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UNIT-II Properties Of Definite Integrals

general formulae.

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b$$

$$= F(b) - F(a)$$

Property -1

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

LHS

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b$$

$$= F(b) - F(a) \rightarrow (1)$$

RHS

$$-\int_b^a f(x)\,dx = -\Big[F(x)\Big]_b^a$$

$$= -\Big[F(a) - F(b)\Big]$$

$$= -F(b) - F(a) \longrightarrow (2)$$

from (1) and (2)

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

22) Property - 4

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

RHS,

$$\Rightarrow \int_0^a f(a-x)\,dx$$

$a - x = y$

$-dx = dy$

$x = 0$, $x = a$

$a - 0 = y$

$y = a$

$a - a = y$

$y = 0$

limit $x = 0$, $x = a$

$y = a$, $y = 0$

$$= \int_a^0 f(y)(-dy)$$

$$= -\int_a^0 f(y)\,dy \rightarrow (1)$$

by. prop - 1

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$= \int_0^a f(y)\,dy$$

by prop - 2

$$\int_a^b f(x)\,dx = \int_a^b f(y)\,dy$$

$$= \int_0^a f(x)\,dx \rightarrow (2)$$

Hence proved.

Property - 5

$$\int_{-a}^a f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(-x)\,dx$$

LHS

$$\Rightarrow \int_{-a}^{a} f(x)\,dx$$

$-a \quad 0 \quad a$

$$\int_{a}^{b} f(x)\,dx = \int_{a}^{c} f(x)\,dx + \int_{c}^{b} f(x)\,dx$$

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

take integral - 1st

$$\int_{-a}^{0} f(x)\,dx$$

take

$x = -y$

$dx = -dy$

$x = 0, \; x = -a$

$0 = -y, \; -a = -y$

$y = 0, \; y = a$

$$= \int_{a}^{0} f(-y) - dy$$

$$= -\int_{a}^{0} f(-y)\,dy$$

By the 1-prop

$$= \int_{0}^{a} f(-y)\,dy$$

By prop - 2

$$= \int_{0}^{a} f(-x)\,dx \longrightarrow (1)$$

put equ (1) in LHS

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx$$

Property - 6

$$\int_{0}^{2a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(2a - x)\,dx$$

$$\int_0^{2a} f(x)dx = \int_0^{a} f(x)dx + \int_0^{2a} f(x)dx$$

$0 \quad a \quad 2a$

take 2nd integral

$$\int_0^{2a} f(x)dx$$

$x = 2a - y$

$dx = -dy$

$x = 2a, \ x = a$

$2a = 2a - y$

$y = 0$

$a = 2a - y$

$y = a$

$$= \int_a^{0} f(2a-y)(-dy)$$

$$= -\int_0^{a} f(2a-y)dy$$

By the 1-prop

$$= \int_0^{a} f(2a-y)dy$$

By prop-2

$$= \int_0^{a} f(2a-x)dx \rightarrow (1)$$

put equ(1) in LHS

$$\int_0^{2a} f(x)dx = \int_0^{a} f(x)dx + \int_0^{a} (2a-x)dx$$

Example 4

$$\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\, dx$$

property - 4

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$I = \int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\, dx \rightarrow (1)$$

$$= \int_0^{\pi/2} \frac{\sin^n(\pi/2 - x)}{\sin^n(\pi/2 - x) + \cos^n(\pi/2 - x)}\, dx$$

$$I = \int_0^{\pi/2} \frac{\cos^n x}{\cos^n x + \sin^n x} \rightarrow (2)$$

By adding equation (1) and (2)

$$2I = \int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\, dx + \int_0^{\pi/2} \frac{\cos^n x}{\cos^n x + \sin^n x}\, dx$$

$$2I = \int_0^{\pi/2} \left[\frac{\sin^n x}{\sin^n x + \cos^n x} + \frac{\cos^n x}{\sin^n x + \cos^n x}\right] dx$$

$$2I = \int_0^{\pi/2} \left[\frac{\sin^n x + \cos^n x}{\sin^n x + \cos^n x}\right] dx$$

$$2I = \int_0^{\pi/2} dx$$

$$2I = [x]_0^{\pi/2}$$

$$2I = \pi/2 - 0$$

$$2I = \pi/2$$

$$I = \pi/2 \times \frac{1}{2}$$

$$I = \pi/4$$

Example : 5

$$\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}\, dx \qquad \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}\, dx \longrightarrow (1)$$

$$= \int_0^{\pi/2} \frac{\sin^2(\pi/2 - x)}{\sin(\pi/2 - x) + \cos(\pi/2 - x)}\, dx$$

$$I = \int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x}\, dx \longrightarrow (2)$$

