Type : 3

1) $\dfrac{1}{(x+1)\sqrt{1-x^2}}\, dx.$

Sol:

put, $(x+1) = \dfrac{1}{t}$

$$x = \frac{1}{t} - 1$$

$$\frac{dx}{dt} = -\frac{1}{t^2}$$

$$dx = -\frac{1}{t^2}\, dt$$

$$= \int \frac{-1/t^2\, dt}{(1/t)\sqrt{1-(1/t-1)^2}}$$

$$= -\int \frac{1/t^2\, dt}{1/t\sqrt{1-(1/t^2+1-2\,1/t(1))}}$$

$$= -\int \frac{1/t^2\, dt}{1/t\sqrt{1-(1/t^2+1-2/t)}}$$

$$= -\int \frac{1/t^2\, dt}{1/t\sqrt{-1/t^2+2/t}}$$

$$= -\int \frac{\frac{1}{t^2}\,dt}{\frac{1}{t}\sqrt{-\frac{1}{t^2}+\frac{2t}{t^2}}}$$

$$= -\int \frac{\frac{1}{t^2}\,dt}{\frac{1}{t}\sqrt{\frac{2t-1}{t^2}}}$$

$$= \int \frac{\cancel{\frac{1}{t^2}}\,dt}{\cancel{\frac{1}{t^2}}\sqrt{2t-1}} \Rightarrow -\int \frac{dt}{\sqrt{2t-1}}$$

$$\frac{dx}{\sqrt{x}} = 2\sqrt{x}$$

$$= -\int \frac{2\cdot dt}{2\sqrt{2t-1}} \Rightarrow -\frac{1}{2}\int \frac{2dt}{\sqrt{2t-1}}$$

$$= -\frac{1}{\cancel{2}}\cdot \cancel{2}\sqrt{2t-1}$$

$$= -\sqrt{2t-1}$$

where,

$$x+1 = \frac{1}{t}$$

$$t = \frac{1}{(x+1)}$$

$$= -\sqrt{2\left(\frac{1}{x+1}\right)-1}$$

$$\frac{dx}{(x-1)\sqrt{x^2+2x-8}}$$

Sol:

put, $(x-1) = \frac{1}{t}$

$$x = \frac{1}{t} + 1$$

$$\frac{dx}{dt} = -\frac{1}{t^2}$$

$$dx = -\frac{1}{t^2}\,dt$$

$$= \int \frac{-1/t^2\,dt}{(1/t)\sqrt{(1/t+1)^2 + 2(1/t+1) - 8}}$$

$$= \int \frac{-1/t^2\,dt}{(1/t)\sqrt{1/t^2 + 1 + 2/t + 2/t + 2 - 8}}$$

$$= \int \frac{-1/t^2\,dt}{1/t\sqrt{1/t^2 + 4/t - 5}}$$

$$= \int \frac{-1/t^2\,dt}{1/t\sqrt{\frac{-5t^2+4t+1}{t^2}}}$$

$$= \int \frac{dt}{\sqrt{-5t^2+4t+1}}$$

$$= \int \frac{dt}{\sqrt{-5\left(t^2 + \frac{4}{5}t + \frac{1}{5}\right)}}$$

$$= -\frac{1}{5}\int \frac{dt}{\left(-t^2 + \frac{4}{5}t + \frac{1}{5}\right)}$$

$$(ax^2 - bx) = (ax - b/2)^2 - (b/2)^2$$

$$\left(t^2 - \frac{4}{5}t\right) = t^2 - \left(\frac{\cancel{4}^{2}}{5} \times \frac{1}{\cancel{2}_{1}}\right)^2 - \left(\frac{\cancel{4}^{2}}{5} \times \frac{1}{\cancel{2}_{1}}\right)^2$$

