{2}\left[D^n(\cos x)\right] - \left[D^n(\cos 5x)\right]$$

$$\frac{1}{2}\cos\left[x+\frac{n\pi}{2}\right] - 5^n\cos\left[5x+\frac{n\pi}{2}\right]$$

2) $\cos^4 x \cos 2x \cos 3x$.

Sol:

$\cos x(\cos 2x \cos 3x)$

$\cos(A+B) + \cos(A-B) = \cos A \cos B - \sin A \sin B +$

$\cos A \cos B - \sin A \sin B$

$\cos(A+B) + \cos(A-B) = 2\cos A \cos B$

$$\frac{\cos(A+B) + \cos(A-B)}{2} = 2\cos A \cos B$$

$$= \cos\left[\frac{\cos(2x+3x)}{2} + \frac{\cos(2x-3x)}{2}\right]$$

$$= \cos x\left[\frac{\cos(5x)}{2} + \frac{\cos(2x)}{2}\right]$$

$$= \tfrac{1}{2}\left[\cos x \cos 5x + \cos 2x\right]$$

$$= \tfrac{1}{2}\left[\cos x \cos 5x + \cos^2 x\right]$$

$$= \tfrac{1}{2}\left[\frac{\sin(6x) - \sin(4x)}{2} + \frac{1-\cos 2x}{2}\right]$$

$$= \tfrac{1}{4}\left[\sin 6x - \sin 4x + 1 - \cos 2x\right]$$

$$= \tfrac{1}{4}\, D^n\left[\sin 6x - \sin 4x\right] + D^n[1] - D^n[\cos 2x]$$

$$= \tfrac{1}{4}\, 6^n \sin x\left(6 + n\pi/2\right) + 4^n \sin\left(4x + \frac{n\pi}{2}\right) + 2^n \cos\left(2x + \right.$$

$$\tfrac{1}{4}\, 2^n \cos\left(2x + n\pi/2\right) + 4^n\left(4x + \frac{n\pi}{2}\right) + 6^n \cos\left(6x + \frac{n\pi}{2}\right.$$

3) $\cos^4 x$.

Sol:

$$= (\cos^2 x)^2$$

$$= \left[\frac{1 + \cos 2\, 2x}{2}\right]^2$$

$$= \tfrac{1}{4}\left(1 + \cos^2 2x + 2\cos 2x\right)$$

$$= \tfrac{1}{4}\left(1 + \frac{1 + \cos 4x}{2} + 2\cos 2x\right)$$

$$= \tfrac{1}{4}\left(\frac{2 + 1 + \cos 4x + 2\cos 2x}{2}\right)$$

$$= \tfrac{1}{8}\left(3 + \cos 4x + 2\cos 2x\right)$$

$$= \tfrac{1}{8} \cdot D^n(3) + D^n(\cos 4x) + D^n(2\cos 2x)$$

$\frac{1}{8} \cdot 4^n \cos\left(4x + n\pi/2\right) + 2^{n-2}\cos\left(2x + n\pi/2\right).$

4) $\cos^2 x \sin^3 x.$

Sol:

$$\cancel{Z}\ \ Z = \cos x + i\sin x$$

$$1/Z = \cos x - i\sin x$$

$$Z + 1/Z = \cos x + i\sin x + \cos x - i\sin x$$

$$= 2i\sin x.$$

$$\cancel{Z^n = \cos^n x + i\sin^n x}.$$

$$Z - 1/Z = \cos x + i\sin x - \cos x + i\sin x$$

$$= 2i\sin x$$

$$Z^n = \cos^n x + i\sin^n x$$

$$Z^n + 1/Z^n = 2\cos^n x, \quad Z^n - 1/Z^n = Zi\sin^n nx.$$

$$\cos^2 x \sin^3 x = (2\cos x)^2 (2i\sin x)^3$$

$$= (Z + 1/Z)^2 (Z - 1/Z)^3$$

$$= (Z + 1/Z)^2 (Z - 1/Z)^2 (Z - 1/Z)$$

$$= \left(Z^2 - 1/Z^2\right)\left(Z^2 - 1/Z^2\right)\left(Z - 1/Z\right)$$

$$= \left(Z^4 + 1/Z^4 - 2\right)\left(Z - 1/Z\right)$$

$$= Z^5 - Z^3 + 1/Z^3 - 1/Z^5 - 2Z + 2/Z$$

$$= \left(Z^5 - 1/Z^5\right) - \left(Z^3 - 1/Z^3\right) - 2\left(Z - 1/Z\right)$$

