$$= \frac{463}{48}$$

Triple Integral:

1) $$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{dz\,dy\,dz}{(x+y+z+1)^3}$$

Sol:

$$= \int\int\int (x+y+z+1)^{-3} dz\,dy\,dz$$

$$= \int\int \left[\frac{(x+y+z+1)^{-3+1}}{-3+1}\right] dy\,dx$$

$$= \int\int \frac{(x+y+z+1)^{-2}}{-2} dy\,dx$$

$$= -\tfrac{1}{2}\int\int \left[(x+y+z+1)^{-2}\right] dy\,dx$$

$$= -\tfrac{1}{2}\int\int (\cancel{x}+\cancel{y}+1-\cancel{x}-\cancel{y}+1)^{-2} (x+y+1)^{-2}$$

$$= -\tfrac{1}{2}\int\int (2)^{-2}(x+y+1)^{-2} dy\,dx$$

$$= -\tfrac{1}{2}\iint \left[\tfrac{1}{4} - (x+y+1) \right]^{-2} dy\,dx$$

$$= -\tfrac{1}{2}\int \left[\tfrac{y}{4} - \frac{(x+y+1)^{-2+1}}{-2+1} \right]_0^{1-x} dx$$

$$= -\tfrac{1}{2}\int \left[\frac{1-x}{4} - \frac{(x+1-x+1)^{-1}}{-2+1} \right] - \left[0 - \frac{(x+1)^{-1}}{-1} \right] dx$$

$$= -\tfrac{1}{2}\int \left[\frac{1-x}{4} - \frac{(x+1-x+1)^{-1}}{-1} \right] - \left[0 - \frac{(x+1)^{-1}}{1} \right] dx$$

$$= -\tfrac{1}{2}\int \left[\frac{1-x}{4} + (2)^{-1} - (x+1)^{-1} \right] dx$$

$$= -\tfrac{1}{2}\int \left[\frac{1-x}{4} + \frac{1}{2} - \frac{1}{x+1} \right] dx$$

$$= -\tfrac{1}{2}\int \left[\frac{1-x+2}{4} - \frac{1}{x+1} \right] dx$$

$$= -\tfrac{1}{2}\int \left[\frac{3}{4} - \frac{x}{4} - \frac{1}{x+1} \right] dx$$

$$= -\tfrac{1}{2}\left[\frac{3}{4}x - \frac{x^2}{8} - \log(x+1) \right]_0^1$$

$$= -\tfrac{1}{2}\left[\frac{3}{4} - \frac{1}{8} - \log(2) \right] - \left[\log(0+1) \right]$$

$$= -\tfrac{1}{2}\left[\frac{3\times 2}{4\times 2} - \frac{1\times 8}{8\times 8} - \log(2) \right] - \left[\log(0) \right]$$

$$= -\tfrac{1}{2}\left[\frac{6}{8} - \frac{1}{8} - \frac{8\log 2}{8} \right]$$

$$= -\frac{1}{2}\left[\frac{5-8\log 2}{8}\right]$$

$$= \frac{8\log 2 - 5}{16}$$

2) $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{a} r^2 \sin\theta \, dr \, d\theta \, d\phi$

Sol:

$$= \iint \left(\frac{r^3}{3}\right)_0^a \sin\theta \, d\theta \, d\phi$$

$$= \frac{a^3}{3}\iint \sin\theta \, d\theta \, d\phi$$

$$= \frac{a^3}{3}\int_0^{2\pi} (-\cos\theta)_0^{\pi/4} \, d\phi$$

$$= \frac{a^3}{3}\int_0^{2\pi} \left(-\cos\frac{\pi}{4} + \cos 0\right) d\phi$$

$$= \frac{a^3}{3}\int_0^{2\pi} \left[-\frac{1}{\sqrt{2}} \, d\phi + \int 1 \, d\phi\right]$$

$$= \frac{a^3}{3}\left\{-\frac{1}{\sqrt{2}}\phi + \phi\right\}_0^{2\pi}$$

$$= \frac{a^3}{3}\left\{-\frac{2\pi}{\sqrt{2}} + 2\pi\right\}$$

$$= \frac{2\pi a^3}{3}\left\{-\frac{1}{\sqrt{2}} + 1\right\}$$

$$= \frac{2\pi a^3}{3}\left\{\frac{-1}{\sqrt{2}} + \frac{\sqrt{2}}{\sqrt{2}}\right\}$$

$$= \frac{2\pi a^3}{3}\left\{\frac{\sqrt{2}-1}{\sqrt{2}}\right\}$$

$$= \frac{\pi a^3}{3}$$

3) Evaluate : $\iiint (x+y+z)\,dx\,dy\,dz$, where the region v is bounded by $x+y+z = a$ $(a>0)$ $x=0$, $y=0$, $z=0$.

Sol:

Given, $x+y+z = a$

$$\boxed{x = a}$$

$$-x+y \neq a \qquad\qquad x+y+z = a$$

$$\boxed{y = a-x} \qquad\qquad \boxed{z = a-x-y}$$

$$= \int_0^a \int_0^{a-x} \int_0^{a-x-y} (x+y+z)\,dz\,dy\,dx$$

$$= \int_0^a \int_0^{a-x} \left(xz + yz + z^2/2\right)_0^{a-x-y} dy\,dx$$

$$= \int_0^a \int_0^{a-x} \left[x(a-x-y) + y(a-x-y) + \frac{(a-x-y)^2}{2}\right] dy\,dx$$

$$= \int_0^a \int_0^{a-x} \left[x(a-x) - xy + y(a-x) - y^2 + \frac{(a-x-y)^2}{2}\right] dy\,dx.$$