$$= \left(t - \frac{2}{5}\right)^2 - \left(\frac{2}{5}\right)^2$$

$$= \left(t - \frac{2}{5}\right)^2 - \frac{4}{25} - \frac{1}{5}$$

$$= \left(t - \frac{2}{5}\right)^2 - \frac{4}{25} - \frac{5}{25}$$

$$= \left(t - \frac{2}{5}\right)^2 - \frac{9}{25}$$

$$= -\frac{1}{\sqrt{5}}\int \frac{dt}{-\left(t - \frac{2}{5}\right)^2 - \frac{3}{5}}$$

$$= -\frac{1}{\sqrt{5}}\int \frac{dt}{\left(\frac{3}{5}\right)^2 - \left(t - \frac{2}{5}\right)^2}$$

$$\frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}(x/a)$$

$$= -\frac{1}{\sqrt{5}} \sin^{-1}\left[\frac{t - 2/5}{3/5}\right]$$

$$= -\frac{1}{\sqrt{5}} \sin^{-1}\left[\frac{5t - 2/5}{3/5}\right]$$

$$= -1/\sqrt{5} \sin^{-1}\left(\frac{5t-2}{3}\right)$$

where,

$$x - 1 = 1/t$$

$$t = 1/x-1$$

$$-1/\sqrt{5} \sin^{-1}\left[\frac{5(1/x-1) - 2}{3}\right]$$

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3) $\dfrac{dx}{(x+2)\sqrt{x+3}}$

Sol:

put $(x+2) = \frac{1}{t}$

$$x = \frac{1}{t} - 2$$

$$\frac{dx}{dt} = -\frac{1}{t^2}$$

$$dx = -\frac{1}{t^2}\, dt$$

$$= \int \frac{-1/t^2\, dt}{(1/t - \cancel{2} + \cancel{2})\sqrt{1/t - 2 + 3}}$$

$$= \int \frac{-1/t^2\, dt}{1/t\sqrt{1/t - 2 + 3}}$$

$$= \int \frac{-1/t^2\, dt}{1/t\sqrt{1/t + 1}}$$

$$= -1/t \int \frac{dt}{\sqrt{1/t + 1}}$$

$$= -\int \frac{dt}{t\sqrt{1/t + 1}}$$

$$= -\int \frac{dt}{\sqrt{t^2/t + t^2}}$$

$$= -\int \frac{dt}{\sqrt{t + t^2}}$$

$$= t^2 + t + (1/2)^2 - (1/2)^2$$

$$= (t + 1/2)^2 - (1/2)^2$$

$$= -\int \frac{dt}{\sqrt{(t + 1/2)^2 - 1/4}}$$

$$= -\cosh^{-1}\left(\frac{t + 1/2}{1/2}\right)$$

$= -\cosh^{-1}(2t+1)$

$t = \frac{1}{x+2}$ //.

Type 4 and 5

1) $\int \frac{dx}{4+5\cos x}$

Sol:

Put. $t = \tan(x/2)$.

$dx = 2dt/1+t^2$

$\cos x = 1-t^2/1+t^2$

$$\int \frac{dx}{4+5\cos x} = 2\int \frac{2dt/1+t^2}{4+5\left(\frac{1-t^2}{1+t^2}\right)}$$

$$= \int \frac{dt}{4(1+t^2)+5(1-t^2)}$$

$$= \int \frac{dt}{9-t^2}$$

$$= 2\int \frac{dt}{3^2-t^2}$$

$$= 2\left(\frac{1}{2\times 3}\right)\log\left(\frac{3+t}{3-t}\right)$$

$$= \frac{1}{3}\log\left[\frac{3+\tan(x/2)}{3-\tan(x/2)}\right]$$

2) $\displaystyle\int_0^{\pi/2} \frac{dx}{9\cos x + 12\sin x} = \frac{\log 6}{15}$