By adding equ (1) and (2)

$$2I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}\, dx + \int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x}\, dx$$

$$2I = \int_0^{\pi/2} \frac{\sin^2 x + \cos^2 x}{\sin x + \cos x}\, dx$$

$$2I = \int_0^{\pi/2} \frac{1}{\sin x + \cos x}\, dx$$

$t = \tan x/2$

$x = \pi/2$, $x = 0$

$t = \tan(\pi/4)$ $\quad \tan(0) = \pi/4$

$t = 1$ $\quad t = 0$

$$\cos x = \frac{1-t^2}{1+t^2} \qquad \sin x = \frac{2dt}{1+t^2} \qquad dx = \frac{2dt}{1+t^2}$$

limit = 0 to 1

$$= \int_0^1 \frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}}$$

$$= \int_0^1 \frac{2dt / 1+t^2}{\frac{2t+1-t^2}{1+t^2}}$$

$$2I = \int_0^1 \frac{2dt}{2t+1-t^2}$$

$$= \int_0^1 \frac{2dt}{-t^2+2t+1}$$

$$= \int_0^1 \frac{2dt}{-(t-1)^2+2}$$

$$= \int_0^1 \frac{2dt}{-(t-1)^2+(\sqrt{2})^2}$$

$$= \int_0^1 \frac{2dt}{(\sqrt{2})^2-(t-1)^2}$$

$$= 2\int_0^1 \frac{dt}{(\sqrt{2})^2-(t-1)^2}$$

$$= 2\left[\frac{1}{2\sqrt{2}} \log \frac{\sqrt{2}+(t-1)}{\sqrt{2}-(t-1)}\right]_0^1$$

$$(ax^2+b)^2 = (ax-b/2)^2-(b/2)^2$$

$$= (t-2/2)^2-(2/2)^2$$

$$= (t-1)-(1)$$

$$= -(t-1)-1-1$$

$$= -(t-1)+2$$

$$= \frac{1}{\sqrt{2}} \log \frac{(\sqrt{2}+1)^2}{2-1}$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \frac{a+x}{a-x}$$

$$2I = \frac{1}{\sqrt{2}} \left[\log \frac{\sqrt{2} + (t-1)}{\sqrt{2} - (t-1)} \right]_0^1$$

$$= \frac{1}{\sqrt{2}} \left[\log \frac{\sqrt{2}+(t-1)}{\sqrt{2}+(1-1)} - \log \frac{\sqrt{2}+(-1)}{\sqrt{2}+1} \right]$$

$$= \frac{1}{\sqrt{2}} \left[\log (1) - \log \frac{\sqrt{2}-1}{\sqrt{2}+1} \right]$$

$$\log(1) = 0$$

$$= \frac{1}{\sqrt{2}} \left[0 - \log \frac{\sqrt{2}-1}{\sqrt{2}+1} \right]$$

$$= \frac{1}{\sqrt{2}} \left(- \log \frac{\sqrt{2}-1}{\sqrt{2}+1} \right)$$

$$x \log a = \log a^x$$

$$= \frac{1}{\sqrt{2}} \left[\log \left(\frac{\sqrt{2}-1}{\sqrt{2}+1} \right) \right]^{-1}$$

$$= \frac{1}{\sqrt{2}} \log \frac{\sqrt{2}+1}{\sqrt{2}-1}$$

$$2I = \frac{1}{\sqrt{2}} \log \frac{\sqrt{2}+1}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1}$$

$$2I = \frac{1}{\sqrt{2}} \log \frac{(\sqrt{2}+1)^2}{2-1}$$

$$2I = \frac{1}{\sqrt{2}} \log \frac{(\sqrt{2}+1)^2}{1}$$

$$\log a^x = x \log a$$

$$2I = \frac{1}{\sqrt{2}} 2 \log (\sqrt{2}+1)$$

$$I = \frac{1}{\sqrt{2}} \log (\sqrt{2}+1)$$

Example 7.