$$= \int_0^a \left[x(a-x)y - x\frac{y^2}{2} + \frac{y^2}{2}(a-x) - \frac{y^3}{3} + \frac{(a-x-y)^3}{6}\cdot(-1) \right]_0^{a-x} dy\,dx$$

$$= \int_0^a \left[x(a-x)y - \frac{xy^2}{2} + \frac{y^2}{2}(a-x) - \frac{y^3}{3} - \frac{(a-x-y)^3}{6} \right]_0^{a-x} dx$$

$$= \int_0^a \left[x(a-x)(a-x) - x\frac{(a-x)^2}{2} + \frac{(a-x)^2}{2}(a-x) - \frac{(a-x)^3}{3} - \frac{(a-x-(a-x))^3}{6} \right] - \left[0-0+0-0-\frac{(a-x-0)^3}{6} \right] dx.$$

$$= \int_0^a \left[x(a-x)^2 - \frac{x(a-x)^2}{2} + \frac{(a-x)^3}{2} - \frac{(a-x)^3}{3} - \frac{(a-\cancel{x}-a+\cancel{x})^3}{6} + \frac{(a-x)^3}{6} \right] dx.$$

$$= \int_0^a \left[x(a-x)^2 - \frac{x(a-x)^2}{2} + \frac{(a-x)^2}{2} - \frac{(a-x)^3}{3} + \frac{(a-x)^3}{6} \right] dx$$

$$= \int_0^a \left[x(a-x)^2 - \frac{x(a-x)^2}{2} + \frac{(a-x)^3}{2} - \frac{(a-x)^3}{6} \right] dx$$

$$= \int_0^a \left[x(a-x)^2 - \frac{x(a-x)^2}{2} + \frac{3(a-x)^3}{6} - \frac{(a-x)^3}{6} \right] dx$$

$$= \int_0^a \left[x(a-x)^2 - \frac{x(a-x)^2}{2} + \frac{\cancel{2}^{1}(a-x)^3}{\cancel{6}_{3}} \right] dx$$

$$= \int_0^a \left[\frac{2x(a-x)^2}{2} - \frac{x(a-x)^2}{2} + \frac{(a-x)^3}{3} \right] dx$$

$$= \int_0^a \left[\frac{x(a-x)^2}{2} + \frac{(a-x)^3}{3} \right] dx$$

$$= \int_0^a \left[\frac{x(a^2+x^2-2ax)}{2} + \frac{(a-x)^3}{3} \right] dx$$

$$= \int_0^a \left[\frac{1}{2}(a^2x + x^3 - 2ax^2) + \frac{(a-x)^3}{3} \right] dx$$

$$= \int_0^a \left[a^2\frac{x^2}{2} + \frac{x^4}{4} - 2a\frac{x^3}{3} \right]_0^a + \left[\frac{(a-x)^4}{12}(-1) \right]_0^a$$

$$= \int_0^a \left[\left(\frac{a^4}{2} + \frac{a^4}{4} - \frac{2a^4}{3} \right) - 0 \right] - \frac{(a-a)^4}{12} + \frac{a^4}{12}$$

$$= \frac{1}{2}\left[\frac{6a^4 + 3a^4 - 8a^4}{12} \right] + \frac{a^4}{12}$$

$$= \frac{1}{2}\left(\frac{a^4}{12} \right) + \frac{a^4}{12}$$

$$= a^4/24 + a^4/12$$

$$= \frac{a^4 + 2a^4}{24} \Rightarrow \frac{\cancel{3}a^4}{\cancel{24}_{8}} \Rightarrow \frac{a^4}{8}.$$

4) $\int_0^1\int_0^1\int_0^1 (x+y+z)\,dx\,dy\,dz$

Sol:

$$= \int_0^1\int_0^1 \left[\frac{x^2}{2} + yx + zx\right]_0^1 dy\,dz$$

$$= \int_0^1\int_0^1 \left[\frac{1}{2} + y + z\right] dy\,dz$$

$$= \int_0^1 \left[\frac{1}{2}y + \frac{y^2}{2} + zy\right] dz$$

$$= \int_0^1 \left[\frac{1}{2} + \frac{1}{2} + z\right] dz$$

$$= \int_0^1 \frac{1}{2}\,dz + \int_0^1 \frac{1}{2}\,dz + \int_0^1 z\,dz$$

$$= \left[\frac{1}{2}z + \frac{1}{2}z + \frac{z^2}{2}\right]_0^1$$

$$= 1/2 + 1/2 + 1/2$$

$$= 3/2.$$

5) $\int_0^a\int_0^x\int_0^y$

Sol:

5) $\int_0^a \int_0^x \int_0^y xyz \, dz \, dy \, dz$

Sol:

$$= \int_0^a \int_0^x \int_0^y xyz \, dz \, dy \, dx$$

$$= \int_0^a \int_0^x \left[xy \left(\frac{z^2}{2} \right) \right]_0^y dy \, dx$$

$$= \int_0^a \int_0^x \left[xy \left(\frac{y^2}{2} \right) \right] dy \, dx$$

$$= \int_0^a \int_0^x \frac{x}{2} (y^3) \, dy \, dx$$

$$= \int_0^a \int_0^x \left[\frac{x}{2} \left(\frac{y^4}{4} \right) \right]_0^x dx$$

$$= \int_0^a x/8 \cdot (x^4) \, dx$$

$$= \int_0^a \frac{1}{8} (x^5) \, dx$$

$$= \left[\frac{1}{8} \left(\frac{x^6}{6} \right) \right]_0^a$$

$$= \frac{1}{8}\left(\frac{a^6}{6}\right)$$

$$= a^6/48 .$$