Sol:

$$\int_0^{\pi/2} \frac{dx}{9\cos x + 12\sin x}$$

$t = \tan(x/2) \quad dx = 2dt/1+t^2 \quad \cos x = 1-t^2/1+t^2$

$\sin x = 2t/1+t^2$

$$= \int_0^{\pi/2} \frac{2dt/1+t^2}{9\left(\frac{1-t^2}{1+t^2}\right) + 12\left(\frac{2t}{1+t^2}\right)}$$

$$= \int_0^{\pi/2} \frac{2dt/\cancel{1+t^2}}{\frac{9(1-t^2)+12(2t)}{\cancel{1+t^2}}}$$

$$= \int_0^{\pi/2} \frac{2dt}{9-9t^2+24t}$$

$$= \int_0^{\pi/2} \frac{2dt}{3(-3t^2+8t+3)}$$

$$= \int_0^{\pi/2} \frac{2dt}{9\left(-t^2+\frac{8}{3}t+1\right)}$$

$$= \frac{2}{9}\int_0^{\pi/2} \frac{dt}{-\left(t^2-\frac{8}{3}t-1\right)}$$

$$(ax^2-bx) = (ax-b/2)^2+(b/2)^2$$

$$= \left(t-\frac{8/3}{2}\right)^2-\left(\frac{8/3}{2}\right)^2$$

$$= \left(t-\frac{8}{3}\times\frac{1}{2}\right)-\left(\frac{8}{3}\times\frac{1}{2}\right)^2$$

$$= \left(t-\frac{8}{6}\right)^2-\left(\frac{8}{6}\right)^2$$

$$= \left(t-\frac{8}{6}\right)^2-\frac{64}{36}-1$$

$$= \left(t-\frac{8}{6}\right)^2-\frac{64-36}{36}$$

$$= (t-8/6)^2-100/36$$

$$= (t-8/6)^2-(10/6)^2$$

$$\int_0^{\pi/2} \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)$$

$$= 2/9\int_0^{\pi/2} \frac{dt}{(t-8/6)^2-(10/6)^2}$$

$$= 2/9\int_0^{\pi/2} \frac{dt}{(10/6)^2-(t-8/6)^2}$$

$$= 2/9 \cdot \frac{1}{2(10/6)}\log\left[\frac{10/6+(t-8/6)}{10/6-(t-8/6)}\right]_0^{\pi/2}$$

$$= \cancel{8}^{1}/\cancel{9}_{3} \cdot \left(\cancel{8}^{1}/\cancel{10}_{5}\right) \log \left[\frac{10+6t-8/6}{10-6t+8/6}\right]_0^{\pi/2}$$

$$= \frac{1}{15} \log \left(\frac{6t+2}{18-6t}\right)$$

$$\therefore t = \tan(x/2)$$

$$= \frac{1}{15} \log \left[\frac{6(\tan x/2)+2}{18-6(\tan x/2)}\right]_0^{\pi/2}$$

cos x

$$= \frac{1}{15} \log \left[(\log 6 \tan x/2 + 2 - \log(18-6\tan x/2)\right]$$

$$= \frac{1}{15}\left[\left(\log 6 \tan \frac{\pi/2}{2} + 2\right) - \left(\log 18 - 6 \tan \frac{\pi/2}{2}\right) - \left(\log 6 \tan 0/2 + 2\right) - \left(\log 18 - 6 \tan 0/2\right)\right]$$

$$= \frac{1}{15}\left\{\log\left(6\tan\frac{\pi}{4}+2\right) - \log\left(18-6\tan\frac{\pi}{4}\right) - \log(6(0)+2) - \log(18-6(0))\right\}$$

$$= \frac{1}{15}\left\{\log 6(1) + \log 2 - \log 18 + \log 6(1) - \log 2 + \log 18\right\}$$

$$= \frac{1}{15}\left\{\log 6(1) + \log 2 - \log 18 + \log 6(1) - \log 2 + \log 18\right\}$$

$$= \frac{1}{15}\left\{\log 6 + \log 6\right\}$$

$= 1/15 \{\log 12\}$

$= \dfrac{\log 12}{15}$

Hence proved.