$$\left(Z^n + 1/Z^n\right) = 2\cos nx, \quad Z^n - 1/Z^n = 2i\sin nx$$

$$= 2i\sin 5x - 2i\sin 3x - 2(2i\sin x)$$

$= 2i(\sin 5x - \sin 3x - 2\sin x)$

$= 2i(\sin 5x - \sin 3x - 2\sin x)$

$(2\cos x)^2 (2i\sin)^3 = 2i(\sin 5x - \sin 3x - 2\sin x)$

$(4\cos^2 x)(8i^3 \sin^3 x) = 2i(\sin 5x - \sin 3x - 2\sin x)$

$8i^3 4\cos^2 x \cdot \sin^3 x = 2i(\sin 5x - \sin 3x - 2\sin x)$

$-16\cos^2 x \sin^3 x = -1/16(\sin 5x - \sin 3x - 2\sin x)$

$\cos^2 x \sin^3 x = 1/16(-\sin 5x + \sin 3x + 2\sin x)$

$D^n(\cos^2 x \sin^3 x) = 1/16\left[D^n(-\sin 5x) + D^n(\sin 3x) + D^n(2\sin x)\right]$

$= 1/6\left[-5^n \sin\left(5x + \frac{n\pi}{2}\right) + 3^n \sin\left(3x + \frac{n\pi}{2}\right) + 2\sin\left(x + n\pi/2\right)\right]$

10) $\log(4 - x^2)$

Sol :

$= \log(4 - x^2)$

$= \log(2^2 - x^2)$

$= \log(2 - x) - \log(2 + x)$

$\log(ax + b) = \frac{(-1)^{n-1}(n-1)!\, an}{(ax+b)^n}$

$= \frac{(-1)^{n-1}(n-1)!(1)^n}{(x+2)^n} - \frac{(-1)^n(n-1)!(1)^n}{(2-x)^n}$

$= \frac{(-1)^{n-1}(n-1)!}{(x+2)^n} - \frac{(n-1)!}{(2-x)^n}$

9) $\frac{1}{x^2-a^2}$

sol:

$$\frac{1}{(x^2-a^2)} = \frac{1}{(x-a)(x+a)}$$

$$= \frac{A}{(x-a)} + \frac{B}{(x+a)}$$

$$= \frac{A(x+a)+B(x-a)}{(x-a)(x+a)}$$

$$\frac{1}{(x^2-a^2)} = \frac{A(x+a)+B(x-a)}{(x^2-a^2)}$$

$$A(x+a)+B(x-a) = 1.$$

$x=-a$,

$$0+B(-2a)=1$$

$$B = -1/2a$$

$x=a$,

$$A(2a)+0=1$$

$$A = 1/2a$$

$$= \frac{(1/2a)}{(x-a)} + \frac{(1/2a)}{(x+a)}$$

$$= \frac{(1/2a)}{(x-a)} - \frac{(1/2a)}{(x+a)}$$

$$= D^n\left(\frac{1/2a}{x-a}\right) - D^n\left(\frac{1/2a}{x+a}\right).$$

$$D^n\left(\frac{1}{ax+b}\right) = \frac{(-1)^n (n)!\, a^n}{(ax+b)^{n+1}}$$

$$= \frac{1}{2a} \frac{(-1)^n (n!) (1)^n}{(x-b)^{n+1}} - \frac{1}{2a} \frac{(-1)^n (n!) (1)^n}{(x-a)^{n+1}}$$

$$= \frac{1}{2a} (-1)^n n! \left( \frac{1}{(x\mp a)^{n+1}} - \frac{1}{(x+a)^{n+1}} \right)$$

$$= \frac{(-1)^n n!}{2a} \left[ \frac{1}{(x-a)^{n+1}} - \frac{1}{(x+a)^{n+1}} \right]$$

