FOURIER SERIES

Definition for Fourier Series.

If $f(x)$ is a periodic function and satisfies Dirichlet conditions [to be described in subsequent article], then it can be represented by an infinite series called Fourier Series as

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \dots + a_n \cos nx + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \dots + b_n \sin nx + \dots$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos nx + b_n \sin nx \right]$$

Where a_0, a_n and b_n are called Fourier Coefficients.

PERIODIC FUNCTIONS

A $f(x)$ [function] is said to have a period T if for all x, $f(x+T) = f(x)$, where T is positive constant. the least value of $T>0$ is called the period of $f(x)$.

DIRICHLET CONDITIONS

Suppose that,

(i) $f(x)$ is defined and single valued except possibly at a finite number of points in $(-L, L)$.

(ii) $f(x)$ is periodic with period $2L$.

(iii) $f(x)$ and $f'(x)$ are piecewise continuous in $(-L, L)$.

then the above series (1) converges to

a.) $f(x)$ if x is a point of continuity.

b.) $\frac{f(x+0) + f(x-0)}{2}$ if x is a point of discontinuity.

Example - 1

Find the fourier Series to represent $x-\pi$ in the interval $(-\pi,\pi)$.

Let $f(x) = x-\pi$

We know that the Fourier Series of $f(x)$ in $(-\pi,\pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow (1)$$

$$a_0 = 1/\pi \int_{-\pi}^{\pi} f(x)\,dx$$

$$= 1/\pi \int_{-\pi}^{\pi} (x-\pi)\,dx$$

$$= 1/\pi \left[\frac{(x-\pi)^2}{2}\right]_{-\pi}^{\pi}$$

$$= -1/\pi \cdot \left[\frac{4\pi^2}{2}\right]$$

$$a_0 = -2\pi \longrightarrow (2)$$

$$a_n = 1/\pi \int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= 1/\pi \left[\int_{-\pi}^{\pi} (x-\pi)\cos nx\,dx\right]$$

$$= 1/\pi \left[(x-\pi)\left[\frac{\sin nx}{n}\right] - (1)\left[-\frac{\cos nx}{n^2}\right]\right]_{-\pi}^{\pi}$$

$$= 1/\pi \left[\frac{\cos n\pi}{n^2} - \frac{\cos n\pi}{n^2}\right]$$

$$a_n = 0 \longrightarrow (3)$$

$$b_n = 1/\pi \int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= 1/\pi \int_{-\pi}^{\pi} (x-\pi)\sin nx\,dx$$

$$= 1/\pi \left[(x-\pi)\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos nx \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x \left[\sin(n+1)x - \sin(n-1)x\right] dx$$

$\because \cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$

$$= \frac{1}{2\pi}\left[x\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right] - \left[-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right]\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left(-\frac{\cos 2(n+1)\pi}{n+1} + \frac{\cos(2(n-1)\pi)}{n-1}\right)\right]$$

$$a_n = -\frac{1}{n+1} + \frac{1}{n-1} \Rightarrow \frac{2}{n^2-1} \quad \text{where } n \neq 1$$

$$a_n = \frac{2}{n^2-1} \longrightarrow (2)$$

When $(n=1)$,

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x \sin 2x \, dx \qquad \because \sin 2x = 2\sin x \cos x$$

$$= \frac{1}{2\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - 1\cdot\left(-\frac{\sin 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[-\pi\right]$$

$$a_n = -\frac{1}{2} \longrightarrow (3)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin nx \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\cos(n-1)x - \cos(n+1)x\right] dx$$

$$= \frac{1}{\pi}\left[\frac{-2\pi \cos n\pi}{n}\right]$$

$$= \frac{-2\cos n\pi}{n}$$

$$b_n = \frac{2(-1)^{n+1}}{n} \longrightarrow (4)$$

Substituting (2), (3) and (4) in (1), We get

$$f(x) = -\pi + 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

Example 2

Express $f(x) = x\sin x$ as a Fourier Series in $0 \le x \le 2\pi$.

