Hence from equation ① & ②

We have,

$$f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y).$$

1.3 F Theorem:

If $f : A \to B$ and if $X \subset B$, $Y \subset B$ then

$$f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y).$$

ie, The inverse image of the intersection of two sets is the intersection of the inverse images.

Proof: Given: $f : A \to B$ and

$X \subset B$, $Y \subset B$.

To prove that:

$$f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y).$$

ie, we have to prove that,

$$f^{-1}(X \cap Y) \subset f^{-1}(X) \cap f^{-1}(Y) \rightarrow ①$$

& $$f^{-1}(X) \cap f^{-1}(Y) \subset f^{-1}(X \cap Y) \rightarrow ②$$

Suppose $a \in f^{-1}(X \cap Y)$.

Then $f(a) \in X \cap Y$.

Hence, both $f(a) \in X$ and $f(a) \in Y$.

So that both $a \in f^{-1}(X)$ and $a \in f^{-1}(Y)$.

It is clear that,

$$a \in f^{-1}(X) \cap f^{-1}(Y)$$

$$\therefore f^{-1}(X \cap Y) \subset f^{-1}(X) \cap f^{-1}(Y)$$

it satisfies the condition ①.

Conversely, let $b \in f^{-1}(X) \cap f^{-1}(Y)$,

then both $b \in f^{-1}(X)$ and $b \in f^{-1}(Y)$.

Hence, both $f(b) \in X$

and $f(b) \in Y$.

So that, $f(b) \in X \cap Y$

Hence $b \in f^{-1}(X \cap Y)$

$\therefore f^{-1}(X) \cap f^{-1}(Y) \subset f^{-1}(X \cap Y)$

it satisfies the condition ②.

From equation ① & ②, we get

$$f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$$

Hence proved.

1.3 G Theorem:

If $f : A \rightarrow B$ and

$X \subset A$, $Y \subset A$, then

$$f(X \cup Y) = f(X) \cup f(Y).$$

ie, The image of the union of two sets is the union of the images.

Proof: Give: $f : A \rightarrow B$ and

$X \subset A$, $Y \subset A$.

To prove that: $f(X \cup Y) = f(X) \cup f(Y)$

ie, We have to prove that

$$f(X \cup Y) \subset f(X) \cup f(Y) \rightarrow ①$$

& $$f(X) \cup f(Y) \subset f(X \cup Y) \rightarrow ②$$

Let $b \in f(X \cup Y)$,

then $b = f(a)$ for some

$a \in X \cup Y$.

Either $a \in X$ or $a \in Y$.

Thus either $b \in f(X)$ or

$b \in f(Y)$.

Hence $b \in f(X) \cup f(Y)$,

it is clear that;

$f(X \cup Y) \subset f(X) \cup f(Y)$

it satisfies the condition ①.

Conversely, if $c \in f(X) \cup f(Y)$

then either $c \in f(X)$ or

$c \in f(Y)$.

Then c is the image

of some point in X or

c is the image of some

point in Y.

Hence c is the image

of some point in $X \cup Y$.

i.e, $c \in f(X \cup Y)$.

Hence, $f(X) \cup f(Y) \subset f(X \cup Y)$

it satisfies the condition ②.

From equation ① & ②, we get

$f(X \cup Y) = f(X) \cup f(Y)$.

Note :. 1.3 H

$f(X \cap Y) = f(X) \cap f(Y)$

for $X \subset A$, $Y \subset A$ this

relation is need not hold.

1.3 I Defn: Composition of Function:

If $f: A \to B$ & $g: B \to C$, then define the function $g \circ f$ by,

$$[g \circ f](x) = g[f(x)] \quad (x \in A).$$

ie, The image of x under $g \circ f$ is defined to be the image of $f(x)$ under g.

The function $g \circ f$ is called the composition of function f with g.

Ex: if $f(x) = 1 + \sin x \ (-\infty < x < \infty)$,

$g(x) = x^2 \ (0 \leq x < \infty)$

then $[g \circ f](x) = g[f(x)]$

$= g[1 + \sin x]$

$= (1 + \sin x)^2$

$= 1 + \sin^2 x + 2\sin x$

$\therefore [g \circ f](x) = 1 + 2\sin x + \sin^2 x \ (-\infty < x < \infty)$

1.4 Real-Valued Functions:

Introduction: A Real-Valued function is a function whose values are real numbers.

ie, it is a function that assigns a real number to each member of its domain.

If X is any set, $f : X \to R$ then f is called as real valued function.

Ex: (1) $f(x) = \sqrt{2-x} + \sqrt{1+x}$,

where $\sqrt{f(x)}$, $f(x) \geq 0$

$2 - x \geq 0$ and $1 + x \geq 0$

$x \geq 2$ and $x \geq -1$

$\Rightarrow -1 \leq x \leq 2$

$\therefore x \in [-1, 2]$

Domain is $[-1, 2]$.