$$\int_0^{\pi} \frac{x+\tan x}{\sec x+\tan x}\,dx$$

$$I = \int_0^{\pi} \frac{\tan x + x}{\sec x+\tan x}\,dx \longrightarrow (1)$$

$$= \int_0^{\pi} \frac{(\pi - x) + \tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)} \qquad \int^{\pi} f(x) = \int f(a-x)$$

$$= \int_0^{\pi} \frac{(\pi - x)(-\tan x)}{-\sec x - \tan x}$$

$$I = \int_0^{\pi} \frac{(\pi - x)\tan x}{\sec x+\tan x} \longrightarrow (2)$$

By adding (1) and (2)

$$2I = \int_0^{\pi} \frac{x\tan x}{\sec x+\tan x} + \int_0^{\pi} \frac{(\pi - x)\tan x}{\sec x+\tan x}$$

$$= \int_0^{2\pi} \frac{x\tan x + (\pi - x)\tan x}{\sec x + \tan x}\,dx$$

$$= \int_0^{\pi} \frac{x\tan x + \pi\tan x - x\tan x}{\sec x+\tan x}\,dx$$

$$2I = \int_0^{\pi} \frac{\pi\tan x}{\sec x+\tan x}\,dx$$

$$2I = \int_0^{\pi} \frac{\pi\frac{\sin x}{\cos x}}{1/\cos x + \frac{\sin x}{\cos x}}\,dx$$

$$2I = \int_0^{\pi} \frac{\pi \sin x / \cos x}{\cos x \frac{1+\sin x}{\cos x}}\,dx$$

$$2I = \int_0 \frac{\pi \sin x}{1+\sin x}\, dx$$

$\sin x = \frac{2t}{1+t^2}$. $\quad dx = \frac{2dt}{1+t^2} \quad \cos x = \frac{1-t^2}{1+t^2}$

$$2I = \pi \int_0^{\pi} \frac{\sin x}{1+\sin x}\, dx$$

$$= \pi \int_0^{\pi} \frac{\sin x}{1+\sin x} \times \frac{1-\sin x}{1-\sin x}\, dx$$

$$= \pi \int_0^{\pi} \frac{\sin x - \sin^2 x}{1 \div \sin^2 x}\, dx$$

$$= \pi \int_0^{\pi} \frac{\sin x - \sin^2 x}{\cos^2 x}\, dx$$

$$= \pi \int_0^{\pi} \left[\frac{\sin x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x}\right] dx$$

$1 + \tan^2 x = \sec^2 x$

$\boxed{\tan^2 x = \sec^2 x - 1}$

$$2I = \pi \int_0^{\pi} \frac{\sin x}{\cos}\left(\frac{1}{\cos x}\right) - \tan^2 x\, dx$$

$$= \pi \int_0^{\pi} \tan x \sec x - \tan^2 x\, dx$$

$$= \pi \int_0^{\pi} \tan x \sec x\, dx - \int_0^{\pi} (\sec^2 x - 1)\, dx$$

$$= \pi \int_0^{\pi} \tan x \sec x\, dx - \int_0^{\pi} \sec^2 x - 1\, dx$$

$$= \pi \int_0^{\pi} \tan x \sec x\, dx - \int_0^{\pi} (\sec^2 x \cancel{-1})\, dx + \int_0^{\pi} dx$$

$$= \pi\ (\sec x)_0^{\pi} - \left[\tan x\right]_0^{\pi} + [x]_0^{\pi}$$

$= \pi [\sec\pi - \sec(0)] - [\tan\pi - \tan 0] + [\pi - 0]$

$= \pi [-1 - 1 - 0 + 0 + \pi]$

$= \pi [\pi - 2]$

$= \pi^2 - 2\pi$

$2I = \pi^2 - 2\pi$

$I = \dfrac{\pi^2 - 2\pi}{2}$

$I = \dfrac{\pi(\pi - 2)}{2}$

$\sec(180) = -1$

$\sec(0) = 1$

$\tan(180) = 0$

$\tan(0) = 0$

Example 8

$$\int_0^1 \frac{\log(1+x)}{1+x^2}\, dx$$

$x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

When, $x = 0$, $\theta = 0$ $\qquad \tan(0) = 1$

$x = 1$, $\theta = \pi/4$ $\qquad \tan(1) = \pi/4$

then,

$$\int_0^1 \frac{\log(1+x)}{1+x^2}\, dx = \int_0^1 \frac{\log(1+\tan\theta)}{1+\tan^2 x} \cdot \sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta$$

$$I = \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta \longrightarrow (1)$$

$\int f(x) = \int f(a-x)$

Prop - 4

$$I = \int_0^{\pi/4} \log(1+\tan(\pi/4-\theta))\, d\theta$$

$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B}$

$$= \int_0^{\pi/4} \log\left[1+\frac{\tan \pi/4 - \tan\theta}{1+\tan\pi/4 \tan\theta}\right] d\theta$$

$$= \int_0^{\pi/4} \log\left[1+\frac{1-\tan\theta}{1+\tan\theta}\right] d\theta$$

$$= \int_0^{\pi/4} \log\left[\frac{1+\tan\theta + 1-\tan\theta}{1+\tan\theta}\right] d\theta$$

$$= \int_0^{\pi/4} \log\left[\frac{2}{1+\tan\theta}\right] d\theta$$

$$I = \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right) d\theta \longrightarrow (2)$$