We know that a Fourier Series for the function $f(x)$ in the interval $[0, 2\pi]$ is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow (A)$$

Here, $f(x) = x\sin x$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \sin x \, dx$$

$$= \frac{1}{\pi}\left[x(-\cos x)\right] - 1\cdot(-\sin x)\Big]_0^{2x}$$

$$= \frac{1}{\pi}\left[-2\pi\right]$$

$$= -2 \qquad [\because \sin 2\pi = 0, \cos 2\pi = 1]$$

$$a_0 = 2 \longrightarrow (1)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx \, dx$$

$$\left[2\sin A \sin B = \cos(A-B) - \cos(A+B)\right]$$

$$= \frac{1}{2\pi}\left[x\left(\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right) - 1\cdot\left(-\frac{\cos(n-1)x}{(n-1)^2} + \frac{\cos(n+1)x}{(n+1)^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\cos 2(n-1)\pi}{(n-1)^2} - \frac{\cos 2(n+1)\pi}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right]$$

$b_n = 0$ [provided $n \neq 1$] $\longrightarrow$ (4)

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin x \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(1-\cos 2x)\, dx$$

$$= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - 1\cdot\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi(2\pi) - \frac{4\pi^2}{2} - \frac{1}{4} + \frac{1}{4}\right]$$

$= \pi \longrightarrow$ (5)

From the equation (A) We get

$$f(x) = \frac{a_0}{2} + a_1\cos x + b_1 \sin x + \sum_{n=2}^{\infty} a_n \cos nx + \sum_{n=2}^{\infty} b_n \sin nx$$

From the equ (1), (2), (3), (4) and (5),

$a_0 = -2$, $a_n = \frac{2}{n^2-1}$, $(n \neq 1)$ $a_1 = \frac{1}{2}$, $b_n = 0$, $b_1 = \pi$

Substitute these Values in (A) We get

$$f(x) = -1 - \frac{1}{2}\cos x + \pi \sin x + \sum_{n=2}^{\infty} \frac{2}{n^2-1}\cos nx$$

$$x\sin x = -1 + \pi \sin x - \frac{1}{2}\cos x + \frac{2}{2^2-1}\cos 2x + \frac{2}{3^2-1}\cos 3x + \cdots$$

Example 3

Find the Fourier Series for $f(x)$ if, $f(x) = -\pi$ in $-\pi < x < 0$
$= x$ in $0 < x < 0$ }

Deduce that $1/1^2 + 1/3^2 + 1/5^2 + \cdots = \pi^2/8$

the fourier Series for the function $f(x)$ in $(-\pi, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow (A)$$

$$a_0 = 1/\pi \int_{-\pi}^{\pi} f(x)\,dx$$

$$= 1/\pi \left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx \right]$$

$$= 1/\pi \left[\int_{-\pi}^{0} -\pi\,dx + \int_{0}^{\pi} x\,dx \right]$$

$$= 1/\pi \left[(-\pi x)_{-\pi}^{0} + \left(\frac{x^2}{2}\right)_0^{\pi} \right]$$

$$= 1/\pi \left[(-\pi x)_{-\pi}^{0} + \left(\frac{x^2}{2}\right)_0^{\pi} \right]$$

$$= 1/\pi \left[-\pi^2 + \frac{\pi^2}{2} \right]$$

$$a_0 = -\pi/2 \longrightarrow (1)$$

$$a_n = 1/\pi \int_{-\pi}^{\pi} f(x) \cos nx\,dx$$

$$= 1/\pi \left[\int_{-\pi}^{0} f(x) \cos nx\,dx + \int_{0}^{\pi} x \cos nx\,dx \right]$$

$$= 1/\pi \left[\int_{-\pi}^{0} -\pi \cos n\pi\,dx + \int_{0}^{\pi} x \cos nx\,dx \right]$$

$$= 1/\pi \left[\left(-\frac{\pi \sin nx}{n} \right)_{-\pi}^{0} + \left(\frac{x \sin nx}{n} - \left(-\frac{\cos nx}{n^2} \right) \right)_0^{\pi} \right]$$

$$= 1/\pi \left[1/n^2 (\cos n\pi - 1) \right]$$

$$= 1/n^2\pi \left[(-1)^n - 1 \right]$$

$a_n = 0$ when n is even
$= \frac{-2}{n^2\pi}$ when n is odd $\Big\} \longrightarrow (2)$

$b_n = 1/\pi \int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$= 1/\pi \left[\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx \right]$