8) $\dfrac{x^2}{(x+1)^2(x+2)}$

Sol:

$$\frac{x^2}{(x+1)^2(x+2)} = \frac{A}{(x+2)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$$

$$\frac{x^2}{(x+1)^2(x+2)} = \frac{A(x+1)^2 + B(x+2) + C((x+2)(x+1)}{(x+2)(x+1)^2}$$

$$\frac{x^2}{(x+1)(x+2)} = \frac{A}{(x+2)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$$

$$\frac{x^2}{(x+1)(x+2)} = \frac{A(x+1)^2 + B(x+2)(x+1) + C(x+2)}{(x+1)(x+2)}$$

$$x^2 = A(x+1)^2 + B(x+2)(x+1) + C(x+2)$$

$x = -2,$

$$(-2)^2 = A(-2+1)^2 + B(0) + (0)$$

$$4 = A \implies A = 4.$$

$(-1)^2 = A(0) + B(-1+2)(-1+1) + C(-1+2)$

$1 = C$

$C = 1$

$x = 1,$

$(+1)^2 = A(1+1)^2 + B(1+2)(1+1) + C(1+2)$

$1 = 4A + 6B + 3C$

$$\frac{x^2}{(x+1)^2(x+2)} = \frac{A}{(x+2)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$$

$A = 4,\ C = 1.$

$1 = 4(4) + 6B + 3(1)$

$1 = 16 + 6B + 3$

$6B = 1 - 3 - 16$

$6B = -2 - 16$

$6B = -18$

$B = -18/6$

$B = 3$

$$= D^n\left(4/_{x+2}\right) + D^n\left(-3/_{x+1}\right) + D^n\left(1/_{(x+1)^2}\right)$$

$$D^n\left(1/_{ax+b}\right) = \frac{(-1)^n(n!)a^n}{(ax+b)^{n+1}}$$

$$= \frac{4(-1)^n n!(1)^n}{(x+2)^{n+1}} + \frac{(-3)(-1)^n n!(1)^n}{(x+1)^{n+1}} + \frac{(-1)^n(n+1)!}{(x+1)^{n+2}}$$

$$\Rightarrow \frac{4(-1)^n n!(+1)^n}{(x+2)^{n+1}} - \frac{3(-1)^n n!}{(x+1)^{n+1}} + \frac{(1)^n(n+1)!}{(x+1)^{n+2}}$$

Section - III

1) $x^2 e^{ax}$

Sol:

$= D^n e^{ax} x^2 + nC_1 D^{n-1} e^{ax} 2x + nC_2 D^{n-2} e^{ax} 2$

$= D^n e^{ax} x^2 + n D^{n-1} e^{ax} 2x + n(n-1) D^{n-2} e^{ax} 2$

$= a^n e^{ax} x^2 + n a^{n-1} e^{ax} 2x + n(n-1) a^{n-2} e^{ax} 2$

$= e^{ax} (a^n x^2 + 2nx a^{n-1} + n(n-1) a^{n-2})$

2) $x^3 \cos x$

Sol:

$= D^n(\cos x) x^3 + nC_1 D^{n-1}(\cos x) 3x^2 + nC_2 D^{n-2}(\cos x)$
$6x + nC_3 D^{n-3} (\cos x) 6$

$= 1^n \cos(x + n\pi/2) x^3 + n(1)^{n-1} \cos\left(x + \frac{(n-1)\pi}{2}\right) 3x^2 +$
$\frac{n(n-1)}{2} 1^{n-2} \cos\left(\frac{x+(n-2)\pi}{2}\right)$
$6x + \frac{n(n-1)(n-2)}{6} 1^{n-3} \cos\left(\frac{x+(n-3)\pi}{2}\right)$

$1^n \cos\left(x + \frac{n\pi}{2}\right) x^3 + n \cdot 1^{n-1} \cos\left(x + \frac{(n-1)\pi}{2}\right) 3x^2 + n(n-1)$
$1^{n-2} \cos\left(x + \frac{(n-2)\pi}{2}\right) 3x + n(n-1)(n-2) 1^{n-3}$
$\cos\left(x + \frac{(n-3)\pi}{2}\right).$

4) $y = a\cos(\log x) + b\sin(\log x)$ S.T $x^2 y_{n+2} + 0(n+1)$
$x y_{n+1} + (n^2+1) y_n = 0.$

Sol:

$y = a\cos(\log x) + b\sin(\log x)$

$$y_1 = \frac{-a\sin\log x}{x} + \frac{b\cos\log x}{x}$$

$$xy_1 = -a\sin\log x + b\cos\log x.$$

Diff again.