(2) $f(x) = \dfrac{1}{6x - x^2 - 5}$ where $6x - x^2 - 5 \neq 0$

$6x - x^2 - 5 \neq 0$

$-x^2 + 6x - 5 \neq 0$

$-x^2 + 5x + x - 5 \neq 0$

$-x(x-5) + 1(x-5) \neq 0$

$(x-5)(1-x) \neq 0$

$x - 5 \neq 0$, $1 - x \neq 0$

$x \neq 5$ $\quad -x \neq -1$

$x \neq 1$

$\therefore x \neq 1, 5$

Domain is $R - \{1, 5\}$

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1.4 B Defn:

If $f : A \to R$ and $g : A \to R$, we define $f+g$ as the function whose value at $x \in A$ is equal to $f(x)+g(x)$.

ie, $(f+g)(x) = f(x)+g(x)$ $[x \in A]$

In set notation we have

$$(f+g) = \{\langle x, f(x)+g(x)\rangle / x \in A\}$$

and $(f+g) : A \to R$.

Similarly, $(f-g)(x) = f(x) - g(x)$ $[x \in A]$

$$(fg)(x) = f(x)\, g(x) \quad [x \in A]$$

if $g(x) \neq 0$, for all $x \in A$ then

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad [x \in A]$$

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1.4 C Defn:

If $f : A \to R$ and c is a real number ($c \in R$), the function cf is defined by

$$(cf)(x) = c\,[f(x)] \quad [x \in A]$$

Ex:

The value of $3f$ at x is 3 times the value of f at x.

ie, $(3f)(x) = 3f(x)$

1.4 D Note:

For a, b real numbers, let $max(a,b)$ denote the larger and $min(a,b)$ denote the smaller of a and b.

Defn: If $f : A \to R$, $g : A \to R$, then $\max(f, g)$ is the function defined by

$$\max(f,g)(x) = \max[f(x), g(x)] \quad x \in A.$$

and

$$\min(f,g)(x) = \min[f(x), g(x)] \quad x \in A.$$

Ex: If $f(x) = \sin x \; (0 \le x \le \pi/2)$

$g(x) = \cos x \; (0 \le x \le \pi/2)$

and $h(x) = \max(f, g)$

put $x = 0$ then $f(x) = \sin 0 = 0$
& $g(x) = \cos 0 = 1$

$\therefore h(x) = \max(0, 1) = 1$

$h(x) = \cos x \; [0 \le x \le \pi/4]$,

put $x = \pi/2$ then

$f(x) = \sin \pi/2 = 1$

$g(x) = \cos \pi/2 = 0$

$\therefore h(x) = \max(1, 0) = 1$

$h(x) = \sin x \; [\pi/4 < x \le \pi/2]$.

Defn: If $f : A \to R$, then $|f|$ is the function defined by

$$|f|(x) = |f(x)| \quad [x \in A].$$

Formulae: If a, b are real numbers, then (i)

$$\max(a,b) = \frac{|a-b| + a+b}{2}$$

$$\min(a,b) = \frac{-|a-b| + a+b}{2}$$

(ii) Above the formulae we have to follow them immediately:

$$\max(f,g) = \frac{|f-g| + f+g}{2}$$

$$\min(f,g) = \frac{-|f-g| + f+g}{2}$$

for real valued functions f, g.

1.4 E Defn:

If $A \subset S$, then χ_A is called the characteristic function of A is defined as:

$$\chi_A(x) = 1 \quad (x \in A),$$

$$\chi_A(x) = 0 \quad (x \in A').$$

It is clear that the set A is charaterized (completely described) by χ_A.

ie, $A = B$ iff $\chi_A = \chi_B$.

The following equations for characteristic functions:

where A, B are subsets of S.

1) $\chi_{A \cup B} = \max(\chi_A, \chi_B)$, $\Longrightarrow$ (1)

2) $\chi_{A \cap B} = \min(\chi_A, \chi_B) = \chi_A \chi_B$.

3) $\chi_{A-B} = \chi_A - \chi_B$ [provided $B \subset A$].

4) $\chi_{A'} = 1 - \chi_A$.

5) $\chi_S = 1$

6) $\chi_\phi = 0$.

1.8

Ex: characteristic function:

Suppose $x \in A \cup B$.

Then $\chi_{A\cup B}(x) = 1$.

But either $x \in A$ or $x \in B$ (or both)

and so either $\chi_A(x) = 1$ (or)

$$\chi_B(x) = 1$$

Thus $\max(\chi_A, \chi_B)(x) = 1$.

Hence,

$$1 = \chi_{A\cup B}(x) = \max(\chi_A, \chi_B)(x) \quad [x \in A\cup B] \rightarrow (2)$$

If $x \notin A \cup B$, i.e., $x \in (A\cup B)'$, then $\chi_{A\cup B}(x) = 0$.