$= 1/\pi \left[\int_{-\pi}^{0} -\pi \sin n\pi\,dx + \int_{0}^{\pi} x \sin nx\,dx \right]$

$= 1/\pi \left[\left(\frac{\pi\cos nx}{n} \right)_{-\pi}^{0} + \left\{ \left(x - \left(\frac{\cos nx}{n} \right) - 1\left(- \frac{\sin nx}{n^2} \right) \right) \right\}_{0}^{\pi} \right]$

$= 1/\pi \left[\pi/n - \pi/n (-1)^n - \frac{\pi(-1)^n}{n} \right]$

$b_n = 1/n \left(1 - 2(-1)^n\right) \longrightarrow (3)$

Substituting (1), (2), (3) in (A)

$f(x) = -\pi/4 + \sum_{n=1,3,5}^{\infty} \frac{-2}{n^2\pi} \cos nx + \sum_{n=1}^{\infty} 1/n \left[1 - 2(-1)^n\right] \sin nx$

$f(x) = -\pi/4 - 2/\pi \left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right) + 3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} \Big) \longrightarrow (3a)$

putting $x = 0$, in the equation (3a)

$f(0) = -\pi/4 - 2/\pi \left(1 + 1/3^2 + 1/5^2 + \dots \right) \longrightarrow (4)$

Here 0 is a point of discontinuity,

Hence, $f(0) = \frac{f(0-0) + f(0+0)}{2}$

$= \frac{-\pi + 0 + 0 + 0}{2}$

$= -\pi/2 \longrightarrow (5)$

from the equation, (4),(5)

$$-\pi/2 = -\pi/4 - 2/\pi\left[1 + 1/3^2 + 1/5^2 + \cdots\right]$$

$$-\pi/4 = -2/\pi\left(1 + 1/3^2 + 1/5^2 + \cdots\right)$$

$$\pi^2/8 = 1 + 1/3^2 + 1/5^2 + \cdots$$

Hence it is proved.

Example 4

Determine the fourier Series expansion of $x+x^2$ in the interval $(-\pi,\pi)$ and hence deduce the sum of series $1/1^2 + 1/2^2 + 1/3^2 + \cdots$

let,
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow (A)$$

$$f(x) = x + x^2 \text{ in } [-\pi,\pi]$$

$$a_0 = 1/\pi \int_{-\pi}^{\pi} f(x)\,dx$$

$$= 1/\pi \int_{-\pi}^{\pi} (x+x^2)\,dx$$

$$= 1/\pi\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= 1/\pi\left[\frac{\pi^2}{2} + \frac{\pi^2}{3}\right] - \left[\frac{\pi^2}{2} - \frac{\pi^3}{3}\right]$$

$$= 1/\pi \cdot \frac{2\pi^3}{3}$$

$$a_0 = 2/3\,\pi^2 \longrightarrow (1)$$

$$a_n = 1/\pi \int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= 1/\pi \int_{-\pi}^{\pi}\left[x+x^2\left(\frac{\sin nx}{n}\right) - (1+2x)\left(\frac{-\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= 1/\pi\left[\frac{(1+2\pi)\cos n\pi}{n^2} - \frac{(1-2\pi)\cos n\pi}{n^2}\right]$$

$x = -\pi$ is a point of discontinuity, therefore

$$f(-\pi) = \frac{f(-\pi-0) + f(-\pi+0)}{2}$$

$$= \frac{(-\pi+\pi^2) + (-\pi+\pi^2)}{2}$$

$$= -\pi + \pi^2$$

Hence, $-\pi + \pi^2 = \pi^2/3 + \sum_{n=1}^{\infty} (-1)^{2n}\, 4/n^2 \longrightarrow (6)$

by adding (5) and (6)

$$2\pi^2 = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^{2n}\, 4/n^2$$

$$\pi^2 = \pi^2/3 + 4\sum_{n=1}^{\infty} 1/n^2$$

$$\sum_{n=1}^{\infty} 1/n^2 = \frac{3\pi^2 - \pi^2}{3 \times 4}$$

$$= \frac{2\pi^2}{12} \Rightarrow \pi^2/6$$

$$1/1^2 + 1/2^2 + 1/3^2 + \dots = \pi^2/6$$

Hence proved.