$$xy_2 + y_1 = \frac{-a\cos\log x}{x} - \frac{b\sin\log x}{x}$$

$$xy_2 + y_1 = \frac{-a\cos\log x - b\sin\log x}{x}$$

Multiply $x$ on both sides,

$$x^2y_2 + y_1x = -a\cos\log x - b\sin\log x$$

$$x^2y_2 + y_1x = -(a\cos\log x + b\sin\log x)$$

$$\Rightarrow x^2y_2 + y_1x = -y$$

$$\Rightarrow x^2y_2 + y_1x + y = 0$$

$$\Rightarrow D^n(x^2y_2 + y_1x + y) = 0$$

$$\Rightarrow D^n(x^2y_2) + D^n(xy_1) + D^ny = 0$$

$$\Rightarrow D^ny_2\,x^2 + nC_1D^{n-1}y_2\,2x + nC_2D^{n-2}y_2\,2 + D^ny_1\,x + nC_1D^{n-1}y_1(1) + D^n(y) = 0.$$

$$\Rightarrow y_{n+2}x^2 + ny_n\,2x + \frac{n(n-1)}{2}y_n\,2 + y_{n+1}x + ny_n + y_n = 0$$

$$\Rightarrow y_{n+2}x^2 + ny_n\,2x + n(n-1)y_n + y_{n+1}x + ny_n + y_n = 0.$$

$$\Rightarrow y_{n+2}x^2 + y_{n+1}x + ny_n\,2x + n^2y_n - ny_n + ny_n + y_n = 0.$$

$$y_{n+2}x^2 + xy_{n+1}(2n+1) + y_n(n^2 - n + n + 1) = 0.$$

$y_{n+2}x^2 + xy_{n+1} + y_n(n^2+1) = 0.$

Hence proved.

5) $y = e^{a\sin^{-1}x}$ show that $(1+x^2)y_{n+2} + xy_{n+1}(2n+1) + (n^2+a^2)y_n = 0$

Sol:

$$y = e^{a\sin^{-1}x}$$

$$y_1 = e^{a\sin^{-1}x} \cdot \frac{a}{\sqrt{1+x^2}}$$ 6

$$\sqrt{1+x^2}\, y_1 = e^{a\sin^{-1}x}\, a$$

squaring on both sides

$$(1+x^2)y_1^2 = (e^{a\sin^{-1}x})^2 a^2$$

$$(1+x^2)y_1^2 = y^2a^2$$

Diff. again

$$(1+x^2)\,2y_1y_2 + 2xy_1^2 - a^2 2y_1y$$

$$(1+x^2)\,2y_1y_2 + 2xy_1^2 = a^2 2y_1y$$

$(\div 2y_1)$.

$$(1+x^2)y_2 + 2xy_1 - a^2y = 0$$

$$D^n(1+x^2)y_2 + D^n xy_1 - D^n a^2y = 0.$$

$$D^n y_2 (1+x^2) + nC_1 D^{n-1}y_2\, 2x + nC_2 D^{n-2}y_2\, 2 + D^n xy_1 + nC_1 D^{n-1}y_1 - D^n a^2 y = 0.$$

$$y_{n+2}(1+x^2) + ny_{n+1}\,2x + \frac{n(n-1)}{2}y_n 2 + y_{n+1}x + ny_n - D^n a^2y = 0.$$

$$y_{n+2}(1+x^2) + ny_{n+1}\,2x + n^2y_n - ny_n + y_{n+1}x + ny_n - a^2y_n = 0$$

$y_{n+2}(1+x^2) + n y_{n+1}\, 2x + n^2 y_n - n y_n + y_{n+1}\, x + n y_n -$

$a^2 y_n = 0.$

$)+1)$ $y_{n+2}(1+x^2) + n y_{n+1}\, 2x + y_{n+1}\, x + n^2 y_n - n y_n + n y_n -$

$a^2 y_n = 0.$

$y_{n+2}(1+x^2) + x y_{n+1}(2n+1) - (n^2 + a^2) y_n = 0.$

Hence proved.