By adding (1) and (2)

$$2I = \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta + \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right) d\theta$$

$$= \int_0^{\pi/4} \log\left[(1+\tan\theta)\left[\frac{2}{1+\tan\theta}\right]\right] d\theta$$

$$= \int_0^{\pi/4} \log(2)\, d\theta$$

$$= \log 2 \int_0^{\pi/4} d\theta$$

$$= \log 2\,(\pi/4 - 0)$$

$$2I = 2 \log \pi/4$$

$$2I = \pi/4 \log 2 \Rightarrow I = \pi/8 \log 2$$

Example 6

$$\int_0^{\pi/4} \log(\sin x)\, dx$$

$$I = \int_0^{\pi/2} \log(\sin x)\, dx \longrightarrow (1)$$

By property - 4

$\int f(x) = \int f(a-x)$

$$= \int_0^{\pi/2} \log(\sin(\pi/4 - x))\, dx$$

$\sin(\pi/4 - \theta) = \cos\theta$

$$I = \int_0^{\pi/2} \log \cos x\, dx \longrightarrow (2)$$

By adding one and (2)

$$2I = \int_0^{\pi/2} \log \sin x\, dx + \int_0^{\pi/2} \log \overset{\cos x}{\cancel{\sin x}}\, dx$$

$$= \int_0^{\pi/2} \log \sin x + \log \sin x \cos x\, dx$$

$$= \int_0^{\pi/2} \log \sin x \cos x\, dx$$

$\sin 2x = 2 \sin x \cos x$

$\frac{\sin 2x}{2} = \sin x \cos x$

formula

$$= \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right) dx$$

$$= \int_0^{\pi/2} [\log \sin 2x - \log 2]\, dx$$

$$= \int_0^{\pi/2} \log \sin 2x\, dx - \int_0^{\pi/2} \log 2\, dx$$

$$= \int_0^{\pi/2} \log \sin 2x \, dx - \log 2 \int_0^{\pi/2} dx$$

$$= \int_0^{\pi/2} \log 2x \, dx = \log 2 \int_0^{\pi/2} dx - \log 2(x)_0^{\pi/2}$$

$$= \int_0^{\pi/2} \log \sin 2x \, dx - \pi/2 \log 2$$

$$= \int_0^{\pi/2} \log \sin 2x \, dx$$

$2x = y$ $\qquad$ $x = \pi/2$

$2dx = dy$ $\qquad$ $2 \times \pi/2 = 4$

$dx = dy/2$ $\qquad$ $y = \pi$

$$= \int_0^{\pi} \log \sin y \, dy/2$$

$x = 0$

$2 \times 0 = y$

$y = 0$

$$= 1/2 \int_0^{\pi} \sin y \cdot \log dy$$

$$= 1/2 \cdot 2 \int_0^{\pi} \log \sin y \, dy$$

$$= \int_0^{\pi/2} \log \sin y \, dy$$

$$\Rightarrow \int_0^{\pi/4} \sin x \, dx = I$$

$$= \int_0^{\pi/2} \log \sin x \, dx - \pi/2 \log 2$$

$$2I = I - \pi/2 \log 2$$

$$2I - I = -\pi/2 \log 2$$

$$I = -\pi/2 \log 2$$

$$I = \pi/2 \log (2)^{-1}$$

$$I = \pi/2 \log (1/2)$$

Example : 9

$$\int_0^{\pi} \theta \sin^5 \theta \, d\theta$$

$$I = \int_0^{\pi} \theta \sin^5 \theta \, d\theta \longrightarrow (1)$$

By frouth prop

$\int f(x) = \int f(a-x)$

$$= \int_0^{\pi} (\pi - \theta) \sin^5 (\pi - \theta) \, d\theta$$

$$I = \int_0^{\pi} (\pi - \theta) \sin^5 \, d\theta \longrightarrow (2)$$

By adding (1) and (2)

$$2I = \int_0^{\pi} \theta \sin^5 \theta \, d\theta + \int_0^{\pi} (\pi - \theta) \sin^5 d\theta$$

$$= \int_0^{\pi} [\theta \sin^5 \theta + (\pi - \theta) \sin^5 \theta] \, d\theta$$

$$= \int_0^{\pi} \pi \sin \theta \, d\theta$$

$$\int_0^{\pi} f(\sin x) dx = 2\int_0^{\pi/2} \sin x \, dx$$

$$= \pi \cdot 2 \int_0^{\pi/2} \sin^5 \theta \, d\theta$$

$$2I = 2\pi \cdot 4/5 \cdot 2/3$$

$$I = 8\pi/15$$