EVEN AND ODD FUNCTION

A function $f(x)$ is said to be even if $f(-x) = f(x)$

Example: $x^2 . \cos x$, $\sin x$, $|x|$, $x \sin x$ are even functions

A function $f(x)$ is odd if $f(-x) = -f(x)$

Example: x^3, $\sin x$, $\tan^3 x$ are odd function.

the product of two even functions or two odd functions in the even function while the product of an even function and an odd function is an odd function.

$$a_n = 4/n^2 (-1)^n \longrightarrow (2)$$

$$b_n = 1/\pi \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= 1/\pi \int_{-\pi}^{\pi} (x+x^2) \sin nx \, dx$$

$$= 1/\pi \left[(x+x^2)\left(\frac{-\cos nx}{n}\right) - (1+2x)\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right) \right]_{-\pi}^{\pi}$$

$$= 1/\pi \left[\frac{(\pi+\pi^2)(-\cos n\pi)}{n} + \frac{2\cos n\pi}{n^3} + \frac{(-\pi+\pi^2)\cos n\pi}{n} - \frac{2\cos n\pi}{n^3} \right]$$

$$= \frac{\cos n\pi}{n\pi} \left[-\pi - \pi^2 - \pi + \pi^2 \right]$$

$$= -2/n (-1)^n \longrightarrow (3)$$

$$b_n = -2/n (-1)^n \qquad \because \cos n\pi = (-1)^n$$

Substituting the values of a_0, a_n and b_n in (A)

$$f(x) = \pi^2/3 + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx + \sum_{n=1}^{\infty} \frac{-2}{n} (-1)^n \sin nx$$

$$= \pi^2/3 + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx + 2\sum_{n=1}^{\infty} \frac{(-1)^n \sin nx}{n}$$

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{4}{n^2} \cos nx - 2/n \sin nx \right] \longrightarrow (4)$$

putting $x = \pi$ in (4)

$$f(\pi) = \pi^2/3 + \sum_{n=1}^{\infty} (-1)^{2n} \frac{4}{n^2}$$

Here π is a point of discontinuity, therefore, $f(x)$ at $x=\pi$ is given by,

$$f(\pi) = \frac{f(\pi-0)+f(\pi+0)}{2}$$

$$= \frac{\pi+\pi^2+\pi+\pi^2}{2}$$

$$= \pi^2 + \pi$$

Here, $$\pi + \pi^2 = \pi^2/3 + \sum_{n=1}^{\infty} (-1)\, 2n\ 4/n^2 \longrightarrow (5)$$

Example 1

If 'a' is neither zero nor an integer, find the Fourier series expansion of period for the function $f(x) = \sin ax$, in $-\pi \le x \le \pi$

$f(x) = \sin ax$ is an odd function

the fourier co-efficients $a_0 = 0$, $a_n = 0$, $f(x)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \, dx \longrightarrow (A)$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} \sin ax \sin nx \, dx$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} (\cos(n-a)x - \cos(n+a)x) \, dx \right]$$

$$= \frac{1}{\pi} \left[\frac{\sin(n-a)x}{n-a} - \frac{\sin(n+a)x}{n+a} \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{\sin(n-a)\pi}{n-a} - \frac{\sin(n+a)\pi}{n+a} \right] \quad \because [\sin 0 = 0]$$

$$= \frac{1}{\pi} \left[\frac{\sin n\pi \cos a\pi - \cos n\pi \sin n\pi}{n-a} - \frac{\sin n\pi \cos a\pi - \cos n\pi \sin n\pi}{n+a} \right]$$

$$= \frac{1}{\pi} \left[\frac{(-1)^n(-\sin a\pi)}{n-a} + \frac{(-1)^{n+1} \sin a\pi}{n+a} \right]$$

$$= \frac{1}{\pi} \left[\frac{(-1)^n(\sin a\pi)}{n-a} + \frac{(-1)^{n+1} \sin a\pi}{n+1} \right]$$

$$= \frac{(-1)^{n+1} \sin a\pi}{\pi} \left[\frac{1}{n-a} + \frac{1}{n+a} \right]$$

$$= (-1)^{n+1} \frac{2\pi \sin a\pi}{\pi(n^2-a^2)}$$

$$b_n = (-1)^{n+1} \frac{2n \sin a\pi}{\pi(n^2-a^2)} \longrightarrow (B)$$

Substituting (B) in (A)