But $x \in A' \cap B'$ ~~by~~ $[\because (A\cup B)' = A' \cap B']$

Hence, $x \in A'$ and $x \in B'$

So that $\chi_A(x) = 0 = \chi_B(x)$.

Thus $\max(\chi_A, \chi_B)(x) = 0$.

Hence,

$$0 = \chi_{A\cup B}(x) = \max(\chi_A, \chi_B)(x) \quad [x \notin A\cup B] \rightarrow (3)$$

$\therefore$ Equations (2) & (3) we have equation (1) is satisfied the characteristic function.

Sec 1.5 Equivalence, Countability

Introduction:

Let S be a set and r a relation between S and itself. We call 'r' an equivalence relation on S if r has the following three properties:

1. **Reflexivity**: Every element of S is related to itself.

2. Symmetry: If S is related to t then t is related to S.

3. Transitivity: If S is related to t and t is related to u, then S is related to u.

Example:

Let $A = \{1, 2, 3, 4\}$ and

$B = \{a, b, c\}$ and

define the following two relations:

1. $r : \{(a,a), (b,b), (a,b), (b,a)\}$
2. $S : 1 \sim 1, 2 \sim 2, 3 \sim 3,$
   $4 \sim 4, 1 \sim 4, 4 \sim 1,$
   $2 \sim 4, 4 \sim 2.$

Another Defn: Equivalence:

Let A, B be sets. It is clear that A is equivalent to B, or that A is equipotent with B, and we write $A \sim B$ if there exists a one-to-One, onto function $f : A \to B$.

Then the relation '$\sim$' is called equivalent the following three conditions are satisfied:

(i) For every set A, we have
$A \sim A$ (Reflexive ~~Symmetry~~)

(ii) If A, B are sets and
$A \sim B$, then $B \sim A$
(Symmetry)

(iii) If A, B, C are sets, if $A \sim B$ and $B \sim C$, then $A \sim C$. [Transitive].

Countability: (countable set)

A countable set is a set with the same cardinality (number of elements) as some subset of the set of natural numbers. A countable set is either a finite set or a countably infinite set.

Defn: Countable set:

A set 'S' is countable

if there exists an injective function f from S to the natural numbers $N = \{1, 2, \ldots\}$

If such an f can be found that is also surjective then S is called countably infinite.

In other words, a set is countably infinite if it has one-to-one correspondence with the natural number set N..

1.5 A Defn:

If $f : A \to B$, then f is called one-to-one if $f(a_1) = f(a_2)$ implies $a_1 = a_2$ $(a_1, a_2 \in A)$.

1.5 B Defn:

If $f : A \to B$ and f is 1-1, then the function f^{-1} is called the inverse function for 'f', is defined as:

if $f(a) = b$, then

$f^{-1}(b) = a$ [b in range of f]

Thus the domain of f^{-1} is the range of f and the range of f^{-1} is A.

Ex:

If $g(x) = x^2$ $(0 \le x < \infty)$, then $g^{-1}(x) = \sqrt{x}$ $(0 \le x < \infty)$.

For, if $b = g(a) = a^2$

then $a = \sqrt{b}$

$a = g^{-1}(b)$.

1.5 C Defn:

If $f : A \xrightarrow{onto} B$ and f is 1-1, then f is called a 1-1 correspondence between A and B.

If there exists a 1–1 correspondence between the sets A and B, then A and B are called equivalent, if its satisfies the ~~Symmetry~~, reflexive, symmetry and transitive conditions.

1.5 D Infinite Set

The set A is said to be infinite if for each positive integer n, A contains a subset with precisely n elements.

It is denoted by

$$I = \{1, 2, \ldots\}$$

is clearly an infinite set.

Ex.:

The set R of all real numbers is also an infinite set.

1.5 E Defn:

The set A is said to be countable (or denumerable) if A is equivalent to the set I of positive integers.

An uncountable set is an infinite set which is not countable.

Thus A is countable

if there exists a 1-1 function f from I onto A. The elements of A are then the images $f(1), f(2), \ldots$ of positive integers.

Ex:

The set of all integers is countable.

Thus, $0, -1, +1, -2, +2, \ldots$ they can be counted and define the function f by

$$f(n) = \frac{n-1}{2} \quad (n = 1, 3, \ldots)$$

$$f(n) = \frac{-n}{2} \quad (n = 2, 4, 6, \ldots)$$

1.5 F Theorem:

If $A_1, A_2, \ldots$ are countable sets, then $\bigcup_{n=1}^{\infty} A_n$ is countable.

ie, The ~~union~~ countable union of countable sets is countable.

Proof:

Let $A_1 = \{a_1^1, a_2^1, a_3^1, \ldots\}$,

$A_2 = \{a_1^2, a_2^2, a_3^2, \ldots\}$,

$\vdots$

$A_n = \{a_1^n, a_2^n, a_3^n, \ldots\}$,

So that a_k^j is the k^{th} element of the set A_j.