6) If $\cos^{-1}(y/n) = n \log(x/b)$ prove that $x^2 y_{n+2} + (2n+1) x y_{n+1} + 2n^2 y_n = 0.$

Sol:

$$\cos^{-1}(y/n) = n(\log(x/b))$$

$$\frac{y_1/n}{\sqrt{1 - y^2/n^2}} = n \frac{1/b}{x/b}$$

$$\frac{y_1/n}{\sqrt{\frac{n^2 - y^2}{n}}} = \frac{n(1/b)}{x/b}$$

$$\frac{y_1}{\cancel{n}} \times \frac{n}{\sqrt{n^2 - y^2}} = n(1/b) \times (b/a)$$

$$\frac{y_1}{\sqrt{n^2 - y^2}} = \frac{n}{x}$$

$$\frac{y_1^2}{(n^2 - y^2)} = \frac{n^2}{x^2}$$

$$x^2 y_1^2 = n^2(n^2 - y^2)$$

$$x^2\, 2y_1\, y_2 + y_1^2\, 2x = n^2(-2 y y_1)$$

$x^2y_2+xy_1+n^2y=0$

$=x^2y_{n+2}+n(2x)y_{n+1}+\frac{n(n-1)}{\cancel{2}}(\cancel{2})y_n+xy_{n+1}+ny_n$
$n^2y_n=0$.

$x^2y_{n+2}+xy_{n+1}(2n+1)+n^2y_n-\cancel{ny_n}+\cancel{ny_n}+n^2y_n$

$x^2y_{n+2}+xy_{n+1}(2n+1)+2n^2y_n=0$.

7) If $y=(\sin^{-1}x)^2$ prove that $(1-x^2)y_{n+2}-(2n+1)$
$xy_{n+1}-n^2y_n=0$.

Sol:

$y=(\sin^{-1}x)^2$

$y_1=2\sin^{-1}x\ \frac{1}{\sqrt{1-x^2}}$

$y_1\sqrt{1+x^2}=2\sin^{-1}x$

$y_1^2(1-x^2)=4(\sin^{-1}x)^2$

$y_1^2(1-x^2)=4y$

$\cancel{2y_1}y_2(1-x^2)+(-\cancel{2}x)y_1^{\cancel{2}}=\cancel{4}\cancel{y_1}$

$y_2(1-x^2)-xy_1-2=0$

$y_{n+2}(1-x^2)+(-2x)ny_{n+1}+\frac{n(n-1)}{\cancel{2}}(-\cancel{2})y_n-(xy_{n+1}+$
$ny_n)$

$y_{n+2}(1-x^2)-xy_{n+1}(2n-1)-n^2y_n+\cancel{ny_n}-\cancel{ny_n}$

$y_{n+2}(1-x^2)-xy_{n+1}(2n-1)-n^2y_n$.

9) If
$y_n$
Sol

Prove that $(x^2-1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0$.

<u>sol</u>:

$$y = (x^2-1)^n$$

Diff $\Rightarrow$

$$y_1 = n(x^2-1)^{n-1} 2x.$$

$$y_1 = 2xn(x^2-1)^n (x^2-1)^{-1}$$

$$y_1 = \frac{2xn(x^2-1)^n}{(x^2-1)}$$

$$y_1(x^2-1) = 2xny$$

Diff $\Rightarrow$

$$y_2(x^2-1) + y_1(2x) = 2n(xy_1 + y)$$

$$y_2(x^2-1) + y_1(2x) = 2nxy_1 + 2ny$$

$$y_2(x^2-1) + 2xy_1 - 2nxy_1 - 2ny = 0.$$

$$y_2(x^2-1) + 2xy_1(1-n) - 2ny = 0.$$

$$= y_{n+2}(x^2-1) + ny_{n+1}(2x) + \frac{n(n-1)}{\cancel{2}}(\cancel{2})y_n + 2(1-n)xy_{n+1}$$
$$+ n2(1-n)y_n - 2ny_n$$

$$= y_{n+2}(x^2-1) + 2xny_{n+1} + \cancel{2ny_{n+1}} - 2nx\cancel{y_{n+1}} + \cancel{2ny_n}$$
$$- 2n^2y_n - \cancel{2ny_n} + n^2y_n - ny_n$$

$$= y_{n+2}(x^2-1) + 2xy_{n+1} - n^2y_n - ny_n$$

$$= y_{n+2}(x^2-1) + 2xy_{n+1} - ny_n(n-1)$$