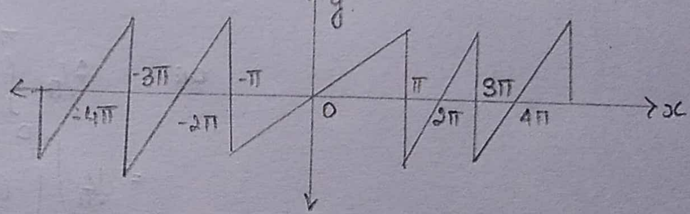
$$f(x) = \sin ax = \frac{2\sin a\pi}{\pi} \sum_{n=1}^{\infty} \frac{n(-1)^{n+1}}{n^2-a^2} \sin nx$$

Example 2

Show that the fourier Series for $f(x) = x, \; -\pi < x < \pi$ is given by,

$$f(x) = 2\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin n\pi}{n}$$

the graph for given function



$f(x) = x$

$f(-x) = -x$

$f(x) = f(-x)$

$= -f(x)$

Hence $f(x)$ is an odd function

$a_0 = 0, \; a_n = 0$

the $f(x)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow (1)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - 1\left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos n\pi}{n} - (-\pi)\left(\frac{-\cos n\pi}{n}\right)\right]$$

$$= \frac{1}{n}(-2\cos n\pi)$$

$$b_n = \frac{2}{n}(-1)^{n+1} \longrightarrow (2)$$

Substituting (2) in (1)

$$f(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin nx}{n}$$

Example : 3

prove that $\sin hax = 2/\pi \left[\sin ha\pi \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot n \sin nx}{n^2+a^2} \right]$ in $(-\pi, \pi)$

Let $f(x) = \sin hax$, $\sin hax$ is clearly odd function

$$f(x) = \sin hax = \frac{e^{ax} - e^{-ax}}{2}$$

$$f(-x) = \frac{e^{-ax} - e^{ax}}{2}$$

$$= -\left[\frac{e^{ax} - e^{-ax}}{2}\right]$$

$$= -f(x)$$

fourier series for $f(x)$ in $(-\pi, \pi)$ is $f(x) = \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (1)$

To find b_n,

$$b_n = 1/\pi \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= 1/\pi \int_{-\pi}^{\pi} \sin hax \; nx \, dx$$

$$= 1/\pi \cdot 2 \int_0^{\pi} \sin hax \sin nx \, dx$$

$\because$ odd function $\times$ odd function = even function

$$= 2/\pi \int_0^{\pi} \frac{e^{ax} - e^{-ax}}{2} \sin nx \, dx$$

$$= 1/\pi \left[\int_0^{\pi} e^{ax} \sin nx \, dx - \int_0^{\pi} e^{-ax} \sin nx \, dx \right]$$

$$= 1/\pi \left[\left\{ \frac{e^{ax}}{a^2+n^2} (a \sin nx - n \cos nx) \right\}_0^{\pi} - \left\{ \frac{e^{-ax}}{a^2+n^2} (a \sin nx - n \cos nx) \right\}_0^{\pi} \right]$$

$$= 1/\pi \left[\frac{-n(-1)^n}{a^2+n^2} e^{ax} + \frac{n}{a^2+n^2} + \frac{n(-1)^n}{a^2+n^2} \cdot e^{ax} - \frac{n}{a^2+n^2} \right]$$

$$= 1/n \frac{(-1)^n \cdot n}{(n^2+a^2)} \left[-e^{a\pi} + e^{-a\pi} \right]$$

$$= \frac{n(-1)^{n+1}}{n(n^2+a^2)} \cdot 2 \sin ha\pi \longrightarrow (2)$$

Substituting (2) in (1), we get

$$f(x) = \sum_{n=1}^{\infty} \frac{2n(-1)^{n+1}}{\pi(n^2+a^2)} \cdot \sinh a\pi \cdot \sin nx$$

$$\sinh ax = \frac{2}{\pi} \sinh a\pi \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot n \sin nx}{n^2+a^2}.$$